L3 - 1.3 Max or Min of a Quadratic Function
MCR3U
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Part 1: Quadratics Review

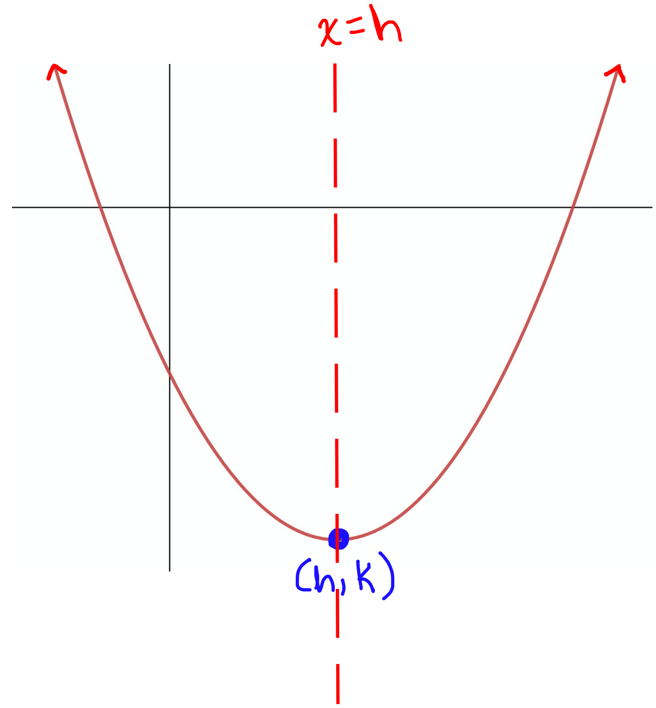
Vertex Form:

$y = a(x-h)^2 + k$

vertex at (h, k)

$a > 0$; opens up
$a < 0$; opens down

axis of symmetry at $x = h$



Factored Form:

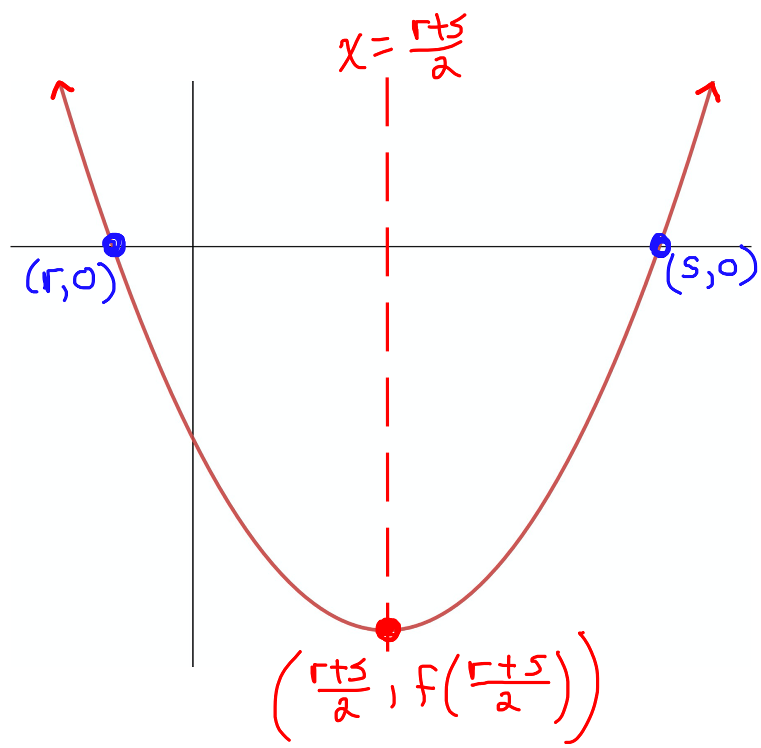
$y = a(x-r)(x-s)$

x-int at $(r,0)$ and $(s,0)$

$a > 0$; opens up
$a < 0$; opens down

axis of symmetry at $x = \frac{r+s}{2}$

vertex at $\left(\frac{r+s}{2}, f\left(\frac{r+s}{2}\right)\right)$



Standard Form:

$\boldsymbol{y = ax^2 + bx + c}$

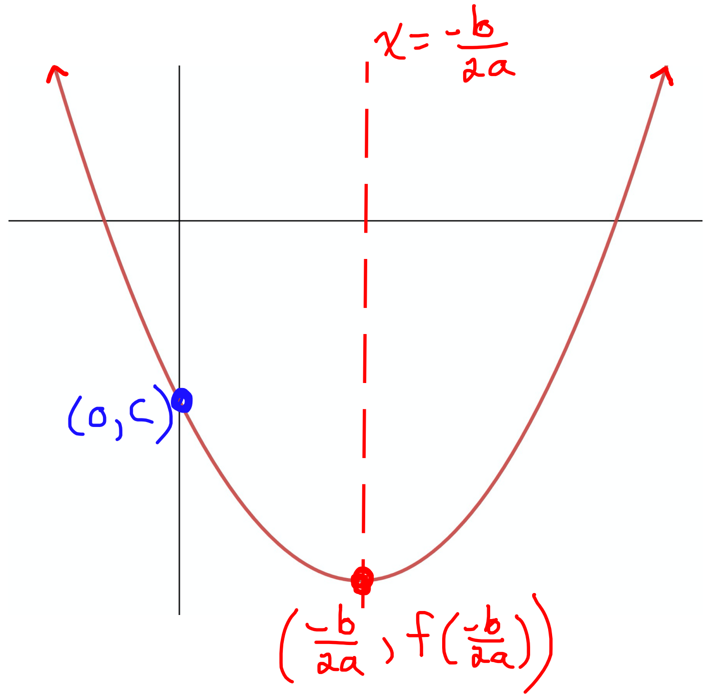
y-int at $(0, c)$

$a > 0$; opens up

$a < 0$; opens down

axis of symmetry at $x = \frac{-b}{2a}$

vertex at $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$



Part 2: Perfect Square Trinomials

Completing the square is a process for changing a standard form quadratic equation into vertex form

$$y = ax^2 + bx + c \rightarrow y = a(x - h)^2 + k$$

Notice that vertex form contains a $(x - h)^2$. A binomial squared can be obtained when factoring a perfect square trinomial:

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$

The process of completing the square involves creating this perfect square trinomial within the standard form equation so that it can be factored to create the vertex form equation.

Let's start by analyzing the following perfect square trinomials. Specifically notice how the middle term is 2 times the product of the square roots of the first and last terms.

$x^2 + 10x + 25$

$10x = 2(\sqrt{x^2})(\sqrt{25})$

$10x = 2(x)(5)$

$10x = 10x$

$x^2 - 12x + 36$

$12x = 2(\sqrt{x^2})(\sqrt{36})$

$12x = 2(x)(6)$

$12x = 12x$

Example 1: Determine the value of k that would make each quadratic a perfect square trinomial. Then factor the trinomial.

a) $x^2 + 14x + k$

$14x = 2(\sqrt{x^2})(\sqrt{k})$

$14\cancel{x} = 2(\cancel{x})(\sqrt{k})$

$\left(\frac{14}{2}\right)^2 = (\sqrt{k})^2$

$k = 49$

b) $x^2 - 24x + k$

$24x = 2(\sqrt{x^2})(\sqrt{k})$

$24\cancel{x} = 2(\cancel{x})(\sqrt{k})$

$\left(\frac{24}{2}\right)^2 = (\sqrt{k})^2$

$k = 144$

Tip: You can calculate the constant term that makes the quadratic a PST by squaring half of the coefficient of the x term.

Note: *this only works when the coefficient of* x^2 *is 1.*

Part 3: Completing the Square

Completing the Square Steps

$$ax^2 + bx + c \rightarrow a(x - h)^2 + k$$

1) Put brackets around the first 2 terms
2) Factor out the constant in front of the x^2 term
3) Look at the last term in the brackets, divide it by 2 and then square it
4) Add AND subtract that term behind the last term in the brackets
5) Move the negative term outside the brackets by multiplying it by the $'a'$ value
6) Simplify the terms outside the brackets
7) Factor the perfect square trinomial

$$a^2 + 2ab + b^2 = (a + b)^2$$

Example 2: Rewrite each quadratic in vertex form by completing the square. Then state the vertex, whether it is a max or min point, and the axis of symmetry.

a) $y = x^2 + 8x + 5$

$y = (x^2 + 8x) + 5$

$y = (x^2 + 8x + \underline{16} - \underline{16}) + 5$

$y = (x^2 + 8x + 16) - 16 + 5$ $\qquad \underline{4} \times \underline{4} = 16$, $\underline{4} + \underline{4} = 8$

$y = (x+4)(x+4) - 11$

$y = (x+4)^2 - 11$

vertex $(-4, -11)$ is a min point

aos at $x = -4$

b) $y = 2x^2 - 12x + 11$

$y = (2x^2 - 12x) + 11$

$y = 2(x^2 - 6x) + 11$

$y = 2(x^2 - 6x + \underline{9} - \underline{9}) + 11$

$y = 2(x^2 - 6x + 9) - 18 + 11$ $\qquad \underline{-3} \times \underline{-3} = 9$, $\underline{-3} + \underline{-3} = -6$

$y = 2(x-3)(x-3) - 7$

$y = 2(x-3)^2 - 7$

vertex at $(3, -7)$ is a min point

aos at $x = 3$

c) $y = -3x^2 + 9x - 13$

$y = -3(x^2 - 3x) - 13$

$y = -3(x^2 - 3x + \frac{9}{4} - \frac{9}{4}) - 13$

$y = -3(x^2 - 3x + \frac{9}{4}) + \frac{27}{4} - \frac{52}{4}$

$y = -3(x - \frac{3}{2})^2 - \frac{25}{4}$ $\qquad \underline{-3/2} \times \underline{-3/2} = \frac{9}{4}$, $\underline{-3/2} + \underline{-3/2} = -3$

vertex at $(\frac{3}{2}, \frac{-25}{4})$ is a max

d) $y = -\frac{2}{3}x^2 + 8x + 5$

$y = -\frac{2}{3}(x^2 - 12x) + 5$

$y = -\frac{2}{3}(x^2 - 12x + \underline{36} - \underline{36}) + 5$

$y = -\frac{2}{3}(x^2 - 12x + 36) + 24 + 5$ $\qquad \underline{-6} \times \underline{-6} = 36$, $\underline{-6} + \underline{-6} = -12$

$y = -\frac{2}{3}(x-6)^2 + 29$

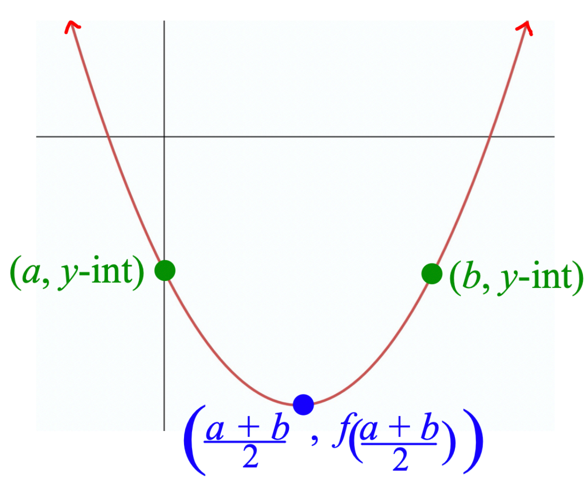
vertex at $(6, 29)$ is a max

aos at $x = 6$

Part 4: Partial Factoring (another method to find the vertex)

Partial Factoring Steps

1) Set the quadratic equal to the y-intercept
2) Solve the equation for x
3) Find the x-value of the vertex by averaging your answers from the previous step
4) Substitute the x-value of the vertex into the original equation and solve for y-value



Example 3: Use partial factoring to find the vertex. Then state if it is a max or min.

a) $y = x^2 + 2x - 6$

$-6 = x^2 + 2x - 6$

$0 = x^2 + 2x$

$0 = x(x+2)$

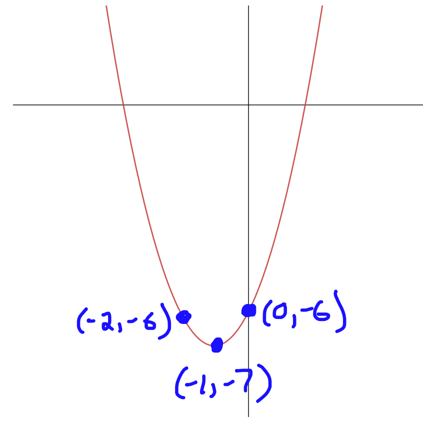
$x_1 = 0 \qquad x + 2 = 0$

$\qquad\qquad x_2 = -2$

$x\text{-vertex} = \frac{0 + (-2)}{2} = -1$

$y\text{-vertex} = (-1)^2 + 2(-1) - 6 = -7$

The vertex at $(-1, -7)$ is a min



b) $y = 4x^2 - 12x + 3$

$3 = 4x^2 - 12x + 3$

$0 = 4x^2 - 12x$

$0 = 4x(x-3)$

$4x = 0 \qquad x - 3 = 0$

$x_1 = 0 \qquad x_2 = 3$

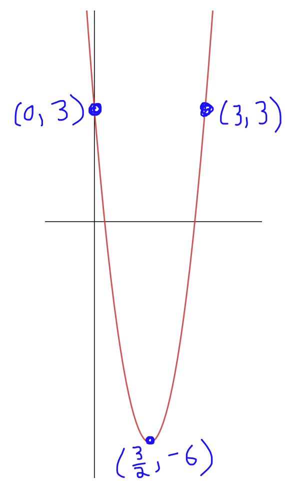
$x\text{-vertex} = \frac{0+3}{2} = \frac{3}{2}$

$y\text{-vertex} = 4\left(\frac{3}{2}\right)^2 - 12\left(\frac{3}{2}\right) + 3$

$= 4\left(\frac{9}{4}\right) - 6(3) + 3$

$= -6$

vertex at $\left(\frac{3}{2}, -6\right)$ is a min.



c) $y = -3x^2 + 9x - 2$

$-2 = -3x^2 + 9x - 2$

$0 = -3x^2 + 9x$

$0 = -3x(x-3)$

$-3x = 0 \qquad x - 3 = 0$

$x_1 = 0 \qquad x_2 = 3$

$x\text{-vertex} = \frac{0+3}{2} = \frac{3}{2}$

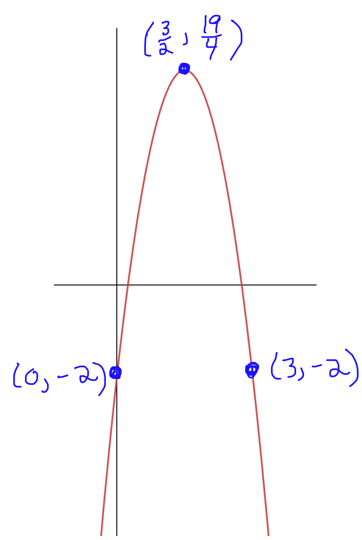
$y\text{-vertex} = -3\left(\frac{3}{2}\right)^2 + 9\left(\frac{3}{2}\right) - 2$

$= -3\left(\frac{9}{4}\right) + \frac{27}{2} - 2$

$= -\frac{27}{4} + \frac{54}{4} - \frac{8}{4}$

$= \frac{19}{4}$

vertex at $\left(\frac{3}{2}, \frac{19}{4}\right)$ is a max



Example 4: Maximizing Revenue

Rachel and Ken are knitting scarves to sell at the craft show. They were planning to sell the scarves for \$10 each, the same as last year when they sold 40 scarves. However, they know that if they adjust the price, they might be able to make mor profit. They have been told that for every 50-cent increase in the price, they can expect to sell four fewer scarves. What selling price will maximize their revenue and what will the revenue be?

Let n = # of \$0.50 increases

cost $= 10 + 0.5n$

number sold $= 40 - 4n$

Revenue $= (\text{cost})(\#\text{ sold})$

$= (10 + 0.5n)(40 - 4n)$

$0 = (10 + 0.5n)(40 - 4n)$

$0 = 10 + 0.5n \qquad 0 = 40 - 4n$

$-10 = 0.5n \qquad 4n = 40$

$n_1 = -20 \qquad n_2 = 10$

n-vertex $= \frac{-20 + 10}{2} = -5$; to get a max revenue they must raise the price by \$0.50 five times.

$R(-5) =$