Chapter 3- Exponential Functions

Lesson Package

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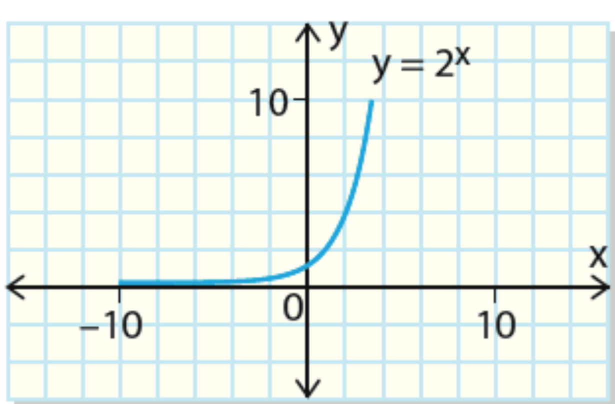

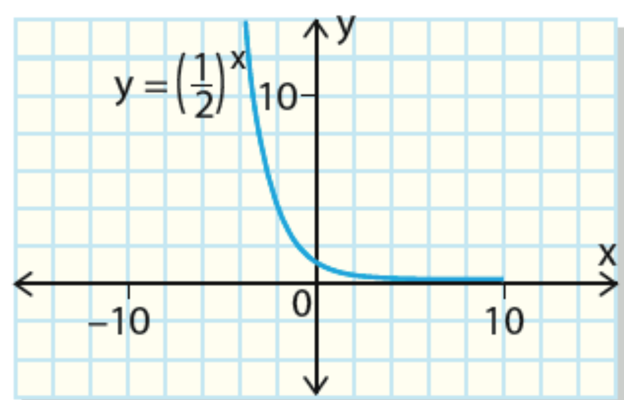

Chapter 3 Outline

Unit Goal: Be able to make connections between the numeric, graphical, and algebraic representations of exponential functions. Be able to identify and represent exponential functions, and solve problems involving exponential functions, including problems arising from real-world applications.

Section	Subject	Learning Goals	Curriculum Expectations
L1	Exponential Growth	- identify exponential functions, including those that arise from real-world applications involving growth	B3.2, B3.3
L2	Exponential Decay	- identify exponential functions, including those that arise from real-world applications involving decay	B3.2, B3.3
L3	Compound Interest	- identify exponential functions, including those that arise from real-world applications involving compound interest	B3.2, B3.3
L4	Properties of Exponential Functions	- be able to identify key properties of exponential functions and distinguish them from linear and quadratic functions by looking at finite differences	A1.4, B2.1,
L5	Transformations of Exponential Functions	- Use knowledge of parameters a, k, d, and c to graph transformed exponential functions	B2.2, B2.3, B2.4, B2.5

Assessments	F/A/O	Ministry Code	P/O/C	KTAC
Note Completion	A		P	
Practice Worksheet Completion	F/A		P	
PreTest Review	F/A		P	
Test – Transformations of Functions	O	A1.4, B2.1, B2.2, B2.3, B2.4, B2.5, B3.2, B3.3	P	K(20%), T(14%), A(9%), C(57%)

L1 – Exponential Growth

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DO IT NOW!

A type of bacteria grows so that it triples in number every day. On the day we begin observations, the bacteria has a population of 100.

a) Make a table to show the population over 5 days.

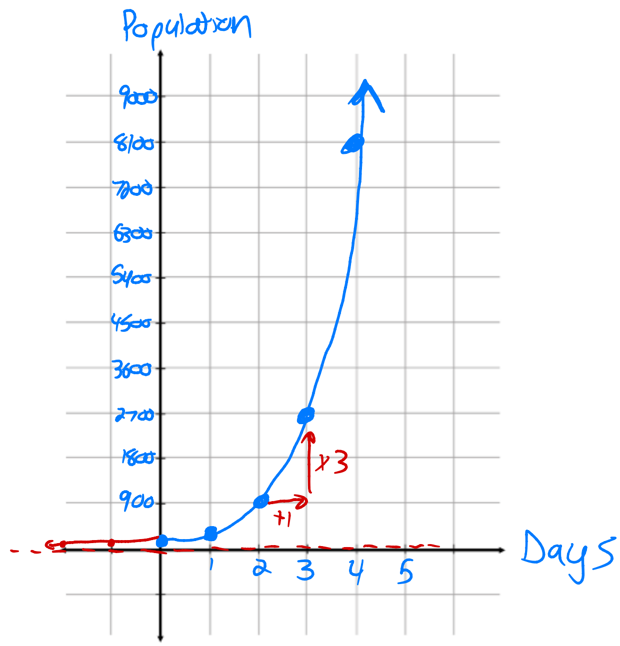
Day	Population
0	100
1	300
2	900
3	2700
4	8100
5	24300

b) Calculate finite differences and indicate any patterns you see

Day	Population	1^{st} Differences	2^{nd} Differences
0	100 +200		
1	×3 300 +600	300 - 100 = 200 +400	
2	×3 900 +1800	×3 900 - 300 = 600 +1200	600 - 200 = 400 ×3
3	×3 2700 +5400	×3 2700 - 900 = 1800 +3600	1800 - 600 = 1200 ×3
4	×3 8100 +16200	8100 - 2700 = 5400 +10800	5400 - 1800 = 3600 ×3
5	×3 24300	24300 - 8100 = 16200	16200 - 5400 = 10800

The finite differences for an exponential relationship have a common RATIO.

c) Graph the relation



d) Write an equation to model this growth

Day (x)	Population (y)
0	$100 \times 3^0 = 100$
1	$100 \times 3^1 = 300$
2	$100 \times 3^2 = 900$
3	$100 \times 3^3 = 2700$
4	$100 \times 3^4 = 8100$
5	$100 \times 3^5 = 24300$

The relationship between days and population is easier to see when we look at the number of times the population has been tripled.

$$\boxed{y = 100(3)^x}$$

General Properties of Exponential Growth

Equation: $y = a(b)^x$

$a =$ initial amount

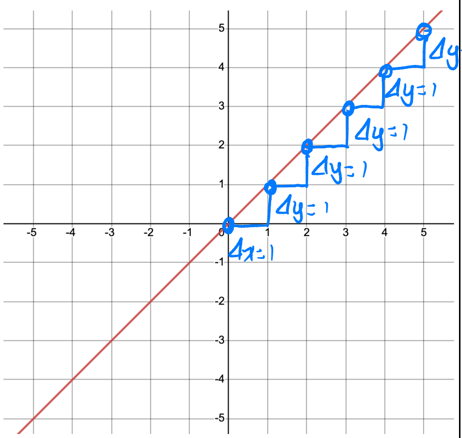
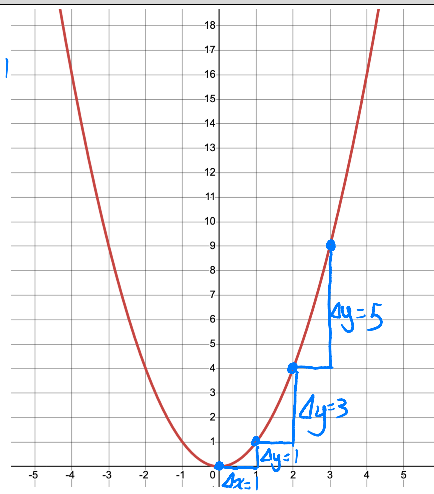
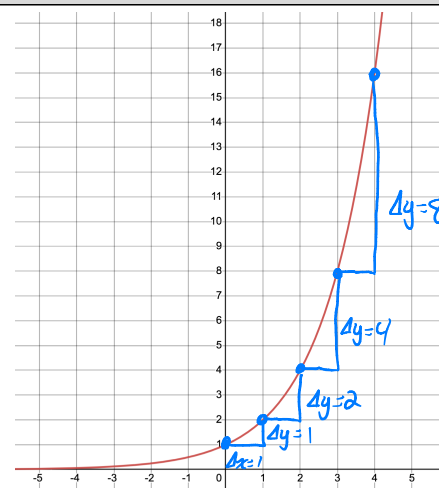
$b =$ growth factor

$y =$ future amount

$x =$ # of growth periods

To calculate x, use the equation: $x = \dfrac{\text{time}}{\text{time of 1 growth period}}$

Summary of Patterns in First Differences for Various Functions

Linear ($y = x$)	Quadratic ($y = x^2$)	Exponential ($y = 2^x$)
		
Constant 1st differences.	The 1st differences are related by addition. 2nd differences would be constant	The finite differences are related by MULTIPLICATION.

Example 1: Your brother tells you a secret. You see no harm in telling two friends. After this second "passing" of the secret, 4 people now know the secret (your brother, you and two friends). If each of these friends now tells two new people, after the third "passing" of the secret, eight people will know. If this pattern of spreading the secret continues, how many people will know the secret after 10 such "passings"?

# of passings (x)	# of ppl that know the secret (y)
0	1
1	2 ($\times 2$)
2	4 ($\times 2$)
3	8 ($\times 2$)

$y = a(b)^x$

$y = 1(2)^x$

$y = 1(2)^{10}$

$\boxed{y = 1024 \text{ people}}$

Example 2:

a) An insect colony has a current population of 50 insects. Its population doubles every 3 days. What is the population after 12 days?

$y = ?$

$a = 50$

$b = 2$

$x = \frac{12}{3} = 4$

$y = a(b)^x$

$y = 50(2)^4 = 50(16) = 800$

$\boxed{y = 800}$

b) The insect colony is actually full of giant, intelligent, mutant insects. They plot that they can overtake the Earth when their population has reached 1 billion. When will we meet our doom? (When does the population reach 1 billion?)

$y = 1\,000\,000\,000$

$a = 50$

$b = 2$

$x = \frac{t}{3}$

$y = a(b)^x$

$\frac{1\,000\,000\,000}{50} = \frac{\cancel{50}(2)^{t/3}}{\cancel{50}}$

$20\,000\,000 = 2^{t/3}$

$\cancel{3} \times \frac{t}{\cancel{3}} = \log_2(20\,000\,000) \times 3$

$\log_2(20\,000\,000) = \frac{\log(20\,000\,000)}{\log(2)}$

$\boxed{t \approx 72.76 \text{ days}}$

Note: a logarithm is a function that solves for an unknown exponent.

Ex:

$\log_3 9 = 2$

base: 3; argument: 9; 2: the exponent that goes on the base to get the argument.

because 2 is the exponent that goes on 3 to get 9.

If exponential growth is given as a percent you can use the equation: $y = a(1+r)^x$

$a =$ initial amount

$r =$ rate of increase (as a decimal)

$x =$ # of growth periods

Example 3: In 2005, there were only 285 Pittsburgh Penguins fans in Oakville. The number of Penguins fans increased by 5% per year after 2005 (this is when Crosby was drafted). How many Penguins fans are now in Oakville this year?

$y = ?$

$a = 285$

$r = 0.05$

$x = 19$

$y = a(1+r)^x$

$y = 285(1+0.05)^{19}$

$\boxed{y \simeq 720 \text{ fans}}$

L2 – Exponential Decay

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General Properties of Exponential Decay

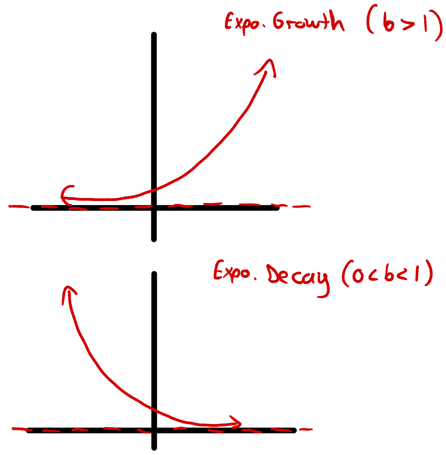
Equation: $y = a(b)^x$

$a =$ initial amount

$b =$ decay factor $(0 < b < 1)$

$y =$ future amount

$x =$ # of decay periods



To calculate x, use the equation: $x = \dfrac{\text{total time}}{\text{time of 1 decay period}}$

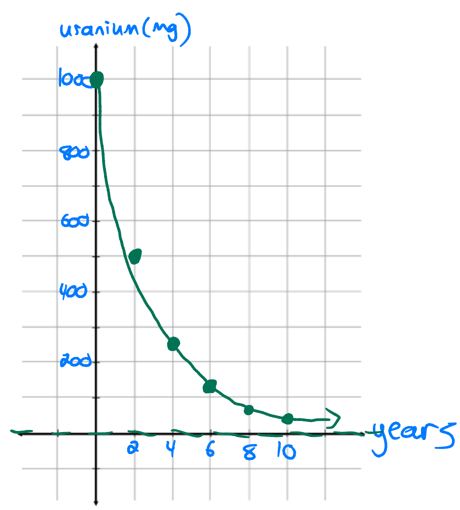
DO IT NOW!

Nuclear power plants use Uranium-239 as a power source. U-239 has a half-life of about 2 years.

a) Complete the chart for the amount of 1000mg sample that will be left after 10 years.

Years (x)	# of half-life periods	Amount of U-239 remaining (y)
0	0	1000 $\times\frac{1}{2}$
2	1	500 $\times\frac{1}{2}$
4	2	250 $\times\frac{1}{2}$
6	3	125 $\times\frac{1}{2}$
8	4	62.5 $\times\frac{1}{2}$
10	5	31.25 $\times\frac{1}{2}$

b) Graph the relation



c) Write an equation to model this growth

$y = a(b)^x$, $x \to \frac{t}{2}$

$y = 1000\left(\frac{1}{2}\right)^x$

$y = 1000\left(\frac{1}{2}\right)^{t/2}$

$A(t) = 1000\left(\frac{1}{2}\right)^{t/2}$

d) How much remains after 25 years?

$A(t) = 1000\left(\frac{1}{2}\right)^{t/2}$

$A(25) = 1000\left(\frac{1}{2}\right)^{25/2}$

$\boxed{A(25) \simeq 0.173 \text{ mg.}}$

Example 1: Plutonium-239 has a half-life of 24 years. Find the amount of a 50mg sample left after 35 years.

$y = ?$

$a = 50$

$b = \frac{1}{2}$

$x = \frac{35}{24}$

$$y = a(b)^x$$

$$y = 50\left(\frac{1}{2}\right)^{35/24}$$

$$\boxed{y \approx 18.2\text{mg}}$$

If exponential decay is given as a percent use the equation: $y = a(1-r)^x$

$a =$ initial amount

$r =$ rate of decrease (use decimal value)

$x =$ # of decay periods $\left(\frac{\text{total time}}{\text{time of 1 decay period}}\right)$

Example 2:

You buy a new car for \$24,000. The value of the car decreases by 16% every year. How much will the car be worth in 8 years?

$y = ?$

$a = 24000$

$r = 0.16$

$x = 8$

$$y = a(1-r)^x$$

$$y = 24000(1-0.16)^8$$

$$y = 24000(0.84)^8$$

$$\boxed{y = \$5949.02}$$

Example 3: An adult takes 400mg of Advil. Each hour, the amount of Advil in the adult's system decreases by about 29%. How much Advil will be left after 4 hours?

$a = 400$

$r = 0.29$

$x = \frac{4}{1} = 4$

$y = ?$

$y = a(1-r)^x$

$y = 400(1-0.29)^4$

$\boxed{y \approx 101.65\text{mg}}$

Example 4: U-239 has a half-life of about 2 years. If you start with a 1000 mg sample, how long will it take to decay to 10 mg?

$a = 1000$

$y = 10$

$b = \frac{1}{2}$

$x = \frac{t}{2}$

$y = a(b)^x$

$10 = 1000\left(\frac{1}{2}\right)^{t/2}$

$0.01 = 0.5^{t/2}$

$\frac{t}{2} = \log_{0.5}(0.01)$

$t = 2\log_{0.5}(0.01)$

$\boxed{t \approx 13.3 \text{ years}}$

L3 – Compound Interest / Initial Amount
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General Properties of Exponential Decay (or growth)

Equation: $y = a(b)^x$

$a =$ initial amount

$b =$ growth $(b>1)$ or decay $(0<b<1)$ factor

$y =$ future amount

$x =$ # of growth/decay periods

To calculate x, use the equation: $x = \dfrac{\text{total time}}{\text{time of 1 growth/decay period}}$

Finding Initial Amount

Example 1: You are going to ship some U-239 which has a half-life of 2 years (24 months). There must be 500g upon arrival. If shipping will take 4 months, how much should you package initially?

$y = 500$

$a = ?$

$b = \frac{1}{2}$

$x = \frac{4}{24} = \frac{1}{6}$

$y = a(b)^x$

$\dfrac{500}{\left(\frac{1}{2}\right)^{1/6}} = \dfrac{a\cancel{\left(\frac{1}{2}\right)^{1/6}}}{\cancel{\left(\frac{1}{2}\right)^{1/6}}}$

$\boxed{a \simeq 561.23\text{ g}}$

Example 2: We (as a class) have been hired by a surgeon to grow a skin graft. It takes 3 days for the amount of skin to double. If we need 2kg of skin in one week, how much should we start with?

$y = 2$

$a = ?$

$b = 2$

$x = \frac{t}{3} = \frac{7}{3}$

$y = a(b)^x$

$\dfrac{2^{3/3}}{2^{7/3}} = \dfrac{a\cancel{(2)^{7/3}}}{\cancel{(2)^{7/3}}}$

$\dfrac{1}{2^{4/3}} = a$

$\boxed{a \simeq 0.4\text{ kg}}$

Compound Interest

Formula: $A = P(1 + \frac{i}{n})^{nt}$

A: Future amount

P: Principle amount

i: interest rate (decimal)

n: number of times interest is compounded each year

t: number of years

Example 3: You have just passed GO and you receive $200. You decide to invest it for 4 years in an account that pays 5% interest per year. How much will you have after 10 years if…

a) the interest is compounded annually?

$$A = P(1+\frac{i}{n})^{nt}$$
$$A = 200(1+\frac{0.05}{1})^{1(4)}$$
$$A = 200(1.05)^4$$
$$A = \$243.10$$

b) the interest is compounded semi-annually?

$$A = 200(1+\frac{0.05}{2})^{2(4)}$$
$$A = 200(1.025)^8$$
$$A = \$243.68$$

c) the interest is compounded monthly?

$$A = 200(1+\frac{0.05}{12})^{12(4)}$$
$$A = 200(1+\frac{0.05}{12})^{48}$$
$$A = \$244.18$$

Example 4: You are about to go to University. When you are done in 4 years, you want to buy a new car. The one you are looking at costs $16,000. If you can find an investment that pays 10.9% interest per year, compounded annually, how much should you invest now?

$$A = P\left(1 + \frac{i}{n}\right)^{nt}$$

$$16000 = P\left(1 + \frac{0.109}{1}\right)^{1(4)}$$

$$\frac{16000}{1.109^4} = \frac{P\cancel{(1.109)^4}}{\cancel{1.109^4}}$$

$$\boxed{P \simeq \$10577.76}$$

L4 – Properties of Exponential Functions
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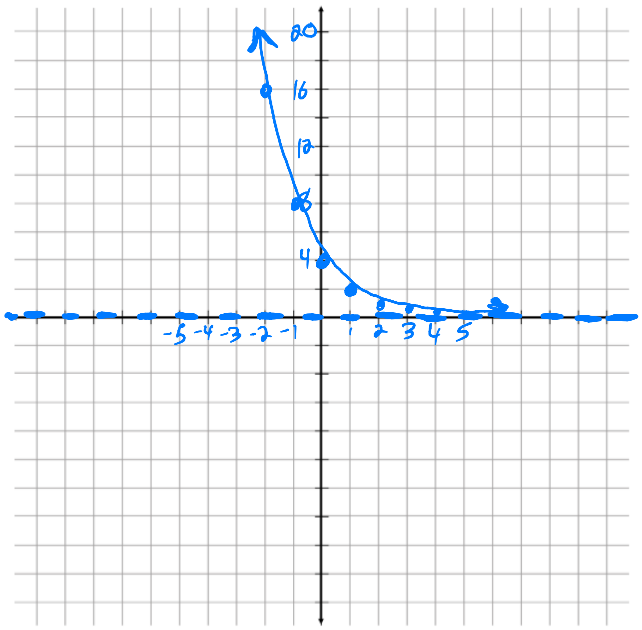
Example 1:

Graph each exponential function. Identify the domain, range, intercepts, intervals of increase/decrease, and the equation of any asymptotes.

a) $y = 4\left(\frac{1}{2}\right)^x$

x	y
-3	32
-2	16
-1	8
0	4
1	2
2	1
3	0.5

$\times\frac{1}{2}$



Domain: $\{X \in \mathbb{R}\}$

Range: $\{Y \in \mathbb{R} \mid y > 0\}$

***x*-int:** NONE

***y*-int:** $(0, 4)$

intervals of increase/decrease:

Decreasing

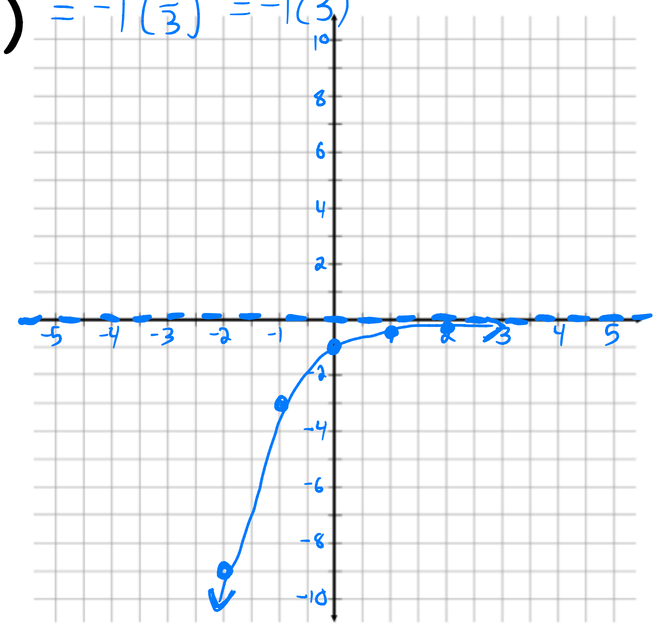
asymptote:

$y = 0$

b) $y = -3^{-x} = -1\left(\frac{1}{3}\right)^x = -1\left(\frac{1}{3}\right)^{-1} = -1(3)^1$

x	y
-2	-9
-1	-3
0	-1
1	-0.33
2	-0.11

$\times\frac{1}{3}$



Domain: $\{X \in \mathbb{R}\}$

Range: $\{Y \in \mathbb{R} \mid y < 0\}$

***x*-int:** NONE

***y*-int:** $(0, -1)$

intervals of increase/decrease:

Increasing

asymptote:

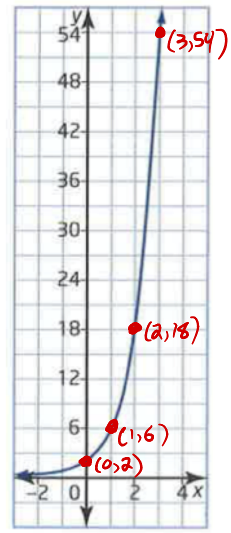
$y = 0$

Example 2:

Write the equation in the form $\boxed{y = ab^x}$ for the graph shown.

Start by determining the growth factor (b). As x changes by 1 unit, what factor does y change by?

x	y	
0	?	$\times 3$
1	6	$\times 3$
2	18	$\times 3$
3	54	



Next, determine the initial value (a) by plugging in the coordinates of one of the points (x, y) on the graph and the growth rate (b), then solve for a.

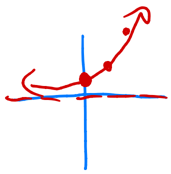
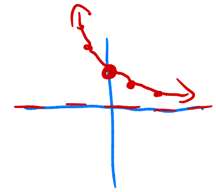
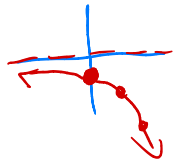
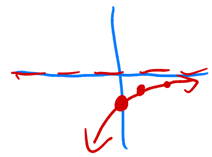
$$y = a(b)^x$$
$$y = a(3)^x$$
$$6 = a(3)^1$$
$$a = \frac{6}{3}$$
$$a = 2$$

Final Equation: $y = 2(3)^x$

Example 3: A radioactive sample has a half-life of 3 days. The initial sample is 200 mg. Write a function to relate the amount remaining, in milligrams, to the time, in days.

$$y = a(b)^x$$
$$y = 200\left(\frac{1}{2}\right)^{t/3}$$
$$A(t) = 200\left(\frac{1}{2}\right)^{t/}$$

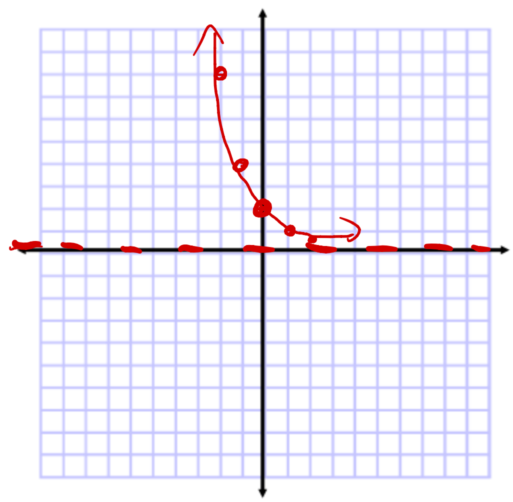
What do you know so far about when a function of the form $y = a(b)^x$ is increasing and when it is decreasing?

$a>0$ $b>1$  Increasing	$a>0$ $0<b<1$  Decreasing
$a<0$ $b>1$  Decreasing	$a<0$ $0<b<1$  Increasing

Example 4: Make a rough sketch of the graph of the following functions based on your knowledge of whether they are increasing or decreasing. $y = a(b)^x$

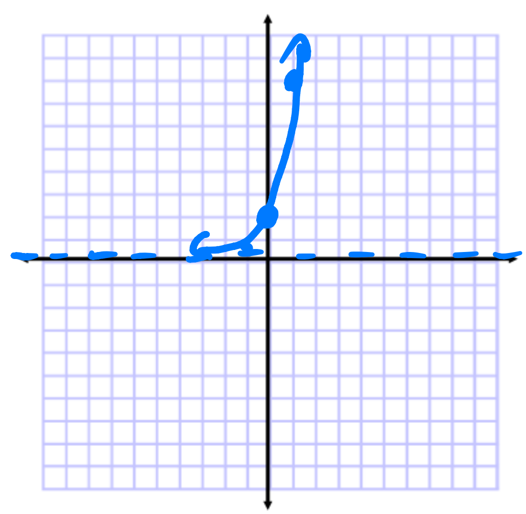
a) $y = 2\left(\frac{1}{2}\right)^x$

Decreasing



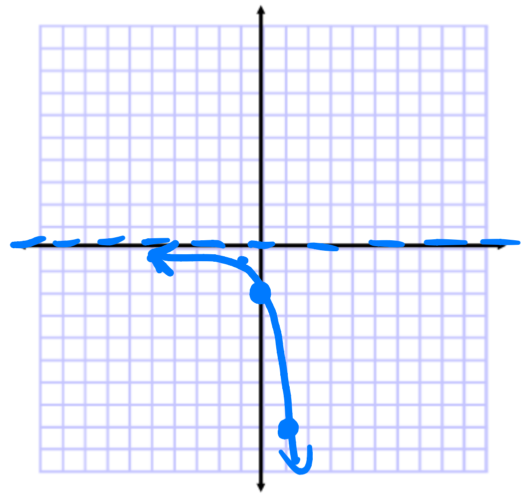
b) $y = 2(4)^x$

Increasing



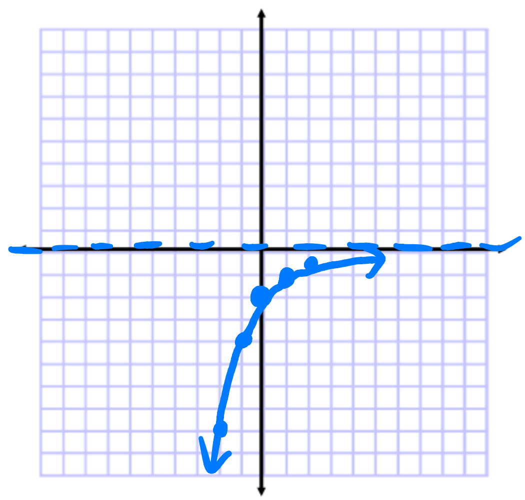
c) $y = -2(4)^x$

Decreasing



d) $y = -2\left(\frac{1}{2}\right)^x$

Increasing



L5 – Transformations of Exponential Functions

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Warm-up: Which of the following graphs are the same? Rule: $(x^a)^b = x^{a \times b}$

$f(x) = 32^x$ $g(x) = 9^x$ $h(x) = 2^{3x} = (2^3)^x = 8^x$ $n(x) = 2^{5x} = (2^5)^x = 32^x$

$p(x) = 3^{3x}$ $q(x) = 3^{2x} = (3^2)^x = 9^x$ $r(x) = 8^x$

Exponential functions can be transformed in the same way as other function. The graph of can be found by performing transformations on the graph of $f(x) = b^x$

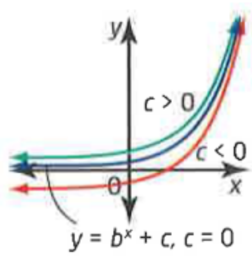
$g(x) = a\, b^{k(x-d)} + c$

<u>Changes to the *y*-coordinates (vertical changes)</u>

***c*: vertical translation** $g(x) = b^x + c$

The graph of $g(x) = b^x + c$ is a vertical translation of the graph of b^x by c units.

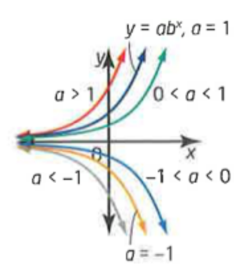
If $c > 0$, the graph shifts **UP**
If $c < 0$, the graph shifts **DOWN**



***a*: vertical stretch/compression** $\boldsymbol{g(x) = a \cdot b^x}$

The graph of $g(x) = \boldsymbol{a} \cdot b^x$ is a vertical stretch or compression of the graph of b^x by a factor of a.

If $a > 1$ OR $a < -1$, **vertical stretch** by a factor of $\lvert a \rvert$
If -1< $a < 1$, **vertical compression** by a factor of $\lvert a \rvert$
If $a < 0$, **vertical reflection** (reflection over the x-axis)

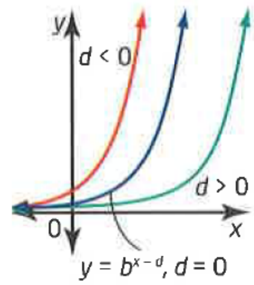


Changes to the *x*-coordinates (horizontal changes)

***d*: horizontal translation** $g(x) = b^{x-d}$

The graph of $g(x) = b^{x-d}$ is a horizontal translation of the graph of b^x by d units.

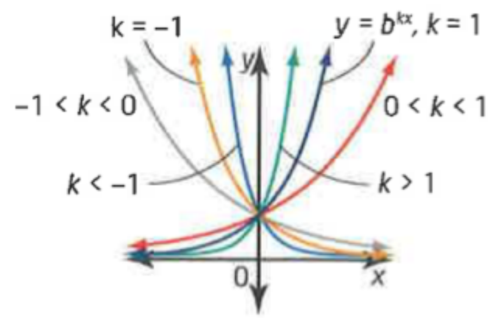
If $d > 0$, the graph shifts **RIGHT**
If $d < 0$, the graph shifts **LEFT**



k: horizontal stretch/compression $g(x) = b^{kx}$

The graph of $g(x) = b^{kx}$ is a horizontal stretch or compression of the graph of b^x by a factor of $\frac{1}{k}$

If $k > 1$ OR $k < -1$, **horizontal compression** by a factor of $\frac{1}{\lvert k \rvert}$
If -1< $k < 1$, **horizontal stretch** by a factor of $\frac{1}{\lvert k \rvert}$
If $k < 0$, **horizontal reflection** (reflection over the y-axis)



Don't forget that the order of the transformations matters!!!

Do the reflections, stretches, and compressions first. Then do the horizontal and vertical shifts.

k d

Example 1: Graph the function $g(x) = 2(2)^{\frac{1}{2}(x-1)}$

a

Step 1: What is the base function?

$y = 2^x$

Step 2: Describe the transformations made to the base function.

a = 2; vertical stretch by a factor of 2 (2y)
k = ½; horizontal stretch by a factor of 2 (2x)
d = 1; shift right 1 unit (x+1)

Step 3: Make a table of values for the base function and the transformed function $g(x)$

$y = 2^x$	
x	y
-3	0.125
-2	0.25
-1	0.5
0	1
1	2
2	4
3	8

×2

×2

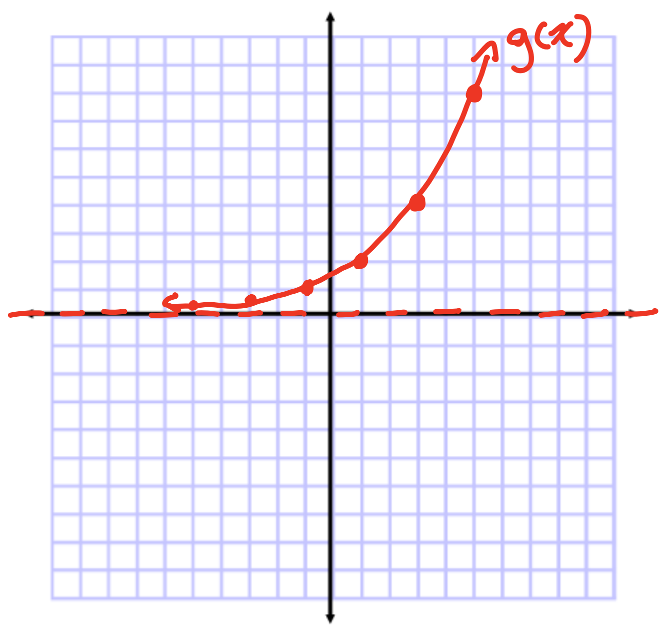
$\frac{x}{k}+d$

$g(x) = \overset{a}{2}(2)^{\overset{k}{\frac{1}{2}}(x-\overset{d}{1})}$

$ay+c$

$g(x) = 2(2)^{\frac{1}{2}(x-1)}$	
$2x+1$	$2y$
-5	0.25
-3	0.5
-1	1
1	2
3	4
5	8
7	16

Step 4: Graph ~~both functions~~



Example 2: Graph the function $g(x) = 3^{2x-4} + 1$ $= 1\,3^{2(x-2)} + 1$

a k d c

Hint 1: *The 'k' value must be common factored out.*

Hint 2: *'c' value is the horizontal asymptote.*

Step 1: What is the base function?

$y = 3^x$

Step 2: Describe the transformations made to the base function.

$k = 2$; horizontal compression by a factor of $\frac{1}{2}$ $\left(\frac{x}{2}\right)$

$d = 2$; shift RIGHT 2 units $(x+2)$

$c = 1$; shift UP 1 unit $(y+1)$

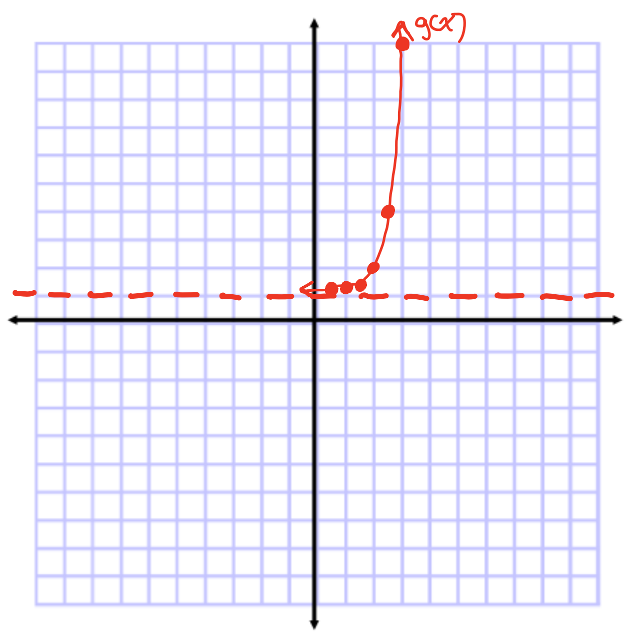
Step 3: Make a table of values for the base function and the transformed function $g(x)$

$y = 3^x$	
x	***y***
-3	0.04
-2	0.11
-1	0.33
0	1
1	3
2	9
3	27

$\frac{x}{k} + d$ $ay + c$

$g(x) = 3^{2(x-2)} + 1$	
$\frac{x}{2} + 2$	$y + 1$
0.5	1.04
1	1.11
1.5	1.33
2	2
2.5	4
3	10
3.5	28

Step 4: Graph the transformed function



Example 3: Graph the function $g(x) = \underset{a}{-2}\left(\frac{1}{2}\right)^{\overset{d}{x-3}} \underset{c}{-\,2}$

Step 1: What is the base function?

$y = \left(\frac{1}{2}\right)^x$

Step 2: Describe the transformations made to the base function.

$a = -2$; vertical stretch by a factor of 2 $(2y)$
vertical reflection $(-y)$

$d = 3$; shift right 3 units $(x+3)$

$c = -2$; shift Down 2 units $(y-2)$

Step 3: Make a table of values for the base function and the transformed function $g(x)$

$y = \left(\frac{1}{2}\right)^x$	
x	***y***
-3	8
-2	4
-1	2
0	1
1	0.5
2	0.25
3	0.125

$\frac{x}{k} + d$ $\quad$ $ay + c$

$g(x) = -2\left(\frac{1}{2}\right)^{x-3} - 2$	
$x+3$	$-2y-2$
0	-18
1	-10
2	-6
3	-4
4	-3
5	-2.5
6	-2.25

Step 4: Graph the transformed function