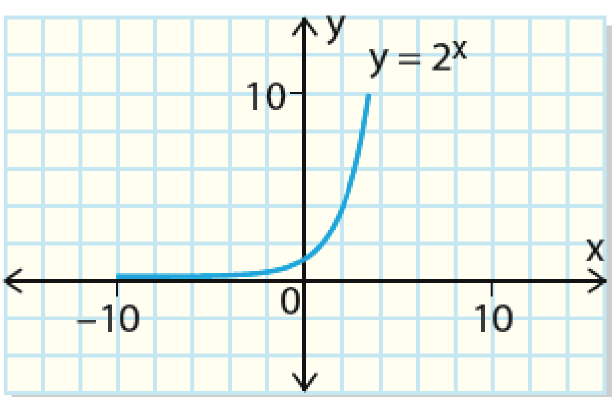
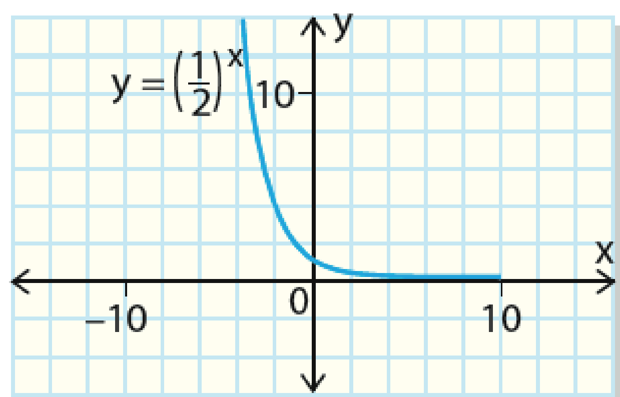
Chapter 3- Exponential Functions

WORKBOOK

MCR3U





3.1 Exponential Growth – Worksheet

SOLUTIONS

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1) An insect colony, with an initial population of 20, quadruples every day.

a) Complete the table.

Day	Population	First Differences	Second Differences
0	20		
1	80	60	180
2	320	240	720
3	1280	960	2880
4	5120	3840	11520
5	20480	15360	

b) Is the relationship between the insect population and the number of days exponential? Explain how you can tell.

The values in each difference column increase by a factor of 4; so it is exponential.

2) Suppose that there is a rumour going around your school that next year all weekends will be extended to three days. Initially, on day 0, five students know the rumour. Suppose that each person who knows the rumour tells two more students the day after they hear about it. Also assume that no-one hears the rumour more than once.

How many people will learn about the rumour...

i) on day 1

Day 1 = $5 \times 2 = 10$

ii) on day 2

Day 2 = $10 \times 2 = 20$

a b

3) A bacteria colony has an initial population of 200. The population triples every week.

a) Write an equation to relate population, *p*, to time, *t*, in weeks.

$p = 200(3)^t$

b) Determine the approximate population after 10 days.

$p = 200(3)^{10/7}$

$p = 960.8$ Approximately 961

c) Determine the approximate population after 12 weeks.

$p = 200(3)^{12}$

$= 106\,288\,200$

a $r = 0.06$

4) An investment of $4000 earns interest, which causes it to increase by 6% per year.

a) Write an equation for the amount, *A*, in dollars, that the investment is worth as a function of time, *n*, in years.

$A = 4000(1.06)^n$

b) Determine the amount that the investment is worth after

i) 2 years

$A = 4000(1.06)^2$

$= \$4494.40$

ii) 10 years

$A = 4000(1.06)^{10}$

$= \$7163.39$

iii) 20 years

$A = 4000(1.06)^{20}$

$= \$12\,828.54$

c) What is the value and meaning of the function at *n* = 0?

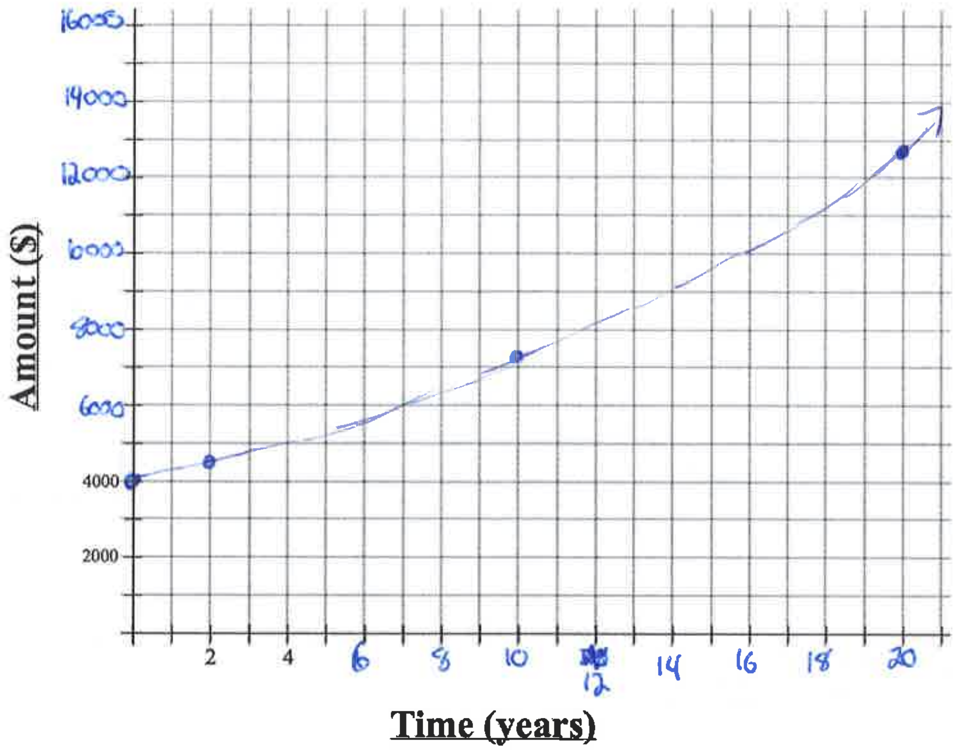
$A = 4000(1.06)^0$

$A = 4000(1)$

$A = \$4000$

This is the initial value.

d) Graph the relation for $n = 0$ to $n = 20$.



e) How long does it take for the investment to be worth $20 000?

$$20\,000 = 4000(1.06)^n$$
$$5 = 1.06^n$$
$$\log 5 = \log(1.06)^n$$
$$\log 5 = n \cdot \log 1.06$$
$$\frac{\log 5}{\log 1.06} = n$$
$$n = 27.62 \text{ years.}$$

5) A colony of bacteria starts with 400 cells and doubles in number every two days. A second colony of bacteria starts with 200 cells and quadruples in number every day. Write an equation to model the population of each colony, *A*, to time, *t*, in days.

Colony 1

$$A = 400(2)^{t/2}$$

Colony 2

$$A = 200(4)^t$$

BONUS: Bacteria A has an initial population of 500 and doubles every day, while bacteria B has an initial population of 50 and triples daily. After how long will the population of B overtake the population of A? What will their populations be at this point?

***(Hint:** product rule for logarithms)*

$500(2)^x = 50(3)^x$

$10(2)^x = 3^x$

$\log[10(2)^x] = \log 3^x$

$\log 10 + \log 2^x = \log 3^x$

$1 + x \cdot \log 2 = x \cdot \log 3$

$1 = x \cdot \log 3 - x \cdot \log 2$

$1 = x(\log 3 - \log 2)$

$\frac{1}{\log 3 - \log 2} = x$

$x = 5.679$ days

$y = 25\,614.23$

Answers

1) a)

Day	Population	First Differences	Second Differences
0	20		
1	80	60	180
2	320	240	720
3	1 280	960	2 880
4	5 120	3 840	11 520
5	20 480	15 360	

b) the values in each difference column increase by a factor of 4.

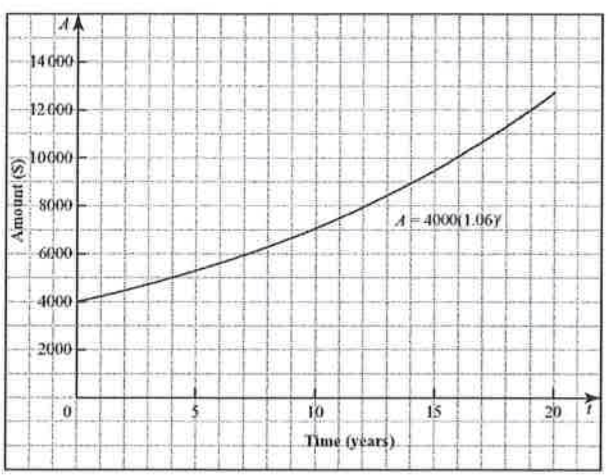
2) i) 10 **ii)** 20

3) a) $p = 200(3)^t$ **b)** 961 **c)** 106 288 200

4) a) $A = 4000(1.06)^n$ **b) i)** \$4494.40 **ii)** \$7163.39 **iii)** \$12 828.54

c) At $n = 0$, $A = \$4000$, which is the initial value of the investment.

d)



e) approximately 27.6 years

5) $A = 400(2)^{\frac{t}{2}}$ and $A = 200(4)^t$

3.2 Exponential Decay – Worksheet

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SOLUTIONS

1) During medical treatment, the number of bacterial cells in a patient decreases by a factor of $\frac{1}{2}$ every day. A patient has 1 000 000 bacterial cells on Monday, the first day of treatment.

b

a

$x=3$

a) How many bacterial cells will remain on Thursday?

$y = 1\,000\,000\left(\frac{1}{2}\right)^3$

$y = 125\,000$

b) How many bacterial cells will remain on Sunday?

$y = 1\,000\,000\left(\frac{1}{2}\right)^6$

$y = 15\,625$

c) On what day will the number of remaining bacterial cells be 1950?

$1950 = 1\,000\,000\left(\frac{1}{2}\right)^x$

$0.00195 = \left(\frac{1}{2}\right)^x$

$\log 0.00195 = \log\left(\frac{1}{2}\right)^x$

$\log 0.00195 = x \cdot \log\left(\frac{1}{2}\right)$

$\frac{\log 0.00195}{\log 0.5} = x$

$x = 9.0$ days $\Rightarrow$ Next Wednesday

2) Tungsten-187 (W-187) is a radioactive isotope that has a half-life of 1 day. Suppose you start with a 100-mg sample...

a) make a table of values that gives the amount of tungsten remaining at the end of each day for the next 4 days.

Time (days)	Amount of W-187 remaining
0	100
1	50
2	25
3	12.5
4	6.25

b) Write an equation in the form $f(x) = ab^x$ to relate the amount of W-187 remaining and time.

$f(x) = 100\left(\frac{1}{2}\right)^x$

c) Sketch a graph of the relation

d) How much W-187 will remain after 1 week?

$f(7) = 100\left(\frac{1}{2}\right)^7$

$= 0.78125$ mg

e) How long will it take for the W-187 to decay to 5% of its initial amount?

$5 = 100\left(\frac{1}{2}\right)^x$

$0.05 = (0.5)^x$

$\frac{\log 0.05}{\log 0.5} = x$

$x = 4.32$ days.

3) Shylo is very excited about her brand new car! Although she paid $\overset{a}{\$20\,000}$ for the car, its resale value will depreciate (decrease) by $\underset{r}{30\%}$ of its current value every year.

a) Write an equation relating the car's depreciated value, *v*, in dollars, to the time, *t*, in years since her purchase.

$V = 20\,000\,(0.7)^t$

b) How much will Shylo's car be worth in...

i) 1 year?

$V = 20\,000(0.7)^1$

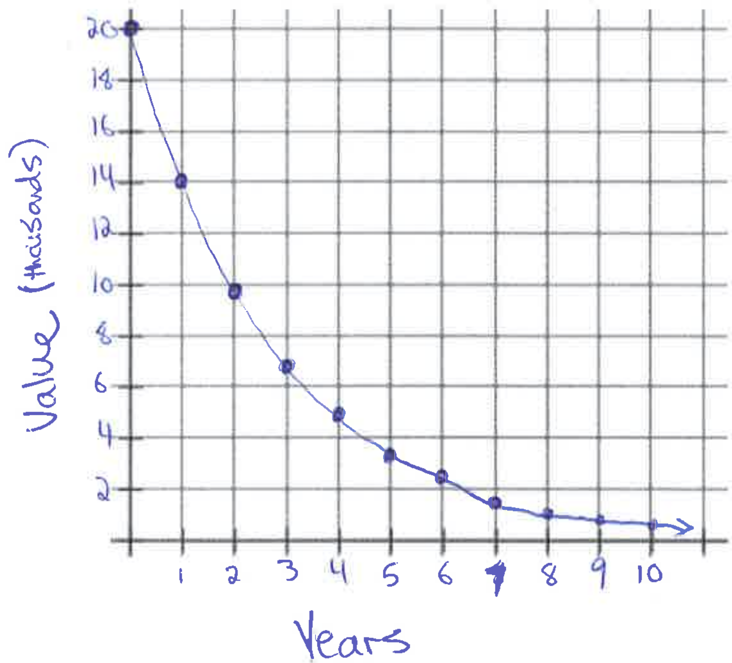
$= \$14000$

ii) 2 years?

$V = 20\,000(0.7)^2$

$= \$9800$

c) Graph the depreciation function.



d) How long will it take for Shylo's car to depreciate to 10% of its original price?

$2000 = 20\,000(0.7)^x$

$0.1 = 0.7^x$

$\frac{\log 0.1}{\log 0.7} = x$

$x \doteq 6.5$ years

4) An isotope of a radioactive substance has a half-life of 23 days. Suppose that you start with an 800-mg sample of the material.

a) Find an equation that models this data $y = 800\left(\frac{1}{2}\right)^{t/23}$

b) Use the equation to determine the amount of substance left after 100 days

$y = 800\left(\frac{1}{2}\right)^{100/23}$

$y = 39.29$ mg.

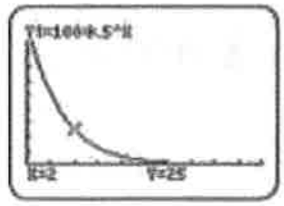
Answers

1) a) 125 000 **b)** 15 625 **c)** next Wednesday

2) a)

Time (days)	Amount of W-187 remaining (mg)
0	100
1	50
2	25
3	12.5
4	6.25

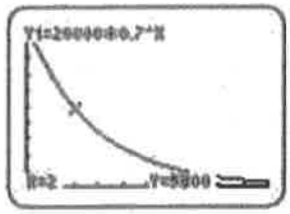
b) $f(x) = 100\left(\frac{1}{2}\right)^x$ c)



d) 0.781 25 mg

e) 4.3 days

3) a) $v(t) = 20000(0.7)^t$ **b) i)** 14 000 **ii)** 9800 **c)**



d) about 6.5 years

4) a) $A = 800\left(\frac{1}{2}\right)^{\frac{t}{23}}$ **b)** 39.29 mg

Compound Interest – Worksheet
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SOLUTIONS

1) Marvin deposits $100 into an account that pays interest at 5% per year, compounded annually.

a) Write an equation that can be used to calculate the amount in his account in the form $A = P(1+i)^n$.

$$A = 100(1.05)^n$$

b) Complete the following table...

Number of Compounding Periods (years)	Amount ($)
0	100
1	105
2	110.25
3	115.76
4	121.55

2) Sadia deposits a $2000 inheritance into an account that earns 4% per year, compounded annually. Find the amount in the account after each time.

a) 3 years

$$A = 2000(1.04)^n$$
$$= 2000(1.04)^3$$
$$= \$2249.73$$

b) 8 years

$$A = 2000(1.04)^8$$
$$= \$2737.14$$

3) Soda invests $500 in an account that earns 7% per year, compounded annually. How long does Soda need to leave her investment in the account in order to double her money?

$$A = 500(1.07)^n$$
$$1000 = 500(1.07)^n$$
$$\log 2 = \log 1.07^n$$
$$\log 2 = n \cdot \log 1.07$$

$$n = \frac{\log 2}{\log 1.07}$$
$$n = 10.24 \text{ years}$$

4) Art Vandelay deposited some money into an account that pays 3% per year, compounded annually. Today the account balance is $660. How much was in the account...

a) 1 year ago

$660 = P(1.03)^1$

$P = \frac{660}{1.03}$

$P = \$640.78$

b) 5 years ago?

$660 = P(1.03)^5$

$P = \frac{660}{(1.03)^5}$

$P = \$569.32$

5) Elaine wants to invest some money that will grow to $1000 in 6 years. If her account pays 4.5% interest, compounded annually, how much should Lydia invest today?

$1000 = \quad (1.045)^6$

$= \frac{1000}{(1.045)^6}$

$= \$767.90$

6) To buy a new guitar, Phoebe borrows $650, which she plans to repay in 5 years. The bank charges 12% per annum, compounded annually.

a) Determine the amount that Phoebe must repay.

$A = 650(1.12)^5$

$A = \$1145.52$

b) How much would she have to pay if the interest was compounded semi-annually instead of annually? (Hint: twice as many compounding periods but the interest rate will need to be cut in half)

$A = ?$

$P = 650$

$i = 0.12 \div 2 = 0.06$

$n = 5 \times 2 = 10$

$A = 650(1.06)^{10}$

$A = \$1164.05$

c) How much would she have to pay if the interest was compounded monthly?

$A = ?$

$P = 650$

$i = 0.12 \div 12 = 0.01$

$n = 5 \times 12 = 60$

$A = 650(1.01)^{60}$

$A = \$1180.85$

Answers

) a) $A = P(1.05)^n$ **b)**

Number of Compounding Periods (years)	Amount ($)
0	100
1	105
2	110.25
3	115.76
4	121.55

2) a) $2249.73 **b)** $2737.14

3) 10.24 years

4) a) $640.78 **b)** $569.32

5) $767.90

6) a) $1145.52 **b)** $1164.05 **c)** $1180.85

3.4 Properties of Exponential Functions – Worksheet SOLUTIONS

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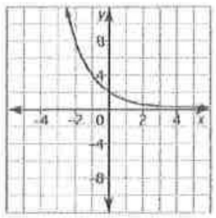
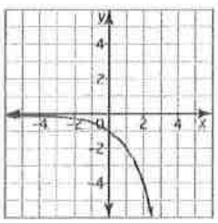
1) Match each graph with its corresponding equation.

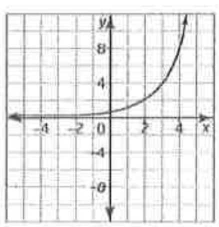
a) B



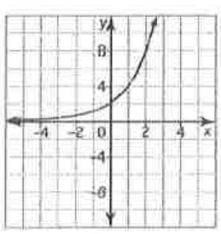
b) D



c) C



d) A



A $y = 2(2)^x$

B $y = 2\left(\frac{1}{2}\right)^x$

C $y = \frac{1}{2}(2)^x$

D $y = -2^x$

2) Given the following exponential equations, state whether they are increasing or decreasing.

a) $y = 3\left(\frac{1}{2}\right)^x$ D

b) $y = -3\left(\frac{1}{2}\right)^x$ I

c) $y = 3(2)^x$ I

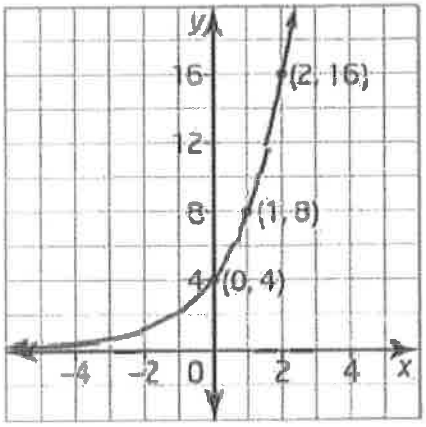
d) $y = -3(2)^x$ D

e) $y = 3(2)^{-x}$

$y = 3\left(\frac{1}{2}\right)^x$

D

3) Write an exponential equation to match the graph shown

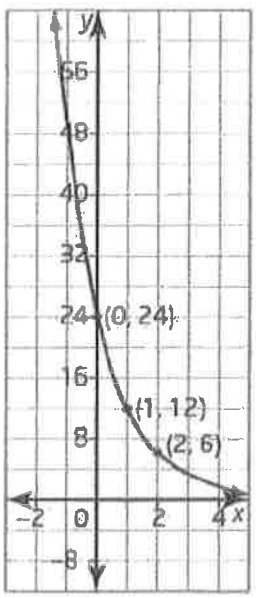


$a = 4$

$b = 2$

$y = 4(2^x)$

4) Write an exponential equation to match the graph shown



$a = 24$

$b = \frac{1}{2}$

$y = 24\left(\frac{1}{2}\right)^x$

5) A radioactive sample with an initial mass of 25 mg has a half-life of 2 days.

a) Write an equation to model this exponential decay where *t* is the time, in days, and *A* is the amount of the substance that remains.

$A = 25\left(\frac{1}{2}\right)^{t/2}$

b) What is the amount of radioactive material remaining after 7 days?

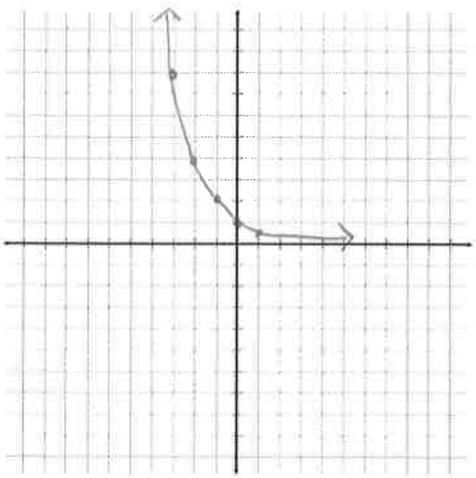
$A = 25\left(\frac{1}{2}\right)^{7/2}$

$A = 2.2\text{ mg}$

6) Graph each function and identify the...

- **i)** domain
- **ii)** range
- **iii)** x- and y-intercepts, if they exist
- **iv)** increasing or decreasing
- **v)** asymptote

a) $f(x) = \left(\frac{1}{2}\right)^x$



i) $D: \{x \in \mathbb{R}\}$

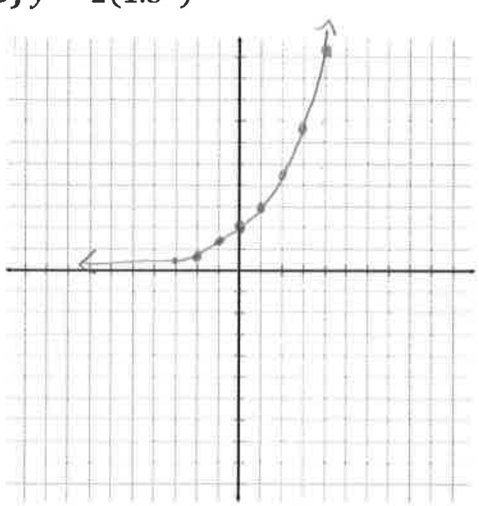
ii) $R: \{y \in \mathbb{R} \mid y > 0\}$

iii) y-intercept at $(0,1)$, no x-intercept.

iv) decreasing

v) $y = 0$ is a horizontal asymptote.

b) $y = 2(1.5^x)$



i) $D: \{x \in \mathbb{R}\}$

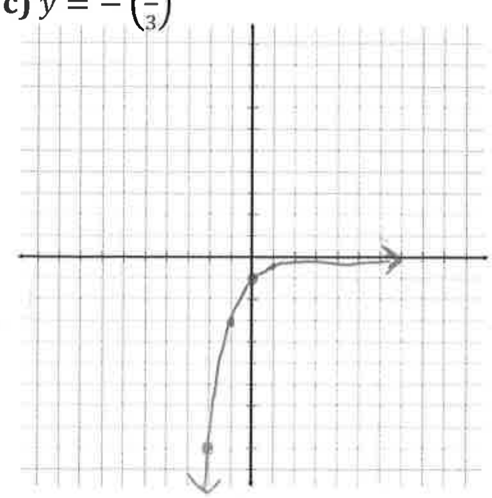
ii) $R: \{y \in \mathbb{R} \mid y > 0\}$

iii) y-int at $(0,2)$, no x-int.

iv) increasing

v) $y = 0$ is a horizontal asymptote.

c) $y = -\left(\frac{1}{3}\right)^x$



i) $D: \{x \in \mathbb{R}\}$

ii) $R: \{y \in \mathbb{R} \mid y < 0\}$

iii) y-int at $(0,-1)$

iv) increasing

v) $y = 0$ is a horizontal asymptote.

Answers

1) a) B **b)** D **c)** C **d)** A

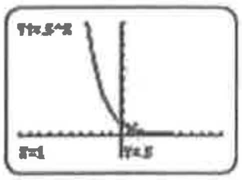
2) a) decreasing **b)** increasing **c)** increasing **d)** decreasing **e)** decreasing

3) $y = 4(2^x)$

4) $y = 24\left(\frac{1}{2}\right)^x$

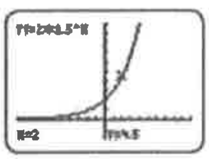
5) a) $A = 25\left(\frac{1}{2}\right)^{\frac{t}{2}}$ **b)** 2.2 mg

6) **a)**



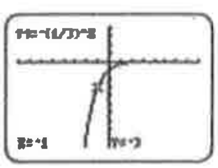
i) $\{x \in \mathbb{R}\}$
ii) $\{y \in \mathbb{R}, y > 0\}$
iii) no x-intercept; y-intercept 1
iv) always decreasing **v)** $y = 0$

b)



i) $\{x \in \mathbb{R}\}$
ii) $\{y \in \mathbb{R}, y > 0\}$
iii) no x-intercept; y-intercept 2
iv) always increasing **v)** $y = 0$

c)



i) $\{x \in \mathbb{R}\}$
ii) $\{y \in \mathbb{R}, y < 0\}$
iii) no x-intercept; y-intercept -1
iv) always increasing **v)** $y = 0$

3.5 Transformations of Exponential Functions - Worksheet

SOLUTIONS

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1) Describe the transformations that map the function $y = 2^x$ onto each of the following functions...

a) $y = 2^x - 2$

- down 2 units $(y-2)$

b) $y = 2^{x+3}$

- left 3 units $(x-3)$

c) $y = 4^x$

$= 2^{2x}$

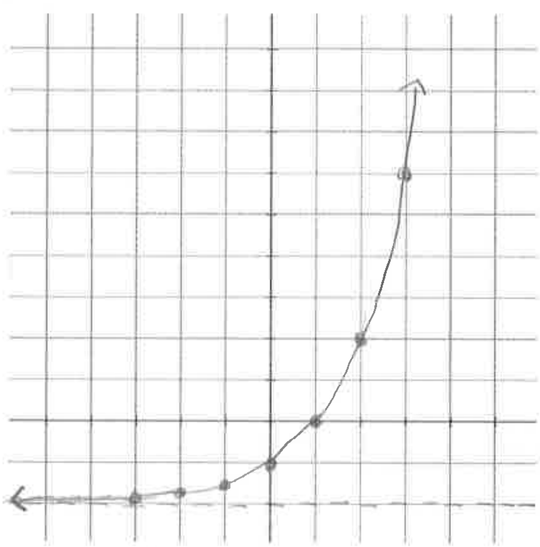
- horizontal compression bafo $\frac{1}{2}$ $(\frac{x}{2})$

d) $y = 3(2^{x-1}) + 1$

- vertical stretch bafo 3 $(3y)$
- right 1 $(x+1)$
- up 1 $(y+1)$

2) Create a sketch of each graph for each equation in question 1. (a table of values may help)

a)



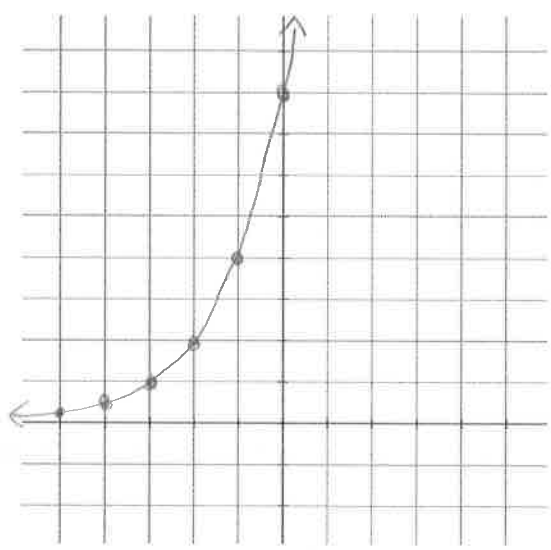
$y = 2^x$

x	y
-3	1/8
-2	1/4
-1	1/2
0	1
1	2
2	4
3	8

$y = 2^x - 2$

x	$y-2$
-3	-1.875
-2	-1.75
-1	-1.5
0	-1
1	0
2	2
3	6

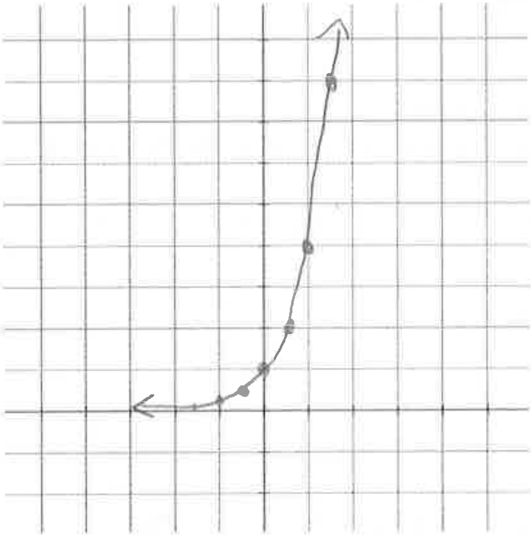
b)



$y = 2^{x+3}$

$x-3$	y
-6	0.125
-5	0.25
-4	0.5
-3	1
-2	2
-1	4
0	8

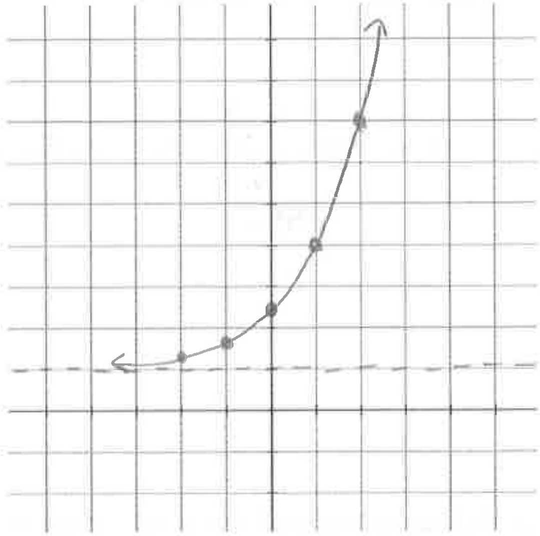
c)



$y = 2^{2x}$

$\frac{x}{2}$	y
-1.5	0.125
-1	0.25
-0.5	0.5
0	1
0.5	2
1	4
1.5	8

d)



$x+1$	$3y+1$
-2	1.375
-1	1.75
0	2.5
1	4
2	7
3	13
4	25

3) Write the equation for the function that results from each transformation applied to the base function $y = 5^x$.

a) translate down 3 units

$y = 5^x - 3$

b) shift right 2 units

$y = 5^{x-2}$

c) translate left $\frac{1}{2}$ unit

$y = 5^{x+0.5}$

d) shift up 1 unit and left 2.5 units

$y = 5^{x+2.5} + 1$

4) Describe the transformations that map the function $y = 8^x$ onto each function.

a) $y = \left(\frac{1}{2}\right)8^x$

vertical compression bafo $\frac{1}{2}$ $\left(\frac{y}{2}\right)$

b) $y = 8^{4x}$

- horizontal compression bafo $\frac{1}{4}$ $\left(\frac{x}{4}\right)$

c) $y = -8^x$

- vertical reflection $(-y)$

d) $y = 8^{-2x}$

- horizontal compression bafo $\frac{1}{2}$ and horizontal reflection $\left(\frac{x}{-2}\right)$

5) Write the equation for the function that results from each transformation applied to the base function $y = 7^x$

a) reflect in the x-axis (vertical reflection)

$y = -7^x$

b) stretch vertically by a factor of 3

$y = 3(7)^x$

) stretch horizontally by a factor of 2.4

$y = 7^{\frac{1}{2.4}x}$

d) reflect in the y-axis and ~~compress~~ stretch vertically by bafo 7

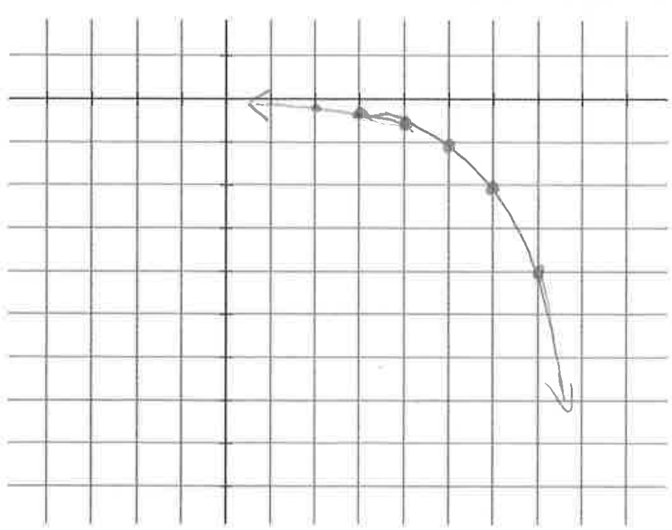
$y = 7(7)^{-x}$

6) Sketch the graph of $y = \left(-\frac{1}{2}\right)2^{x-4}$ by using $y = 2^x$ as the base and applying transformations.

$y = 2^x$

x	y
-3	1/8
-2	1/4
-1	1/2
0	1
1	2
2	4
3	8

x+4	$\frac{y}{-2}$
1	-0.0625
2	-0.125
3	-0.25
4	-0.5
5	-1
6	-2
7	-4



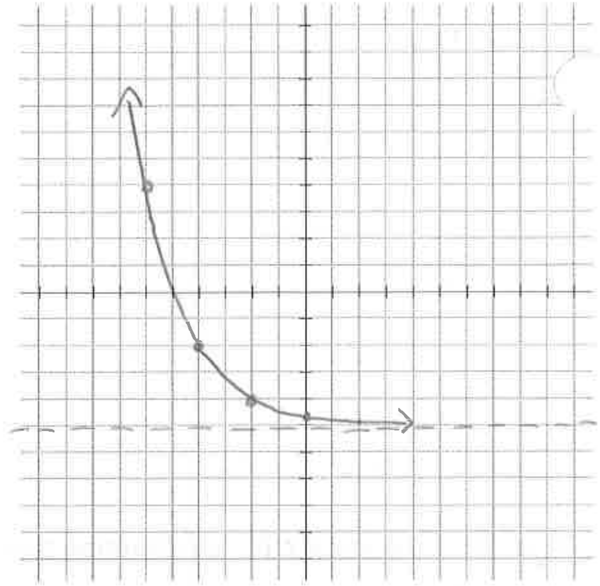
7) Sketch the graph of $y = 3^{-0.5x-1} - 5$ by using $y = 3^x$ as the base and applying transformations.

$y = 3^x$

x	y
-3	1/27
-2	1/9
-1	1/3
0	1
1	3
2	9
3	27

$y = 3^{-0.5(x+2)} - 5$

$-2x-2$	$y-5$
4	-4.96
2	-4.88
0	-4.67
-2	-4
-4	-2
-6	4
-8	22



8) a) Rewrite $y = 9^x$ using a base of 3. Describe how you can graph this function by transforming the graph of $y = 3^x$.

$y = 9^x$

$= (3^2)^x$

$= 3^{2x}$

horizontal compression by a factor of $\frac{1}{2}$

b) Rewrite $y = 9^x$ using a base of 81. Describe how you can graph this function by transforming the graph of $y = 81^x$.

$y = 9^x$

$= (81^{1/2})^x$

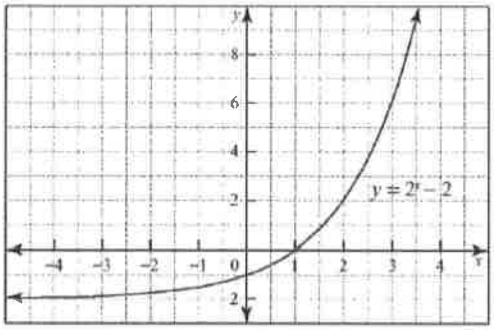
$= 81^{\frac{1}{2}x}$

horizontal stretch by a factor of 2.

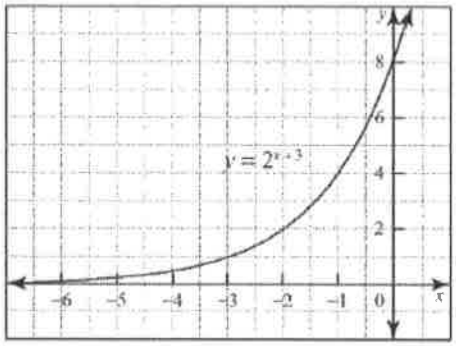
Answers

1) a) translate 2 units down **b)** translate 3 units left **c)** horizontal compression by a factor of $\frac{1}{2}$
d) vertical stretch by a factor of 3, a translation 1 unit to the right and 1 unit up

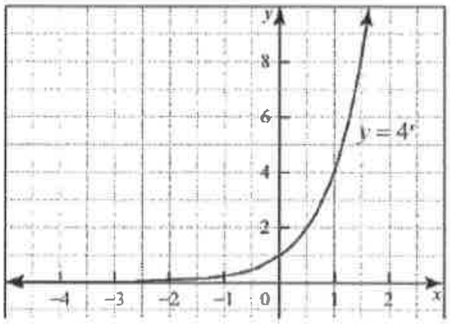
2) a)



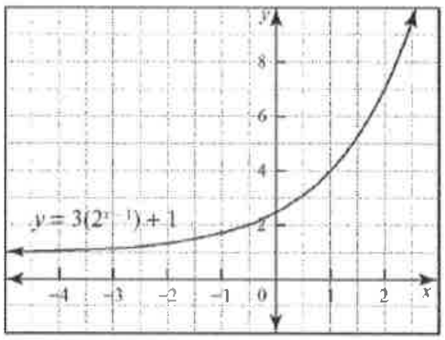
b)



c)



d)

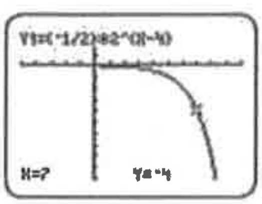


3) a) $y = 5^x - 3$ **b)** $y = 5^{x-2}$ **c)** $5^{x+\frac{1}{2}}$ **d)** $y = 5^{x+2.5} + 1$

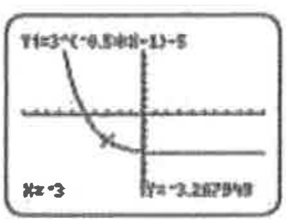
4) a) vertical compression by a factor of $\frac{1}{2}$ **b)** horizontal compression by a factor of $\frac{1}{4}$
c) vertical reflection (reflection in the x-axis)
d) horizontal reflection (reflection in the y-axis) and horizontal compression by a factor of $\frac{1}{2}$

5) a) $y = -7^x$ **b)** $y = 3(7^x)$ **c)** $y = 7^{\frac{x}{2.4}}$ **d)** $y = 7(7^{-x})$

6)



7)



8) a) $y = 3^{2x}$; horizontal compression of the graph of $y = 3^x$ by a factor of $\frac{1}{2}$

b) $y = 81^{\frac{1}{2}x}$; horizontal stretch of the graph of $y = 81^x$ by a factor of 2