Chapter 6 – Discrete Functions – Review
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SOLUTIONS

Write the required formulas:

Arithmetic Term Formula: $t_n = a + (n-1)d$

Arithmetic Series Formula: $S_n = \frac{n}{2}[2a + (n-1)d]$ OR $S_n = \frac{n}{2}(a + t_n)$

Geometric Term Formula: $t_n = a \cdot r^{n-1}$

Geometric Series Formula: $S_n = \frac{a(r^n - 1)}{r-1}$

Section 1: Sequences

1) Determine whether each sequence is arithmetic, geometric, or neither.

a) $-1, 9, 19, 29, \ldots$ $+10$

arithmetic

b) $3, 12, 19, 44, \ldots$

neither

c) $-2, 6, -18, 54, \ldots$ $\times(-3)$

geometric

2) For each arithmetic sequence, write the first three terms, then calculate term 13.

a) $t_n = 2n - 10$

$t_1 = 2(1) - 10 = -8$

$t_2 = 2(2) - 10 = -6$

$t_3 = 2(3) - 10 = -4$

$t_{13} = 2(13) - 10 = 16$

b) $t_n = 40 + 13n$

$t_1 = 40 + 13(1) = 53$

$t_2 = 40 + 13(2) = 66$

$t_3 = 40 + 13(3) = 79$

$t_{13} = 40 + 13(13) = 209$

3) For each geometric sequence write the first four terms.

a) $t_n = 3\left(\sqrt{2}^{n-1}\right)$

$t_1 = 3\left(\sqrt{2}^{1-1}\right) = 3$

$t_2 = 3\left(\sqrt{2}^{2-1}\right) = 3\sqrt{2}$

$t_3 = 3\left(\sqrt{2}^{3-1}\right) = 6$

$t_4 = 3\left(\sqrt{2}^{4-1}\right) = 3(2)^{3/2} = 3\sqrt{2^3} = 3\sqrt{8}$

b) a = 800, r = -1/4

$t_n = 800\left(-\frac{1}{4}\right)^{n-1}$

$t_1 = 800\left(-\frac{1}{4}\right)^{1-1} = 800$

$t_2 = 800\left(-\frac{1}{4}\right)^{2-1} = -200$

$t_3 = 800\left(-\frac{1}{4}\right)^{3-1} = 50$

$t_4 = 800\left(-\frac{1}{4}\right)^{4-1} = -12.5$

4) For each arithmetic sequence, determine the values of a and d and the formula for the general term. Then, write the next four terms.

a) $3, \overset{-2}{1}, -1, -3, \ldots$

$a = 3$

$d = -2$

$t_n = 3 + (n-1)(-2)$

$t_5 = -5$

$t_6 = -7$

$t_7 = -9$

$t_8 = -11$

b) $\frac{2}{3}, \overset{+\frac{1}{4}}{\frac{11}{12}}, \frac{7}{6}, \frac{17}{12}$

$a = \frac{2}{3}$

$d = \frac{1}{4}$

$t_n = \frac{2}{3} + (n-1)\left(\frac{1}{4}\right)$

$t_5 = \frac{17}{12} + \frac{1}{4} = \frac{20}{12} = \frac{5}{3}$

$t_6 = \frac{5}{3} + \frac{1}{4} = \frac{23}{12}$

$t_7 = \frac{23}{12} + \frac{1}{4} = \frac{26}{12} = \frac{13}{6}$

$t_8 = \frac{13}{6} + \frac{1}{4} = \frac{29}{12}$

5) Write the first three terms of each geometric sequence

a) $f(n) = 2(-1)^n$

$t_1 = 2(-1)^1 = -2$

$t_2 = 2(-1)^2 = 2$

$t_3 = 2(-1)^3 = -2$

b) $t_n = -3(2)^{n+1}$

$t_1 = -3(2)^{1+1} = -12$

$t_2 = -3(2)^{2+1} = -24$

$t_3 = -3(2)^{3+1} = -48$

6) For each geometric sequence, determine the values of a and r and the formula for the general term.

a) $64, 32, 16, 8, \ldots$

$a = 64$

$r = \frac{1}{2}$

$t_n = 64\left(\frac{1}{2}\right)^{n-1}$

b) $-4000, 1000, -250, 62.5, \ldots$

$\times\left(-\frac{1}{4}\right)$

$a = -4000$

$r = -\frac{1}{4}$

$t_n = -4000\left(-\frac{1}{4}\right)^{n-1}$

7) Determine a and d, write the general term formula t_n, and calculate t_{21}, given $t_8 = 72$ and $t_{14} = 54$.

$t_8 = 72$

$72 = a + (8-1)d$

① $72 = a + 7d$

$t_{14} = 54$

$54 = a + (14-1)d$

② $54 = a + 13d$

① $72 = a + 7d$
② $54 = a + 13d$

$18 = -6d$

$d = -3$

$54 = a + 13d$

$54 = a + 13(-3)$

$54 = a - 39$

$a = 93$

$t_n = 93 + (n-1)(-3)$

$t_{21} = 93 + (21-1)(-3)$

$t_{21} = 93 - 60$

$t_{21} = 33$

8) Determine a and r, write the general term formula t_n, and calculate t_8, given $t_5 = 72$ and $t_3 = 8$.

$t_5 = 72$

$72 = a(r)^{5-1}$

$72 = ar^4$

Isolate 'a'

$a = \frac{72}{r^4}$

$t_3 = 8$

$8 = a(r)^{3-1}$

$8 = ar^2$

sub $a = \frac{72}{r^4}$ into $8 = ar^2$

$8 = \left(\frac{72}{r^4}\right)(r^2)$

$8 = \frac{72}{r^2}$

$r^2 = \frac{72}{8}$

$r = 3$

$a = \frac{72}{r^4}$

$a = \frac{72}{(3)^4}$

$a = \frac{72}{81}$

$a = \frac{8}{9}$

$t_n = \frac{8}{9}(3)^{n-1}$

$t_8 = \frac{8}{9}(3)^{8-1}$

$t_8 = 1944$

9) Determine the number of terms in each sequence. Prove mathematically.

a) $5, 8, 11, \ldots, 62$ (+3)

$a = 5$
$d = 3$
$t_n = 62$
$n = ?$

$t_n = a + (n-1)d$

$62 = 5 + (n-1)(3)$

$57 = (n-1)(3)$

$19 = n - 1$

$n = 20$

b) $-4, 12, -36, \ldots, -19\,131\,876$ ($\times(-3)$)

$a = -4$
$r = -3$
$t_n = -19\,131\,876$
$n = ?$

$t_n = a \cdot r^{n-1}$

$-19\,131\,876 = (-4)(-3)^{n-1}$

$4\,782\,969 = (-3)^{n-1}$

$(-3)^{14} = (-3)^{n-1}$

$14 = n - 1$

$n = 15$

Section 2: Series

10) Find the sum of the first 28 terms of the arithmetic series that starts as: $-6 - 1 + 4 + 9 + \ldots$ (+5)

$a = -6$
$d = 5$
$n = 28$
$S_{28} = ?$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{28} = \frac{28}{2}[2(-6) + (28-1)5]$

$S_{28} = 14[123]$

$= 1722$

11) Find the sum of the first 11 terms of the series that begins with: $1024 + 512 + 256 + \ldots$ ($\times \frac{1}{2}$)

$a = 1024$
$r = \frac{1}{2}$
$n = 11$
$S_{11} = ?$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$S_{11} = \frac{1024\left[\left(\frac{1}{2}\right)^{11} - 1\right]}{\frac{1}{2} - 1}$

$S_{11} = \frac{1024\left(\frac{1}{2048} - \frac{2048}{2048}\right)}{-\frac{1}{2}}$

$S_{11} = \frac{1024\left(\frac{-2047}{2048}\right)}{-\frac{1}{2}}$

$= -2048\left(\frac{-2047}{2048}\right)$

$= 2047$

12) Find S_{14} given $t_n = 5 + \frac{3}{2}n$ arithmetic

$t_1 = 5 + (\frac{3}{2})(1) = \frac{10}{2} + \frac{3}{2} = \frac{13}{2}$

$t_{14} = 5 + (\frac{3}{2})(14) = 5 + 3(7) = 26$

$S_n = \frac{n}{2}(a + t_n)$

$S_{14} = \frac{14}{2}(\frac{13}{2} + 26)$

$S_{14} = 7(\frac{13}{2} + \frac{52}{2})$

$S_{14} = 7(\frac{65}{2})$

$S_{14} = \frac{455}{2}$

13) Find S_7 given $t_n = 5(2^{n-1})$. geometric

$a = 5$

$r = 2$

$n = 7$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$S_7 = \frac{5(2^7 - 1)}{2 - 1}$

$S_7 = 5(127)$

$S_7 = 635$

14) What is the sum of the arithmetic series $2\sqrt{3} \overset{+3\sqrt{3}}{+} 5\sqrt{3} + 8\sqrt{3} + \cdots + 83\sqrt{3}$

$a = 2\sqrt{3}$

$d = 3\sqrt{3}$

$t_n = 83\sqrt{3}$

$n = ?$

$t_n = a + (n-1)d$

$83\sqrt{3} = 2\sqrt{3} + (n-1)(3\sqrt{3})$

$81\sqrt{3} = (n-1)(3\sqrt{3})$

$27 = n - 1$

$n = 28$

$S_n = \frac{n}{2}(a + t_n)$

$S_{28} = \frac{28}{2}(2\sqrt{3} + 83\sqrt{3})$

$= 14(85\sqrt{3})$

$= 1190\sqrt{3}$

15) Determine the sum of the geometric series $\overset{\times 2}{5 + 10} + 20 + \cdots + 2560$

$a = 5$

$r = 2$

$t_n = 2560$

$n = ?$

$t_n = a \cdot r^{n-1}$

$2560 = 5(2)^{n-1}$

$512 = 2^{n-1}$

$2^9 = 2^{n-1}$

$9 = n-1$

$n = 10$

$S_n = \dfrac{a(r^n - 1)}{r-1}$

$S_{10} = \dfrac{5(2^{10} - 1)}{2-1}$

$S_{10} = 5(1023)$

$S_{10} = 5115$

16) The 23rd term in an arithmetic sequence is 95. The 31st term is 127. Find the sum of the first 31 terms in the series.

$t_{23} = a + (23-1)d$

① $95 = a + 22d$

$t_{31} = a + (31-1)d$

② $127 = a + 30d$

$$\begin{aligned} 127 &= a + 30d \\ - \quad 95 &= a + 22d \\ \hline 32 &= 8d \end{aligned}$$

$d = 4$

$95 = a + 22d$

$95 = a + 22(4)$

$a = 7$

$S_n = \frac{n}{2}(a + t_n)$

$S_{31} = \frac{31}{2}(7 + 127)$

$S_{31} = \frac{31}{2}(134)$

$S_{31} = 31(67)$

$S_{31} = 2077$

17) The sum of the first 10 terms in an arithmetic sequence is 145. The 5th term in the series is 16. Find the sum of the first 20 terms in the series.

$S_{10} = \frac{10}{2}[2a + (10-1)d]$

$145 = 5[2a + 9d]$

① $29 = 2a + 9d$

$$\begin{aligned} ① \quad 29 &= 2a + 9d \\ 2 \times ② \quad 32 &= 2a + 8d \\ \hline -3 &= d \end{aligned}$$

$t_5 = a + (5-1)d$

① $16 = a + 4d$

$16 = a + 4(-3)$

$16 = a - 12$

$a = 28$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{20} = \frac{20}{2}[2(28) + (20-1)(-3)]$

$S_{20} = 10(56 - 57)$

$S_{20} = 10(-1)$

$S_{20} = -10$

18) The third term of a geometric series is 24 and the fourth term is 36. Determine the sum of the first 10 terms.

$r = \frac{36}{24} = \frac{3}{2}$

$t_3 = a\cdot\left(\frac{3}{2}\right)^{3-1}$

$24 = a\left(\frac{3}{2}\right)^2$

$24 = a\left(\frac{9}{4}\right)$

$\frac{24}{\left(\frac{9}{4}\right)} = a$

$a = \frac{96}{9}$

$a = \frac{32}{3}$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$S_{10} = \frac{\frac{32}{3}\left[\left(\frac{3}{2}\right)^{10} - 1\right]}{\frac{3}{2} - 1}$

$S_{10} = \frac{\frac{32}{3}\left(\frac{59049}{1024} - \frac{1024}{1024}\right)}{\frac{1}{2}}$

$S_{10} = \frac{\frac{32}{3}\left(\frac{58025}{1024}\right)}{\frac{1}{2}}$

$S_{10} = \frac{64}{3}\left(\frac{58025}{1024}\right)$

$S_{10} = \frac{3713600}{3072}$

$S_{10} = \frac{58025}{48}$

19) A lottery awards prizes such that the grand winner gets \$100 000, and the next winner gets one half what the previous winner did. If the lottery pays out a total of 11 prizes, how much money must they pay out?

$a = 100\,000$

$r = \frac{1}{2}$

$n = 11$

$S_{11} = \frac{100\,000\left[\left(\frac{1}{2}\right)^{11} - 1\right]}{\frac{1}{2} - 1}$

$S_{11} = \frac{100\,000\left(\frac{1}{2048} - \frac{2048}{2048}\right)}{-\frac{1}{2}}$

$= -200\,000\left(\frac{-2047}{2048}\right)$

$= \$\,199\,902.34$

Section 3: Recursive Functions

20) Write the first 4 terms of the sequence, given the recursive formula.

a) $t_n = 13 - 2t_{n-1}, t_1 = 4$

$t_1 = 4$

$t_2 = 13 - 2(4) = 5$

$t_3 = 13 - 2(5) = 3$

$t_4 = 13 - 2(3) = 7$

$4, 5, 3, 7$

b) $f(n) = f(n-1) + 3n, f(1) = 1$

$f(1) = 1$

$f(2) = 1 + 3(2) = 7$

$f(3) = 7 + 3(3) = 16$

$f(4) = 16 + 3(4) = 28$

$1, 7, 16, 28$

21) Determine a recursion formula for each sequence.

a) -2, 7, 16, 25, ...

$t_n = t_{n-1} + 9$; $t_1 = -2$

b) 1, -3, 9, -27, ...

$t_1 = 1$; $t_n = -3 \cdot t_{n-1}$

22) Expand the following using Pascal's Triangle.

a) $(3x + 5x^2)^4$

$= 1(3x)^4 + 4(3x)^3(5x^2)^1 + 6(3x)^2(5x^2)^2 + 4(3x)^1(5x^2)^3 + 1(5x^2)^4$

$= 81x^4 + 540x^5 + 1350x^6 + 1500x^7 + 625x^8$

b) $(2x^2 + y)^7$

$= 1(2x^2)^7 + 7(2x^2)^6(y)^1 + 21(2x^2)^5(y)^2 + 35(2x^2)^4(y)^3 + 35(2x^2)^3(y)^4 + 21(2x^2)^2(y)^5 + 7(2x^2)^1(y)^6 + 1(y)^7$

$= 128x^{14} + 448x^{12}y + 672x^{10}y^2 + 560x^8y^3 + 280x^6y^4 + 84x^4y^5 + 14x^2y^6 + y^7$

23) What is the 5^{th} term in the expansion of $(5xy + y)^6$?

$= 15(5xy)^2(y)^4$

$= 15(25)x^2y^2y^4$

$= 375x^2y^6$

Answers

1) a) arithmetic   b) neither   c) geometric

2)

3)

4) a) $t_n = -2n + 5$; the next four terms are -5, -7, -9, -11   b) $t_n = \frac{3n+5}{12}$; next four terms are $\frac{5}{3}, \frac{23}{12}, \frac{13}{6}, \frac{29}{12}$

5) a) -2, 2, -2   **b)** -12, -24, -48

6) a) $t_n = 64\left(\frac{1}{2}\right)^{n-1}$ **b)** $t_n = -4000\left(-\frac{1}{4}\right)^{n-1}$

7) $t_n = 93 + (n-1)(-3)$; $t_{21} = 33$

8) $t_n = \frac{8}{9}(3)^{n-1}$; $t_8 = 1944$

9) a) 20 terms   **b)** 15 terms

10) 1722

11) 2047

12) $\frac{455}{2}$

13) 635

14) $1190\sqrt{3}$

15) 5115

16) 2077

17) -10

18) $\frac{58\,025}{48}$

19) 199 902.34

20) a) 4, 5, 3, 7   **b)** 1, 7, 16, 28

21) a) $t_1 = -2$, $t_n = t_{n-1} + 9$ **b)** $t_1 = 1$, $t_n = -3t_{n-1}$

22) a) $81x^4 + 540x^5 + 1350x^6 + 1500x^7 + 625x^8$ b)

b) $128x^{14} + 448x^{12}y + 672x^{10}y^2 + 560x^8y^3 + 280x^6y^4 + 84x^4y^5 + 14x^2y^6 + y^7$

23) $375x^2y^6$