Chapter 2(part 2)
Transformations

Lesson Package

MCR3U

$$g(x) = af[k(x - d)] + c$$

Chapter 2 (part 2) Outline

Unit Goal: Be able to demonstrate an understanding of functions, their representations, and their inverses, and make connections between the algebraic and graphical representations of functions using **transformations.**

Section	Subject	Learning Goals	Curriculum Expectations
L1	Intro to Transformations	- understand the roles of parameters a k, d, and c in functions of the form $f(x) = af[k(x-d)] + c$	A1.8, A1.9
L2	Transformations of x^2	- apply understanding of the roles of parameters a k, d, and c in transformation of the graph of $f(x) = x^2$	A1.8, A1.9
L3	Transformations of $\sqrt{x}$	- apply understanding of the roles of parameters a k, d, and c in transformation of the graph of $f(x) = \sqrt{x}$	A1.8, A1.9
L4	Transformations of $\frac{1}{x}$	- apply understanding of the roles of parameters a k, d, and c in transformation of the graph of $f(x) = \frac{1}{x}$	A1.8, A1.9
L5	Inverse of a Function	- Be able to algebraically determine the equation of the inverse of a function. - Understand the relationship between the domain and range of a function and the domain and range of its inverse.	A1.4, A1.5, A1.6, A1.7

Assessments	**F/A/O**	**Ministry Code**	**P/O/C**	**KTAC**
Note Completion	A		P	
Practice Worksheet Completion	F/A		P	
PreTest Review	F/A		P	
Test - Functions	O	A1.1, A1.2, A2.1, A2.2, A2.3, A2.5, A3.2	P	K(21%), T(34%), A(10%), C(34%)

Graphs of Common Functions

and

Intro to Transformations

In this chapter you will learn about transformations of functions. There are three main functions that we will use to learn about transformations:

1. $f(x) = x^2$ (quadratic functions)

2. $f(x) = \sqrt{x}$ (radical or square root functions)

3. $f(x) = \dfrac{1}{x}$ (rational functions)

Note: *the equations given for each type of function are considered the base or parent functions of their respective families of functions. All transformations of these functions will be compared to these base functions.*

Before learning about transformations, you must understand what the base functions look like and be able to generate the key points for the graph of each function.

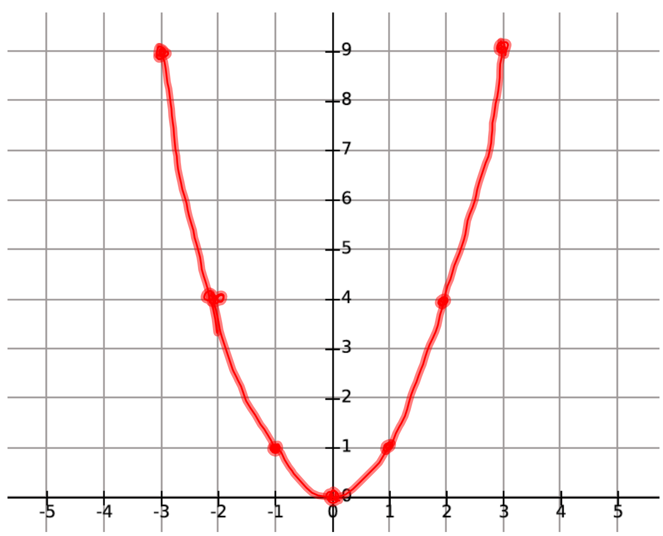
Quadratic Functions

Base Function: $f(x) = x^2$

Key Points:

x	y $f(x)$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

Graph of Base Function



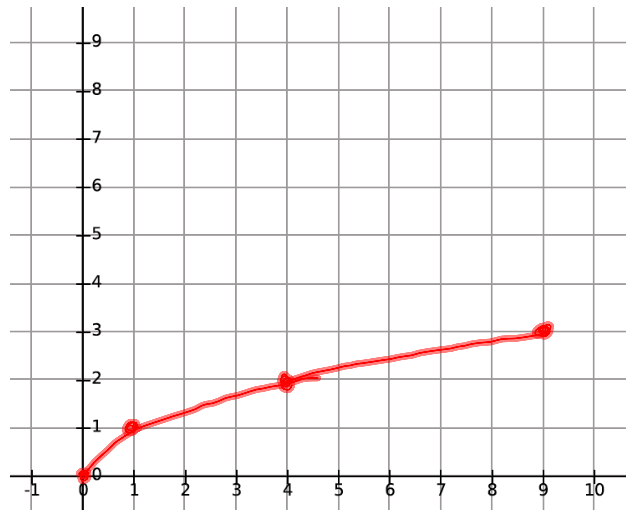
Radical (square root) Functions

Base Function: $f(x) = \sqrt{x}$

Graph of Base Function

Key Points:

x	y
0	0
1	1
4	2
9	3



Rational Functions

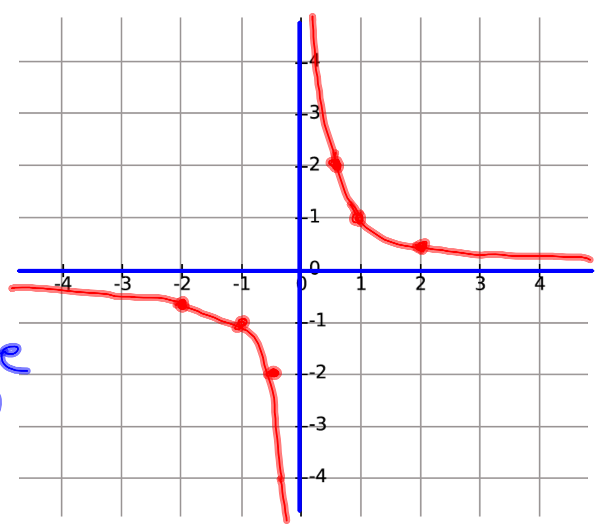
Base Function: $f(x) = \frac{1}{x}$

Graph of Base Function

Key Points:

x	y
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
0	undefined
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$

vertical asymptote at $x=0$



Asymptotes

Asymptote: *a line that a curve approaches more and more closely but never touches.*

The function $f(x) = \frac{1}{x}$ has two asymptotes:

Vertical Asymptote: Division by zero is undefined. Therefore the expression in the denominator of the function can not be zero. Therefore $x \neq 0$. This is why the vertical line $x = 0$ is an asymptote for this function.

Horizontal Asymptote: For the range, there can never be a situation where the result of the division is zero. Therefore the line $y = 0$ is a horizontal asymptote. For all functions where the denominator is a higher degree than the numerator, there will by a horizontal asymptote at $y = 0$.

Transformations of Functions

Transformation:

A change made to a figure or a relation such that the figure or the graph of the relation is shifted or changed in shape.

Translations, stretches, and reflections are types of transformations.

The general function:

$f(x)$ parent function you are transforming

$$g(x) = af[k(x - d)] + c$$

a transformed function

takes f(x) and performs transformations to it

Changes to the *y*-coordinates (vertical changes)

c: vertical translation $\quad g(x) = f(x) + c$

The graph of $g(x) = f(x) + c$ is a vertical translation of the graph of $f(x)$ by c units.

If c > 0, the graph shifts **up**
If c < 0, the graph shifts **down**

a: vertical stretch/compression $\quad g(x) = af(x)$

The graph of $g(x) = af(x)$ is a vertical stretch or compression of the graph of $f(x)$ by a factor of a.

If $a > 1$ or $a < -1$, **vertical stretch** by a factor of a.
If $-1 < a < 1$, **vertical compression** by a factor of a.
If $a < 0$, **vertical reflection** (reflection over the x-axis)

Note: a vertical stretch or compression means that distance from the x-axis of each point of the parent function changes by a factor of a.

Note: for a vertical reflection, the point (x, y) becomes point $(x, -y)$

Changes to the *x*-coordinates (horizontal changes)

d: horizontal translation $\quad g(x) = f(x - d)$

The graph of $g(x) = f(x - d)$ is a horizontal translation of the graph of $f(x)$ by d units.

If d > 0, the graph shifts **right**
If c < 0, the graph shifts **left**

k: horizontal stretch/compression

The graph of $g(x) = f(kx)$ is a horizontal stretch or compression of the graph of $f(x)$ by a factor of $\frac{1}{k}$

If $k > 1$ or $k < -1$, **compressed horizontally** by a factor of $\frac{1}{k}$
If $-1 < k < 1$, **stretched horizontally** by a factor of $\frac{1}{k}$
If $k < 0$, **horizontal reflection** (reflection in the y-axis)

Note: a vertical stretch or compression means that distance from the y-axis of each point of the parent function changes by a factor of $1/k$.

Note: for a horizontal reflection, the point (x, y) becomes point $(-x, y)$

Order of Transformations

1. stretches, compressions, reflections

2. translations

$a \rightarrow k \rightarrow d \rightarrow c$

Example 1: List the transformations and the order in which they should be done to a function $f(x)$.

a) $g(x) = -f(x)$ $a = -1$

vertical reflection (change sign of all y-values)

b) $g(x) = 2f(1/3x)$ $a = 2$ $k = \frac{1}{3}$

vertical stretch by a factor of 2 (multiply y-coordinates by 2)

horizontal stretch by a factor of 3 (multiply x-coordinates by 3)

c) $g(x)=3f(x+2)-1$ $a=3$ $d=-2$ $c=-1$

vertical stretch by a factor of 3 (multiply y-coordinates by 3)

shift left 2 units (x-coordinates - 2)

shift down 1 unit (y-coordinates - 1)

d) $g(x)=1/4\,f[2(x-1)]$ $a=\frac{1}{4}$ $k=2$ $d=1$

vertical compression by a factor of 1/4 (divide y-coordinates by 4)

horizontal compression by a factor of 1/2 (divide x-coordinates by 2)

shift right 1 unit (x-coordinates plus 1)

e) $g(x)= -5f[-1/4(x+2)]+7$ $a=-5$ $k=-\frac{1}{4}$

$d=-2$ $c=7$

vertical stretch by a factor of 5 (multiply y-coordinates by 5)

vertical reflection (change sign of y-coordinates)

horizontal stretch by a factor of 4 (multiply x-coordinates by 4)

horizontal reflection (change sign of x-coordinates)

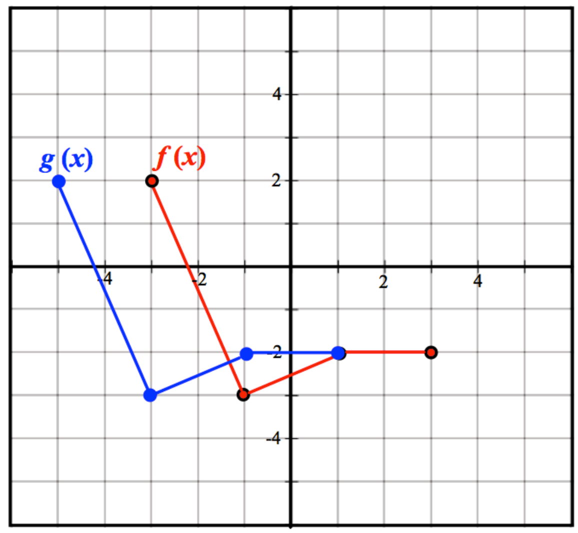
shift left two units (x-coordinates - 2)

shift up 7 units (y-coordinates + 7)

Example 2: List the transformations and the order in which they should be done to the function $f(x)$. Use the given graph of $f(x)$ to sketch the graph of $g(x)$

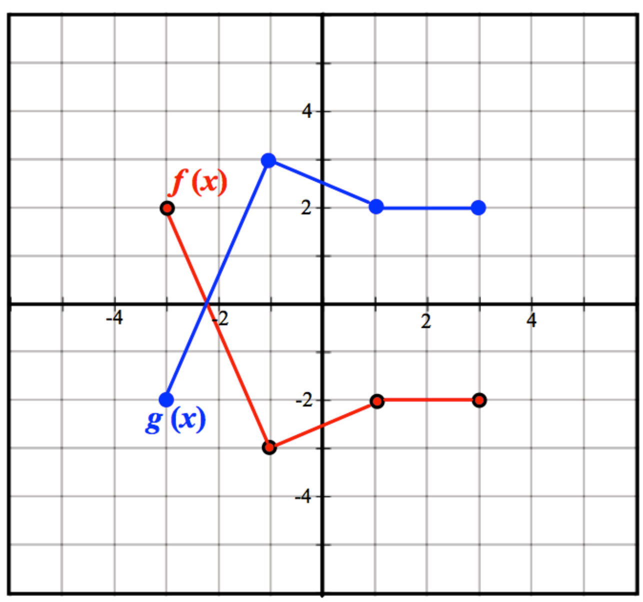
a) $g(x) = f(x+2)$

shift left 2 units



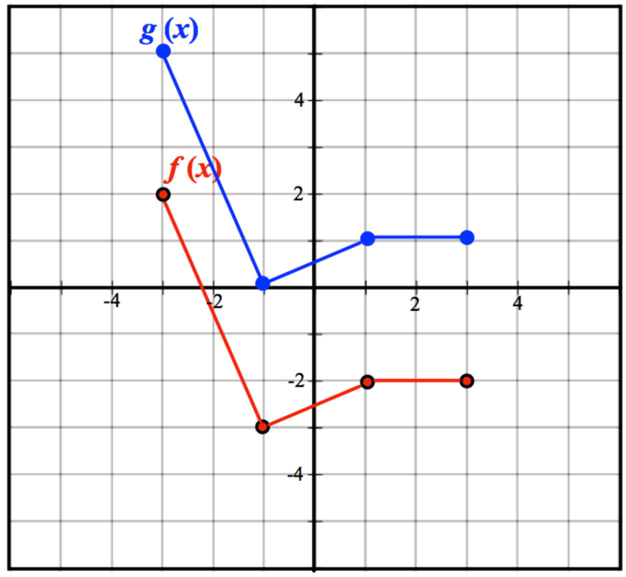
b) $g(x) = -f(x)$

vertical reflection over the x-axis



c) $g(x) = f(x) + 3$

shift up 3 units



d) $g(x) = f(2x) - 1$

It may help to make a list of ***image points*** *(any point that has been transformed from a point on the original figure or graph)*

horizontal compression by a factor of 1/2 (x/2)

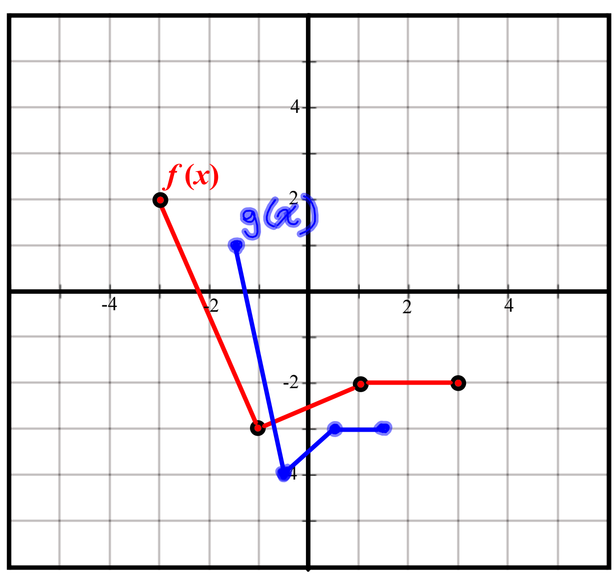
shift down 1 unit (y - 1)

$\boxed{f(x)} \longrightarrow \boxed{g(x)}$

x	y
-3	2
-1	-3
1	-2
3	-2

$\frac{x}{2}$	$y-1$
-1.5	1
-0.5	-4
0.5	-3
1.5	-3

↑
image points



Transformations of Quadratic Functions

Transformations of Functions

Transformation:

A change made to a figure or a relation such that the figure or the graph of the relation is shifted or changed in shape.

Translations, stretches, and reflections are types of transformations.

The general function:

$f(x)$ parent function you are transforming

$$g(x) = af[k(x-d)] + c$$

a transformed function

takes f(x) and performs transformations to it

Changes to the *y*-coordinates (vertical changes)

c: vertical translation $\quad g(x) = f(x) + c$

The graph of $g(x) = f(x) + c$ is a vertical translation of the graph of $f(x)$ by c units.

If c > 0, the graph shifts **up**
If c < 0, the graph shifts **down**

a: vertical stretch/compression $\quad g(x) = af(x)$

The graph of $g(x) = af(x)$ is a vertical stretch or compression of the graph of $f(x)$ by a factor of a.

If $a > 1$ or $a < -1$, **vertical stretch** by a factor of a.
If $-1 < a < 1$, **vertical compression** by a factor of a.
If $a < 0$, **vertical reflection** (reflection over the x-axis)

Note: a vertical stretch or compression means that distance from the x-axis of each point of the parent function changes by a factor of a.

Note: for a vertical reflection, the point (x, y) becomes point $(x, -y)$

Changes to the *x*-coordinates (horizontal changes)

d: horizontal translation $\quad g(x) = f(x - d)$

The graph of $g(x) = f(x - d)$ is a horizontal translation of the graph of $f(x)$ by d units.

If d > 0, the graph shifts **right**
If c < 0, the graph shifts **left**

k: horizontal stretch/compression

The graph of $g(x) = f(kx)$ is a horizontal stretch or compression

of the graph of $f(x)$ by a factor of $\frac{1}{k}$

If $k > 1$ or $k < -1$, **compressed horizontally** by a factor of $\frac{1}{k}$
If $-1 < k < 1$, **stretched horizontally** by a factor of $\frac{1}{k}$
If $k < 0$, **horizontal reflection** (reflection in the y-axis)

Note: a vertical stretch or compression means that distance from the y-axis of each point of the parent function changes by a factor of $1/k$.

Note: for a horizontal reflection, the point (x, y) becomes point $(-x, y)$

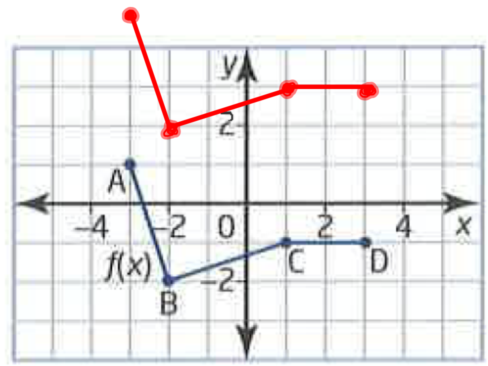
DO IT NOW!

a) Complete the table of values for the function $f(x)$ and $g(x)$. Then use the table of values to plot image points and graph the function $g(x)$

(x, y) $(x, y+4)$

$f(x) : (x, f(x))$	$g(x) : (x, f(x) + 4)$
A(-3, 1)	A'(-3, 5)
B(-2, -2)	B'(-2, 2)
C(1, -1)	C'(1, 3)
D(3, -1)	D'(3, 3)

↑ base function ↑ image points



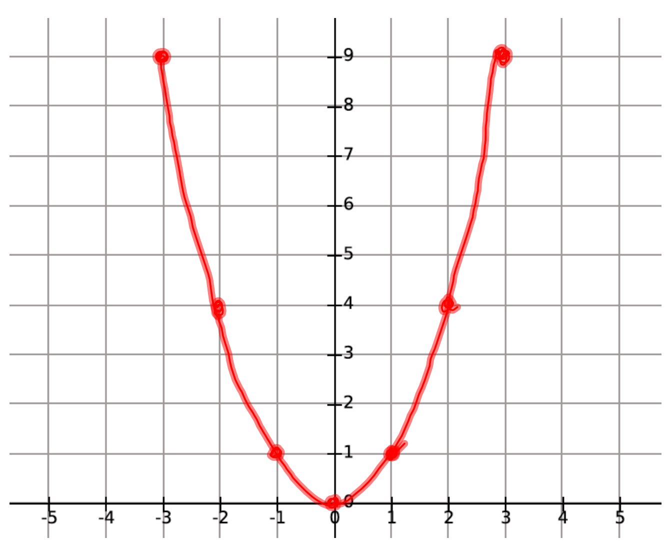
Quadratic Functions

Base Function:

Key Points:

x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

Graph of Base Function



<u>Order of Transformations</u>

1. stretches, compressions, reflections

2. translations

$a \rightarrow k \rightarrow d \rightarrow c$

Example 1: If $f(x) = x^2$, describe the changes and write the transformed function:

a) $g(x) = 2f(x)$

vertical stretch b.a.f.o. 2

$g(x) = 2x^2$

b) $g(x) = f(2x)$

horizontal compression b.a.f.o. $\frac{1}{2}$

$g(x) = (2x)^2$

c) $g(x) = f(x)+4$

shift up 4 units

$g(x) = x^2+4$

d) $g(x)=f(x+3)$

shift left 3 units

$g(x) = (x+3)^2$

e) $g(x) = -f(x)$

vertical reflection

(flip over x-axis)

$g(x) = -x^2$

f) $g(x) = f(-x)$

horizontal reflection

(flip over y-axis)

$g(x) = (-x)^2$

Example 2: For each of the following functions, describe the transformations to $f(x) = x^2$ in order and write the transformed equation.

a) $g(x)=-2\,f\,\overbrace{[\,-3(x + 3)]}\,-1$ plug in for x into x^2

vertical stretch by a factor of 2

vertical reflection

horizontal compression by a factor of 1/3

horizontal reflection

shift left 3 units and down 1 unit

$g(x) = -2[-3(x+3)]^2 - 1$

b)

$$y = \tfrac{1}{2} f[-3(x-2)] + 5$$

vertical compression by a factor of 1/2

horizontal compression by a factor of 1/3

horizontal reflection

shift right 2 units and up 5 units

$$g(x) = \frac{1}{2}[-3(x-2)]^2 + 5$$

Example 3: for each of the following functions…

i) make a table of values for the parent function

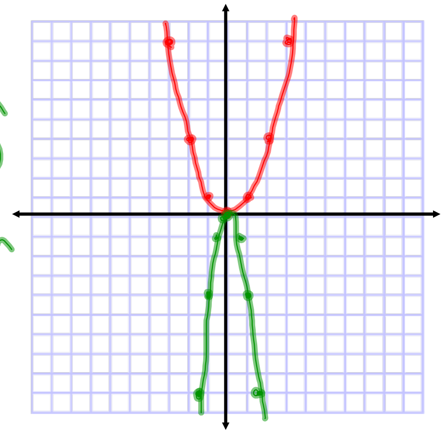
ii) graph the parent function $f(x) = x^2$

iii) describe the transformations

iv) make a table of values of image points

v) graph the transformed function and write it's equation

a) $g(x) = -f(2x)$



$a = -1$; vertical reflection (multiply y-values by -1)

$k = 2$; horizontal compression by a factor of $\frac{1}{2}$ (divide x-values by 2)

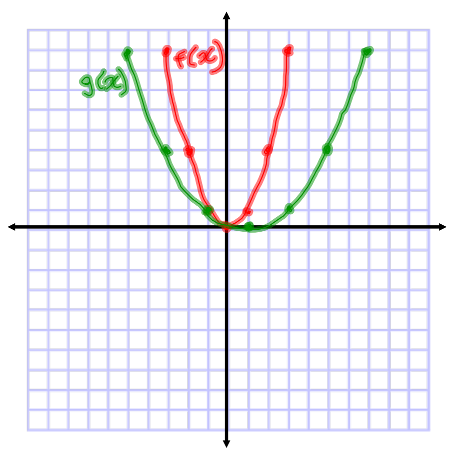
$f(x) = x^2$

x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

$g(x) = -(2x)^2$

$\frac{x}{2}$	$-y$
-1.5	-9
-1	-4
-0.5	-1
0	0
0.5	-1
1	-4
1.5	-9

b) $g(x) = f[-1/2(x-1)]$



horizontal stretch bafo 2 $(2x)$

horizontal reflection $(-x)$

shift right 1 unit. $(x+1)$

$f(x) = x^2$

x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

$g(x) = [-\frac{1}{2}(x-1)]^2$

$-2x+1$	y
7	9
5	4
3	1
1	0
-1	1
-3	4
-5	9

c) $g(x) = -2f[-3(x+3)] - 1$

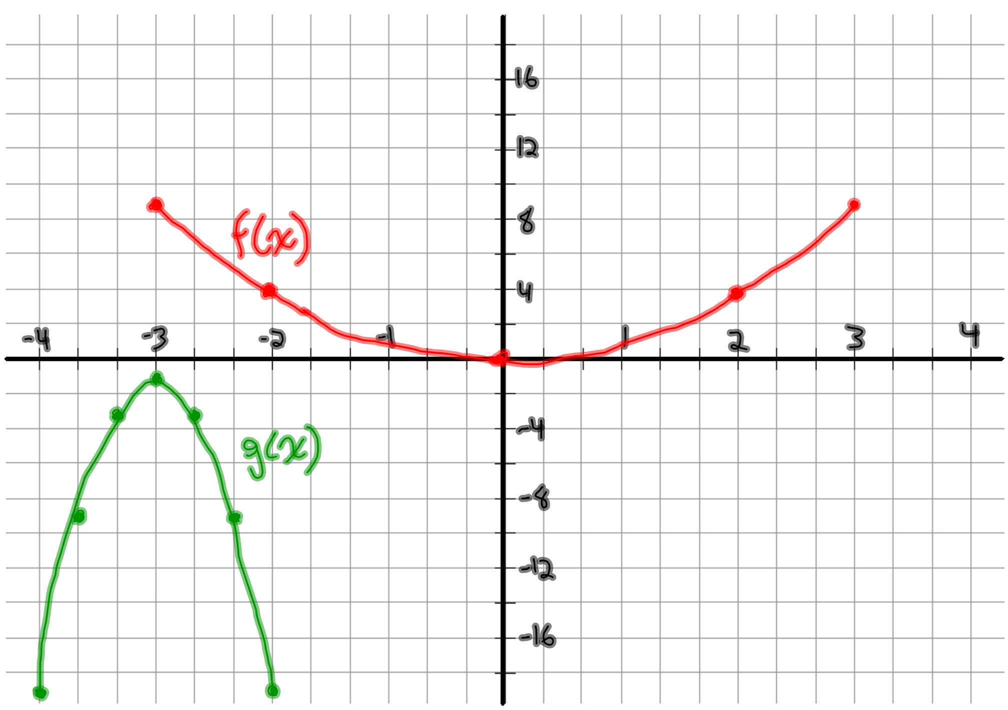
vertical stretch bafo 2. $(2y)$
vertical reflection $(-y)$
horizontal compression bafo $\frac{1}{3}$ $(\frac{x}{3})$
horizontal reflection $(-x)$
shift left 3 units and down 1. $(x-3)$ $(y-1)$

$f(x) = x^2$

x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

$g(x) = -2[-3(x+3)]^2 - 1$

$\frac{-x}{3} - 3$	$-2y-1$
-2	-19
-2.3	-9
-2.67	-3
-3	-1
-3.3	-3
-3.67	-9
-4	-19



Complete Worksheet

Transformations of $\sqrt{x}$

<u>Transformations of Functions</u>

Transformation:

A change made to a figure or a relation such that the figure or the graph of the relation is shifted or changed in shape.

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The general function:

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Changes to the *y*-coordinates (vertical changes)

c: vertical translation $g(x) = f(x) + c$

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a: vertical stretch/compression $g(x) = af(x)$

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Note: a vertical stretch or compression means that distance from the x-axis of each point of the parent function changes by a factor of a.

Note: for a vertical reflection, the point (x, y) becomes point $(x, -y)$

Changes to the *x*-coordinates (horizontal changes)

d: horizontal translation $g(x) = f(x - d)$

The graph of $g(x) = f(x - d)$ is a horizontal translation of the graph of $f(x)$ by d units.

If d > 0, the graph shifts **right**
If d < 0, the graph shifts **left**

k: horizontal stretch/compression

The graph of $g(x) = f(kx)$ is a horizontal stretch or compression

of the graph of $f(x)$ by a factor of $\frac{1}{k}$

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If $k < 0$, **horizontal reflection** (reflection in the y-axis)

Note: a vertical stretch or compression means that distance from the y-axis of each point of the parent function changes by a factor of $1/k$.

Note: for a horizontal reflection, the point (x, y) becomes point $(-x, y)$

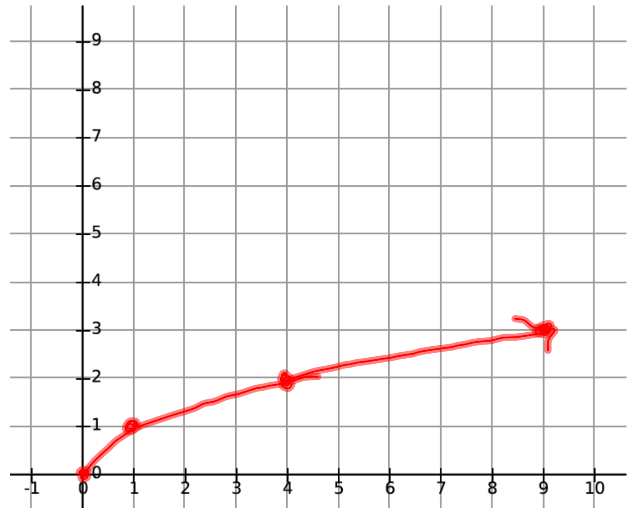
Radical (square root) Functions

Base Function: $f(x) = \sqrt{x}$

Graph of Base Function

Key Points:

x	y
0	0
1	1
4	2
9	3



Example 1: Using the parent function $f(x) = \sqrt{x}$, describe the transformations and write the equation of the transformed function $g(x)$.

$$g(x) = -2\, f\left[-\tfrac{1}{3}(x+6)\right] - 5$$

- *Vertical stretch bafo 2*
- *vertical reflection (reflection across the x-axis)*
- *Horizontal stretch bafo 3*
- *Horizontal reflection (reflection across the y-axis)*
- *Phase shift 6 units left*
- *Translate 5 units down*

$$g(x) = -2\sqrt{-\frac{1}{3}(x+6)} - 5$$

Example 2: for each of the following functions…

i) make a table of values for the parent function

ii) graph the parent function $f(x) = \sqrt{x}$

iii) describe the transformations

iv) make a table of values of image points

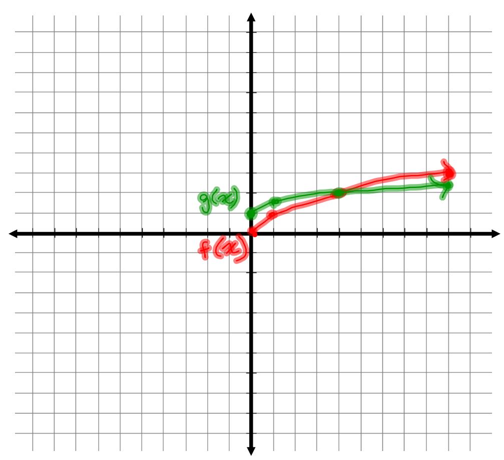
v) graph the transformed function and write it's equation

a)

$$g(x) = \frac{1}{2}f(x) + 1$$

vertical compression by a factor of $\frac{1}{2}$ $\left(\frac{y}{2}\right)$

shift up 1 unit $(y+1)$



$f(x) = \sqrt{x}$

x	y
0	0
1	1
4	2
9	3

$g(x) = \frac{1}{2}\sqrt{x} + 1$

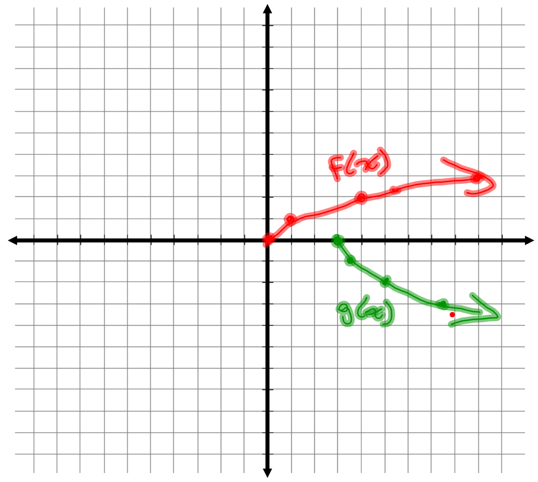
x	$\frac{y}{2}+1$
0	1
1	1.5
4	2
9	2.5

b) $g(x) = -f[2(x-3)]$

vertical reflection $(-y)$

horizontal compression by a factor of $\frac{1}{2}$ $\left(\frac{x}{2}\right)$

shift right 3 units $(x+3)$



$f(x) = \sqrt{x}$

x	y
0	0
1	1
4	2
9	3

$g(x) = -\sqrt{2(x-3)}$

$\frac{x}{2}+3$	$-y$
3	0
3.5	-1
5	-2
7.5	-3

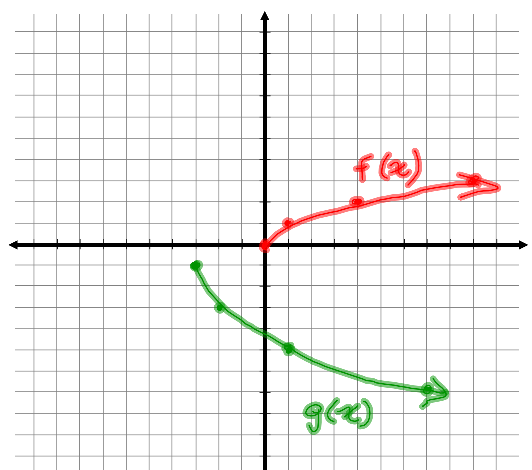
c) $g(x) = -2f(x+3) - 1$

vertical stretch by a factor of 2 $(2y)$

vertical reflection $(-y)$

shift left 3 units $(x-3)$

shift down 1 unit $(y-1)$



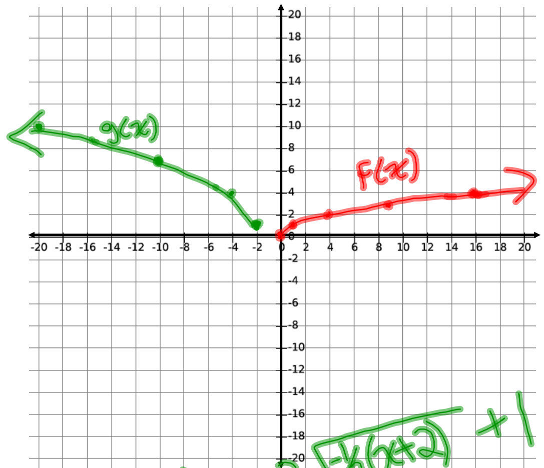
$f(x) = \sqrt{x}$

x	y
0	0
1	1
4	2
9	3

$g(x) = -2\sqrt{x+3} - 1$

$x-3$	$-2y-1$
-3	-1
-2	-3
1	-5
6	-7

d) $g(x) = 3f\left[-\frac{1}{2}(x+2)\right] + 1$



$g(x) = -3\sqrt{-\frac{1}{2}(x+2)} + 1$

vertical stretch by a factor of 3 (3y)

horizontal stretch by a factor of 2 (2x)

horizontal reflection (-x)

shift left 2 units (x-2)

shift up 1 unit (y+1)

$f(x) = \sqrt{x}$

x	y
0	0
1	1
4	2
9	3

$-2x-2$	$3y+1$
-2	1
-4	4
-10	7
-20	10

Complete Worksheet

Transformations of $f(x) = \frac{1}{x}$

Transformations of Functions

Transformation:

A change made to a figure or a relation such that the figure or the graph of the relation is shifted or changed in shape.

Translations, stretches, and reflections are types of transformations.

The general function:

$f(x)$ parent function you are transforming

$$g(x) = af[k(x - d)] + c$$

a transformed function

takes f(x) and performs transformations to it

Changes to the *y*-coordinates (vertical changes)

c: vertical translation $g(x) = f(x) + c$

The graph of $g(x) = f(x) + c$ is a vertical translation of the graph of $f(x)$ by c units.

If $c > 0$, the graph shifts **up**
If $c < 0$, the graph shifts **down**

a: vertical stretch/compression $g(x) = af(x)$

The graph of $g(x) = af(x)$ is a vertical stretch or compression of the graph of $f(x)$ by a factor of a.

If $a > 1$ or $a < -1$, **vertical stretch** by a factor of a.
If $-1 < a < 1$, **vertical compression** by a factor of a.
If $a < 0$, **vertical reflection** (reflection over the x-axis)

Note: a vertical stretch or compression means that distance from the x-axis of each point of the parent function changes by a factor of a.

Note: for a vertical reflection, the point (x, y) becomes point $(x, -y)$

Changes to the *x*-coordinates (horizontal changes)

d: horizontal translation $g(x) = f(x - d)$

The graph of $g(x) = f(x - d)$ is a horizontal translation of the graph of $f(x)$ by d units.

If $d > 0$, the graph shifts **right**
If $d < 0$, the graph shifts **left**

k: horizontal stretch/compression

The graph of $g(x) = f(kx)$ is a horizontal stretch or compression of the graph of $f(x)$ by a factor of $\frac{1}{k}$

If $k > 1$ or $k < -1$, **compressed horizontally** by a factor of $\frac{1}{k}$
If $-1 < k < 1$, **stretched horizontally** by a factor of $\frac{1}{k}$
If $k < 0$, **horizontal reflection** (reflection in the y-axis)

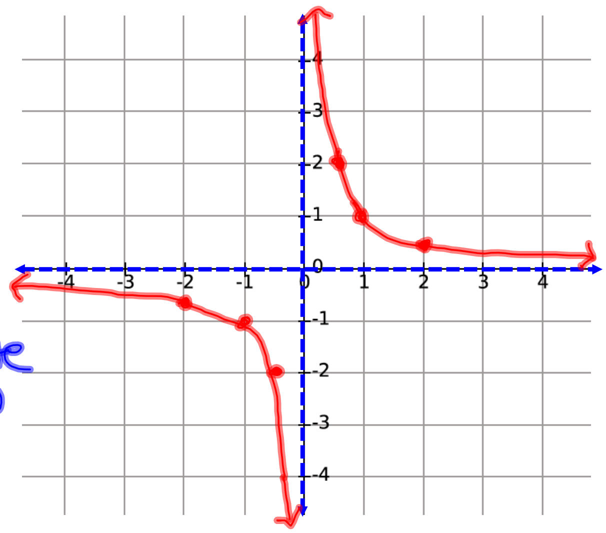
Note: a vertical stretch or compression means that distance from the y-axis of each point of the parent function changes by a factor of $1/k$.

Note: for a horizontal reflection, the point (x, y) becomes point $(-x, y)$

Rational Functions

Base Function: $f(x) = \frac{1}{x}$

Graph of Base Function



Key Points:

x	y
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
0	undefined
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$

vertical asymptote at $x=0$

Asymptotes

Asymptote: *a line that a curve approaches more and more closely but never touches.*

The function $f(x) = \frac{1}{x}$ has two asymptotes:

Vertical Asymptote: Division by zero is undefined. Therefore the expression in the denominator of the function can not be zero. Therefore $x \neq 0$. This is why the vertical line $x = 0$ is an asymptote for this function.

Horizontal Asymptote: For the range, there can never be a situation where the result of the division is zero. Therefore the line $y = 0$ is a horizontal asymptote. For all functions where the denominator is a higher degree than the numerator, there will by a horizontal asymptote at $y = 0$.

Example 1: Describe the combination of transformations that must be applied to the base function $f(x) = \frac{1}{x}$ to obtain the transformed function. Then, write the corresponding equation.

a) $g(x)=4f(x - 3) + 0.5$

vertical stretch by a factor of 4

shift right 3 units

shift up 0.5 units

$$g(x) = 4\left(\frac{1}{x-3}\right) + 0.5$$

$$\downarrow$$

$$g(x) = \frac{4}{x-3} + 0.5$$

b) $g(x)= f[-2(x + 0.5)] - 1$

horizontal compression by a factor of 1/2

horizontal reflection

shift left 0.5 units

shift down 1 unit

$$g(x) = \frac{1}{-2(x+0.5)} - 1$$

Example 2: for each of the following functions…

i) make a table of values for the parent function $f(x) = 1/x$

ii) describe the transformations

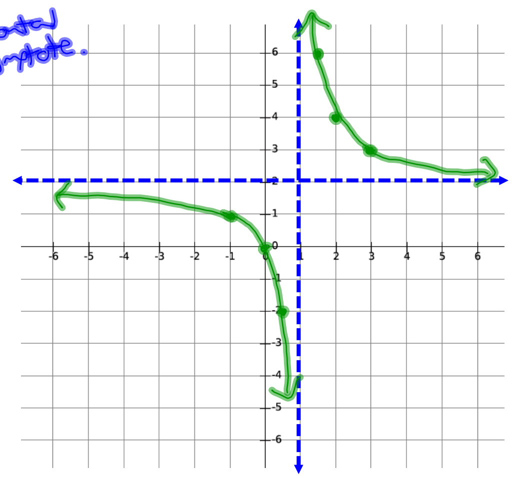
iii) make a table of values of image points

iv) graph the transformed function and write it's equation

a) $g(x)=2f(x - 1) + 2$

horizontal asymptote.

vertical asymptote



vertical stretch by a factor of 2 $(2y)$

shift right 1 unit $(x+1)$

shift up 2 units $(y+2)$

$f(x) = \frac{1}{x}$

x	y
-2	-½
-1	-1
-½	-2
0	undefined
½	2
1	1
2	½

$x+1$	$2y+2$
-1	1
0	0
0.5	-2
1	undefined
1.5	6
2	4
3	3

$g(x) = 2\left(\frac{1}{x-1}\right)+2$

↓

$g(x) = \frac{2}{x-1} + 2$

b) $g(x) = -f[2(x+0.5)] - 1$ ← horizontal asymptote

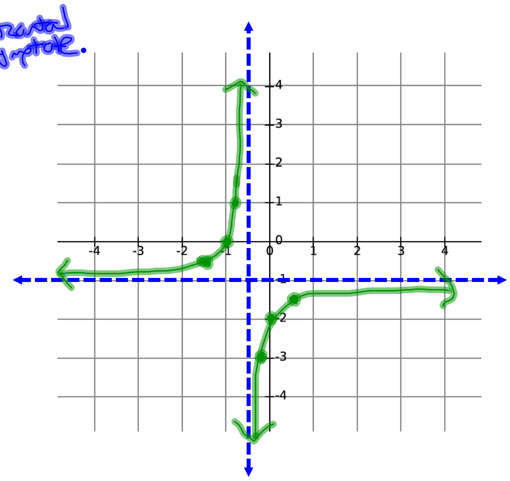
↖ vertical asymptote

vertical reflection $(-y)$

horizontal compression by a factor of 1/2 $\left(\frac{x}{2}\right)$

shift left 0.5 units $(x-0.5)$

shift down 1 unit $(y-1)$



x	*y*
-2	-½
-1	-1
-½	-2
0	undefined
½	2
1	1
2	½

$\frac{x}{2}-0.5$	$-y-1$
-1.5	-0.5
-1	0
-0.75	1
-0.5	undefined
-0.25	-3
0	-2
0.5	-1.5

$$g(x) = -1\left(\frac{1}{2(x+0.5)}\right) - 1$$

↓

$$g(x) = \frac{-1}{2(x+0.5)} - 1$$

Complete Worksheet

2.6 - Inverse of a Function

Inverse of a function:

- The inverse of a function f is denoted as f^{-1}
- The function and its inverse have the property that if $f(a) = b$, then $f^{-1}(b) = a$
- So if $f(5) = 13$, then $f^{-1}(13) = 5$.

- More simply put: The inverse of a function has all the same points as the original function, except that the x's and y's have been reversed.

It is important to note that $f^{-1}(x)$ is read as "the inverse of f at x". The -1 does not behave like an exponent.

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

To draw an inverse, all you need to do is swap the x and y coordinates of each point.

Finding Inverses by Numerically

Example 1: The table shows ordered pairs belonging to a function $f(x)$. Determine $f^{-1}(x)$, then state the domain and range of $f(x)$ and its inverse.

$f(x)$	$f^{-1}(x)$
(-5, 0)	(0, -5)
(-4, 2)	(2, -4)
(-3, 5)	(5, -3)
(-2, 6)	(6, -2)
(0, 7)	(7, 0)

$f(x)$

D: $\{x \in \mathbb{R} \mid x = -5, -4, -3, -2, 0\}$

R: $\{y \in \mathbb{R} \mid y = 0, 2, 5, 6, 7\}$

$f^{-1}(x)$

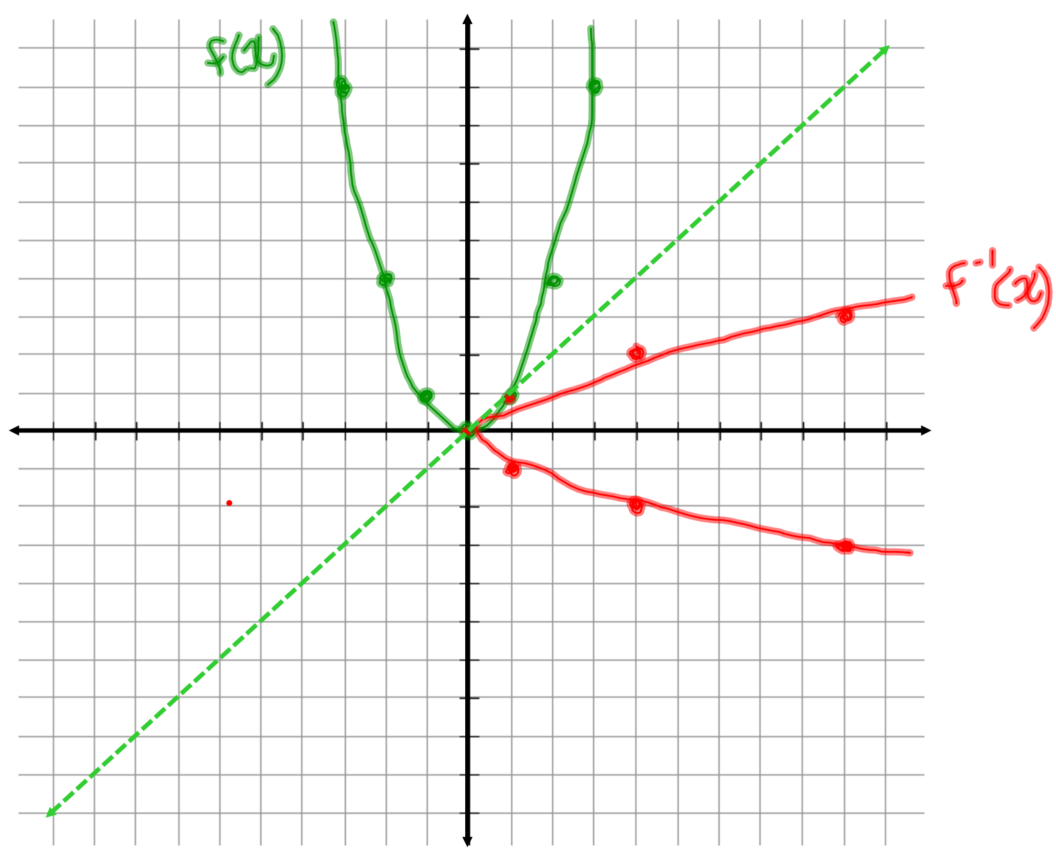
D: $\{x \in \mathbb{R} \mid x = 0, 2, 5, 6, 7\}$

R: $\{y \in \mathbb{R} \mid y = -5, -4, -3, -2, 0\}$

Example 2:

a) Graph the function $f(x) = x^2$ and its inverse $f^{-1}(x)$

$f(x)$	$f^{-1}(x)$
(-3,9)	(9,-3)
(-2,4)	(4,-2)
(-1,1)	(1,-1)
(0,0)	(0,0)
(1,1)	(1,1)
(2,4)	(4,2)
(3,9)	(9,3)



b) state the domain and range of both functions

$f(x)$

D: $\{x \in \mathbb{R}\}$

R: $\{y \in \mathbb{R} \mid y \geq 0\}$

$f^{-1}(x)$

D: $\{x \in \mathbb{R} \mid x \geq 0\}$

R: $\{y \in \mathbb{R}\}$

Note: *the domain and range of inverse functions are the reverse of each other.*

Example 3:

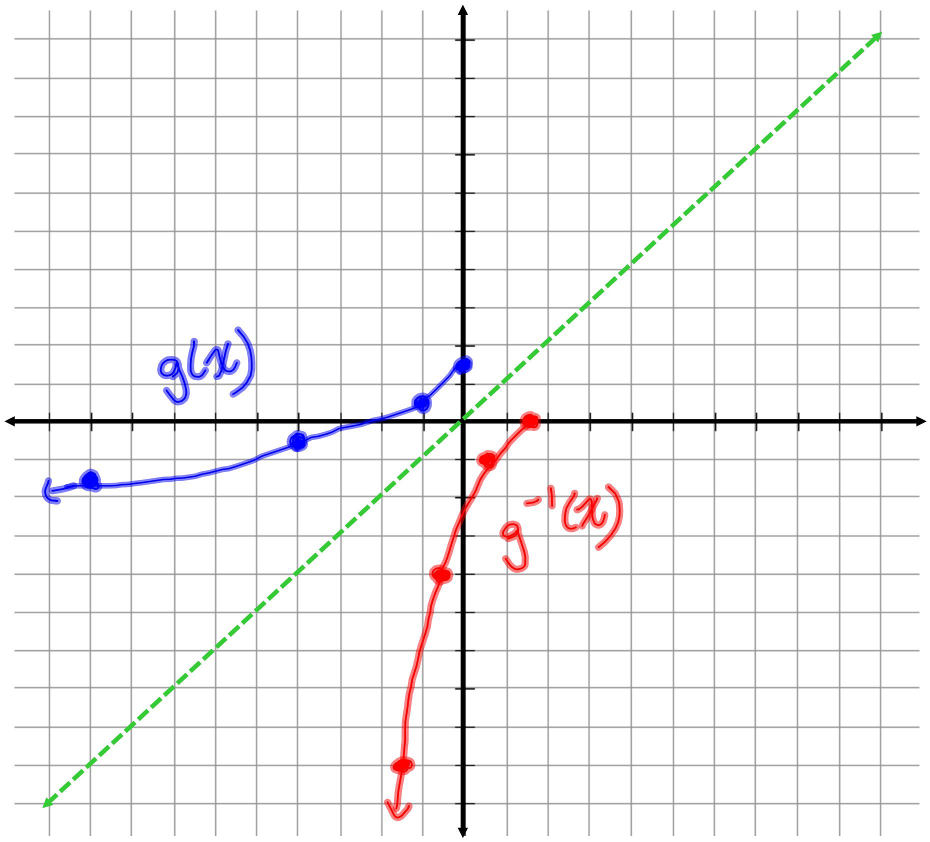
Sketch the graph of $g(x) = -2\sqrt{(-\frac{1}{2}x)} + 3$ then graph $g^{-1}(x)$.

$f(x) = \sqrt{x}$ $\longrightarrow$ $g(x) = -2\sqrt{(-\frac{1}{2}x)} + 3$ $\rightarrow$ $g^{-1}(x)$

x	*y*
0	0
1	1
4	2
9	3

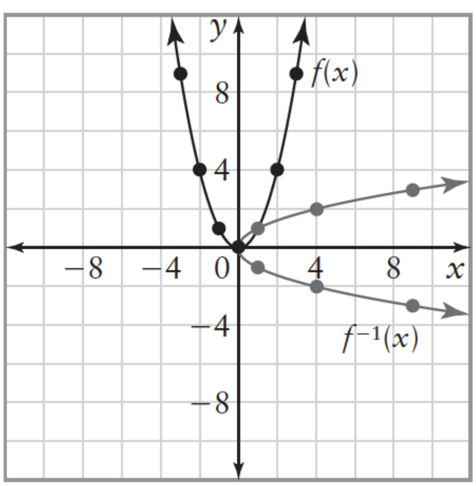
$-2x$	$-2y+3$
0	3
-2	1
-8	-1
-18	-3

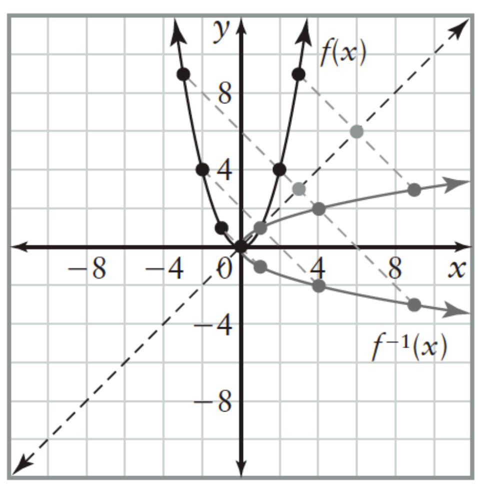
x	*y*
3	0
1	-2
-1	-8
-3	-18



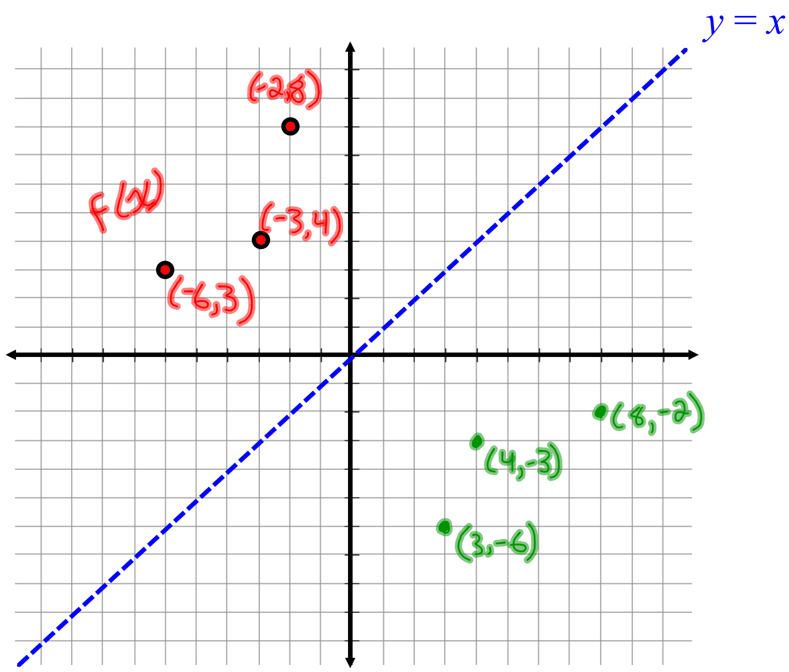
Finding Inverses by Graphing

The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected in the line $y = x$. This is true for all functions and their inverses. If you find the midpoint of each pair of points from example 2 and connect them you can prove this theorem.





Example 4: Sketch the inverse of the $f(x)$



Finding Inverses Algebraically

Algebraic Method for finding the inverse:

1. Replace $f(x)$ with "y"
2. Switch the x and y variables
3. Isolate for y
4. replace y with $f^{-1}(x)$

Example 5: Find the inverse of the following functions…

a) $g(x) = \dfrac{(3x)}{4}$

$y = \dfrac{3x}{4}$

$x = \dfrac{3y}{4}$

$4x = 3y$

$\dfrac{4x}{3} = y$

$g^{-1}(x) = \dfrac{4x}{3}$

b) $h(x) = 4x + 3$

$y = 4x + 3$

$x = 4y + 3$

$x - 3 = 4y$

$\dfrac{x-3}{4} = y$

$h^{-1}(x) = \dfrac{x-3}{4}$

c) $f(x) = x^2 - 1$

$y = x^2 - 1$

$x = y^2 - 1$

$x + 1 = y^2$

$\pm\sqrt{x+1} = y$

$f^{-1}(x) = \pm\sqrt{x+1}$

d) $h(x) = \dfrac{4x+3}{5}$

$y = \dfrac{4x+3}{5}$

$x = \dfrac{4y+3}{5}$

$5x = 4y + 3$

$\dfrac{5x-3}{4} = y$

$h^{-1}(x) = \dfrac{5x-3}{4}$

e) $f(x) = 2x^2 + 16x + 29$

> **Note:** for algebraic inverses of quadratic functions, before interchanging x and y's you must re-write in vertex form.

$y = (2x^2 + 16x) + 29$

$y = 2(x^2 + 8x + 16 - 16) + 29$

$y = 2(x^2 + 8x + 16) - 32 + 29$

$y = 2(x+4)^2 - 3$

$x = 2(y+4)^2 - 3$

$\frac{x+3}{2} = (y+4)^2$

$\pm\sqrt{\frac{x+3}{2}} = y + 4$

$-4 \pm \sqrt{\frac{x+3}{2}} = y$

$f^{-1}(x) = -4 \pm \sqrt{\frac{x+3}{2}}$

f) $r(x) = \sqrt{(x)} + 2$

$y = \sqrt{x} + 2$

$x = \sqrt{y} + 2$

$x - 2 = \sqrt{y}$

$(x-2)^2 = y$

$r^{-1}(x) = (x-2)^2$