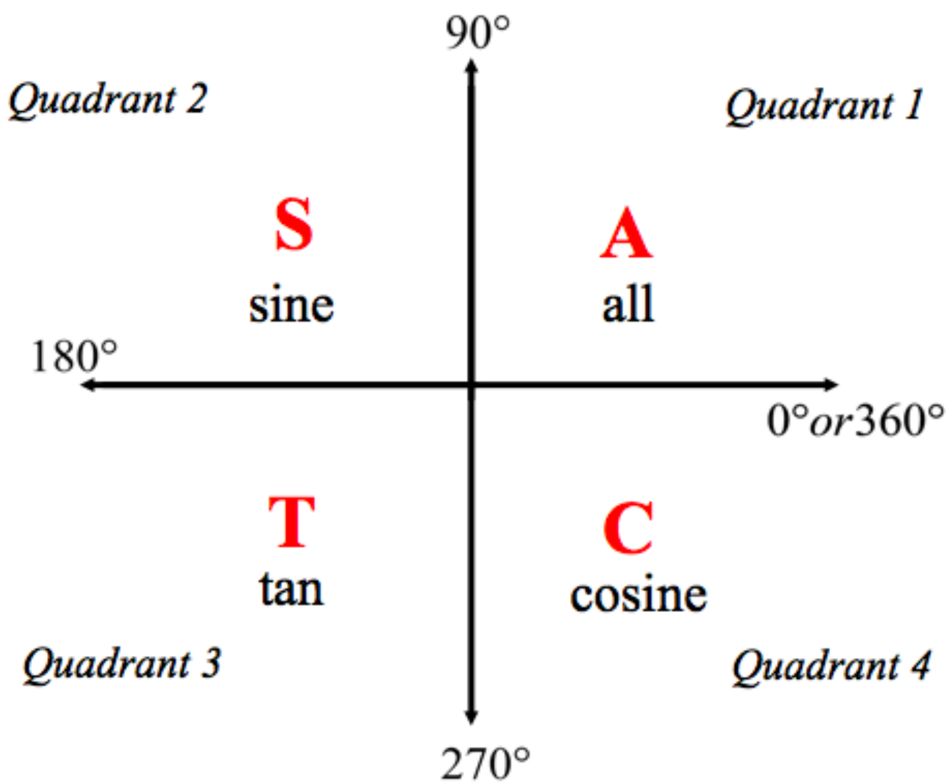
Chapter 4- Trigonometry

Lesson Package

MCR3U



Chapter 4 Outline

Unit Goal: Be able to determine the values of the trigonometric ratios for angles less than 360º; prove simple trigonometric identities; and solve problems using the primary trigonometric ratios, the sine law, and the cosine law.

Section	Subject	Learning Goals	Curriculum Expectations
L1	Special Angles 1	- determine the exact values of the sine, cosine, and tangent of the special angles: 0º, 30º, 45º, 60º, and 90º	D1.1
L2	Special Angles 2	- determine the exact values of the sine, cosine, and tangent of the special angles: 0º, 30º, 45º, 60º, and 90º	D1.1, D1.2
L3	Related and Co-Terminal Angles	- determine the measures of two angles from 0º to 360º for which the value of a given trigonometric ratio is the same	D1.3
L4	Reciprocal Trig Ratios	- define the secant, cosecant, and cotangent ratios for angles in a right triangle in terms of the sides of the triangle	D1.4
L5	Problems in 2-Diminesions	- solve problems involving right triangles and oblique triangles in two-dimensional settings using the primary trigonometric ratios, the cosine law, and the sine law	D1.6
L6	Problems in 3-Diminesions	- solve problems involving right triangles and oblique triangles in three-dimensional settings using the primary trigonometric ratios, the cosine law, and the sine law	D1.7
L7	Ambiguous Case of Sine	- solve problems in two-dimensional settings that involve the ambiguous case of sine	D1.6
L8	Trig Identities 1	- prove trig identities using the Pythagorean identity, the quotient identity, and reciprocal identities	D1.5
L9	Trig Identities 2	- prove trig identities using the Pythagorean identity, the quotient identity, and reciprocal identities	D1.5

Assessments	F/A/O	Ministry Code	P/O/C	KTAC
Note Completion	A		P	
Practice Worksheet Completion	F/A		P	
PreTest Review	F/A		P	
Test – Trig Geometry	O	D1.1, D1.2, D1.3, D1.4, D1.5, D1.6, D1.7	P	K(21%), T(34%), A(10%), C(34%)

L1 – Trig Review and Special Angles
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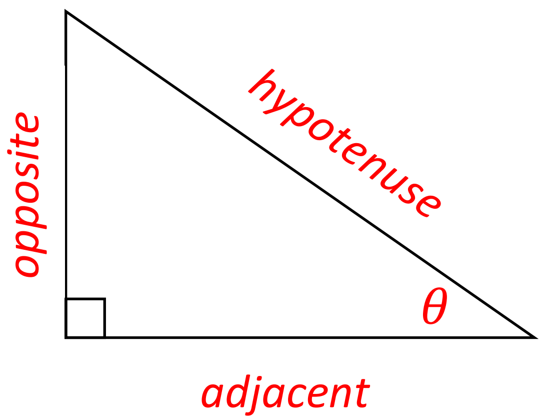
Part 1: Trig Review

Your main takeaway from grade 10 trigonometry should have been:

If we know a right triangle has an angle of θ, all other right triangles with an angle of θ are ____________ and therefore have __________________ ratios of corresponding sides.

There are three primary trigonometric ratios for right angled triangles. ______, _________, and ___________.



$\sin\theta =$

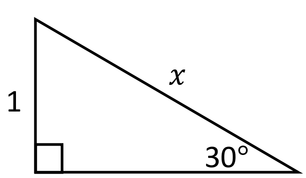
$\cos\theta =$

$\tan\theta =$

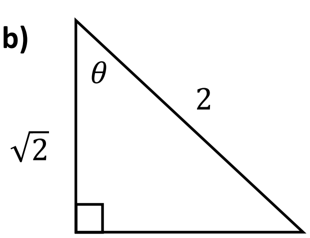
Acronym: SOHCAHTOA S $\frac{O}{H}$ C $\frac{A}{H}$ T $\frac{O}{A}$

Example 1: Find the indicated missing side or angle of each triangle

a)



b)

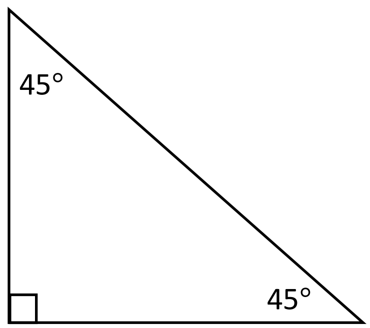


Part 2: Special Angles

There are 2 special triangles:

i) isosceles: 45° - 45° - 90°
ii) half equilateral: 30° - 60° - 90°

i)

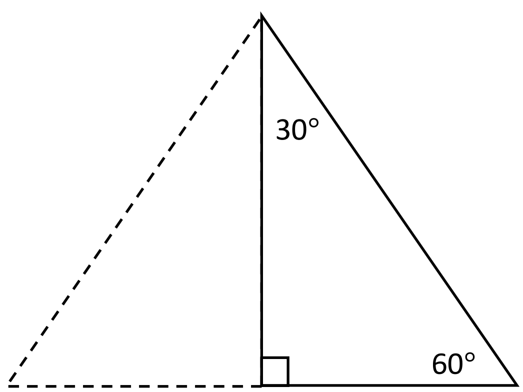


$\sin 45° =$

$\cos 45° =$

$\tan 45° =$

ii)



$\sin 30° =$

$\cos 30° =$

$\tan 30° =$

$\sin 60° =$

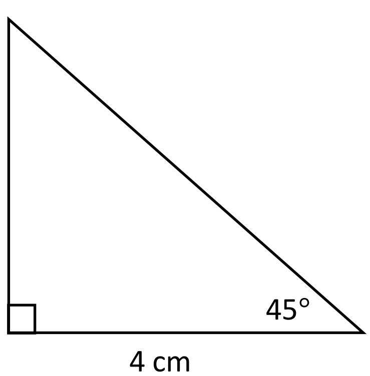
$\cos 60° =$

$\tan 60° =$

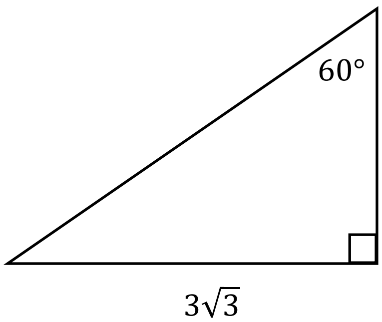
All sized right triangles with these angles are SIMILAR and therefore will have the same ratios of corresponding sides. Therefore, we can use these 2 special triangles to get ________ values for trig ratios involving a 30°, 45°, or 60° reference angle AND we don't need a calculator!

Example 2: Use special triangles to find the EXACT values of all sides and angles

a)



b)



Example 3: Determine the exact value of…

a) $(\sin 45°)(\cos 45°) + (\sin 30°)(\sin 60°)$

b) $\frac{\sin^2 30°}{1-\cos 30°}$

Part 3: Rationalizing the Denominator

Fractions should be simplified so that the denominator contains only rational numbers.

Example 4: Rationalize the denominator for each of the following expressions

a) $\frac{1}{\sqrt{2}}$

b) $\frac{3}{1+\sqrt{5}}$

L2 – Trig Ratios for Angles Greater than 90°

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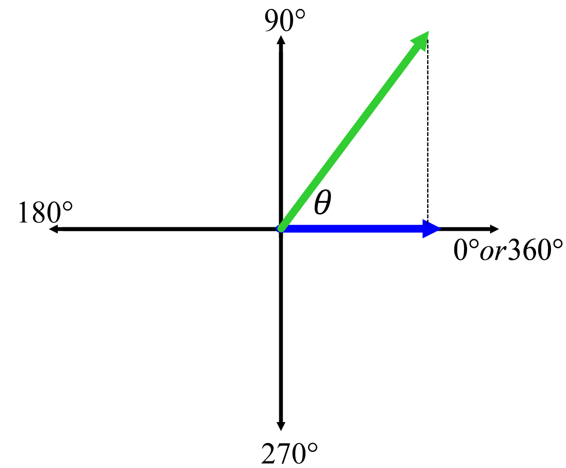
Part 1: Reference Angles

_______________: Always lies on the positive x-axis at 0°. Meets the terminal arm at the origin.

_______________: The arm that rotates around the origin counter clockwise to form a positive angle or clockwise for a negative angle.

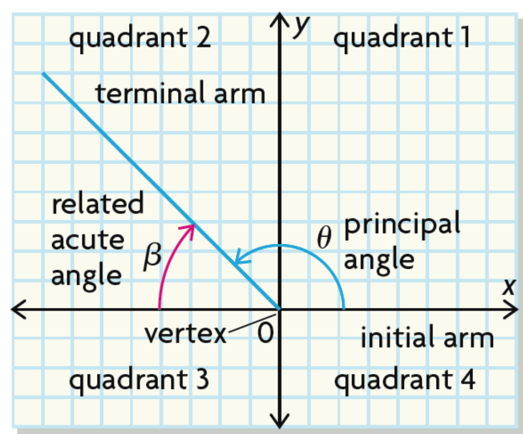
___________ is measured from the initial arm to the terminal arm.



_______________: The counter clockwise angle between the initial arm and the terminal arm of an angle in standard position. It's value is between 0° and 360°.

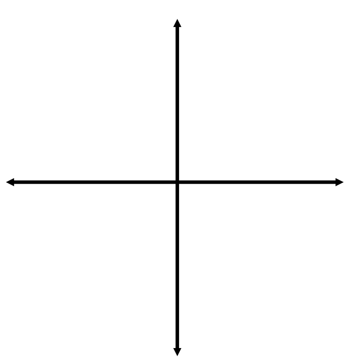
________________________________: The acute angle between the terminal arm of an angle in standard position and the closest x-axis when the terminal arm lies in quadrant 2, 3, or 4.

The reference angle helps us determine the exact trig ratios when we are given obtuse angles.

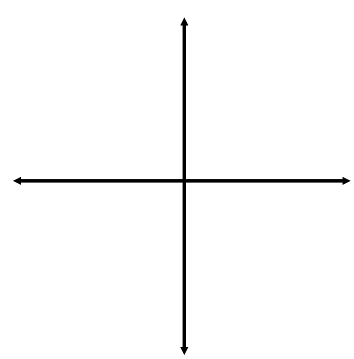


Example 1: Find the reference angle for each of the following principal angles

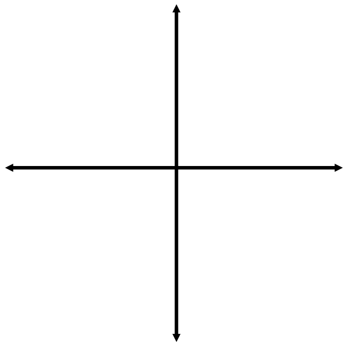
a) $250°$



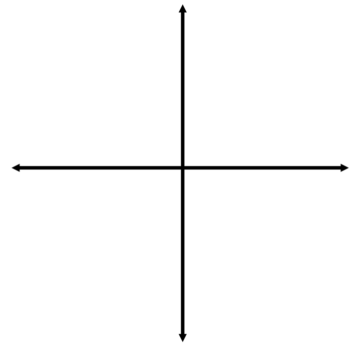
b) $120°$



c) $300°$



d) $225°$



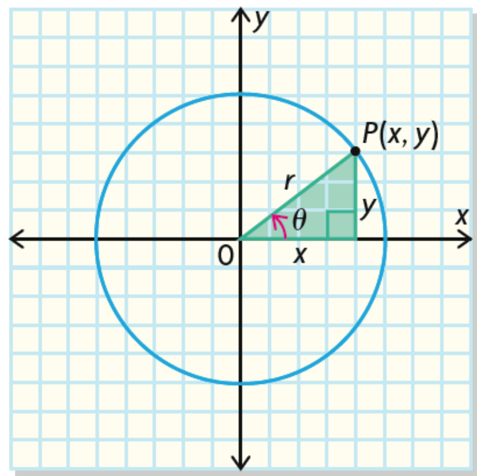
Part 2: Evaluating Trig Ratios for Any Angle

For any point $P(x, y)$ in the Cartesian plane, the trigonometric ratios for angles in standard position can be expressed in terms of x, y, and r.

$\sin\theta =$

$\cos\theta =$

$\tan\theta =$



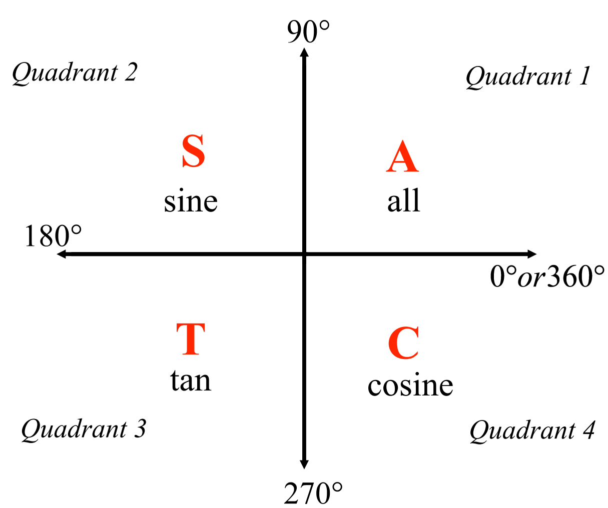
The CAST rule is an easy way to remember which primary trig ratios are positive in which quadrant. Since r is always positive, the sign of each primary ratio depends on the signs of the coordinates of the point (x, y).

In Q1, _____ ratios are positive because both x and y are positive.

In Q2, only __________ is positive, since x is negative and y is positive.

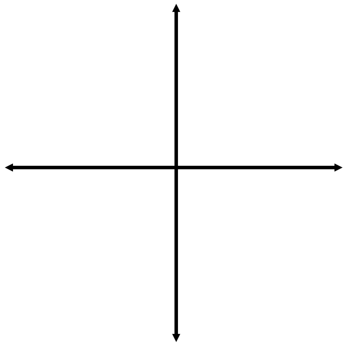
In Q3, only __________ is positive, since both x and y are negative.

In Q4, only __________ is positive, since x is positive but y is negative.

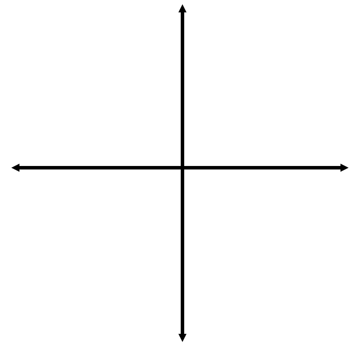


Example 2: Find the EXACT value of each of the following

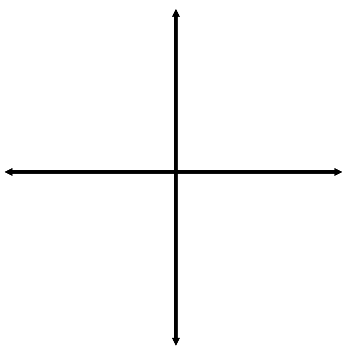
a) $\sin 45°$



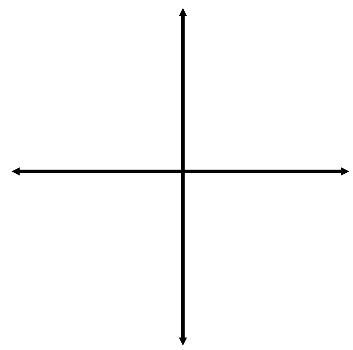
b) $\sin 210°$



c) $\cos 240°$

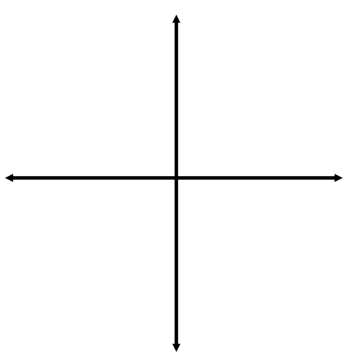


d) $\tan 315°$

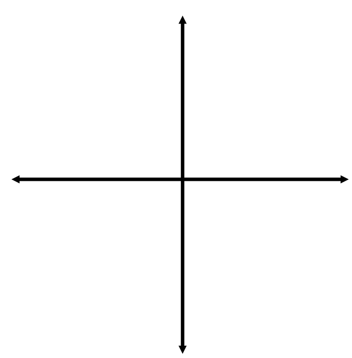


Example 3: Each point lies on the terminal arm of angle θ in standard position. Determine each of the primary trig ratios for angle θ.

a) (5, -12)



b) (-8, 3)



Part 3: Unit Circle

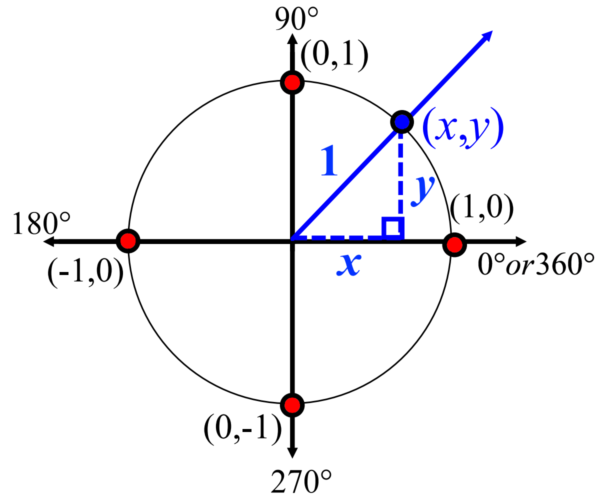
The unit circle, a circle with a radius of **1 unit**, is very useful since the x and y coordinates of where the terminal intersects it tell us the Cosine and Sine ratios respectively.

For any point $P(x, y)$ in the Cartesian plane that intersects the ***unit circle***, the trigonometric ratios for angles can be expressed in terms of x, y, and r.

$\sin\theta =$

$\cos\theta =$

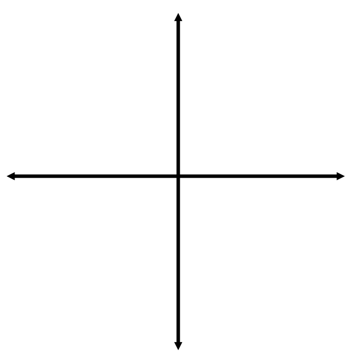
$\tan\theta =$



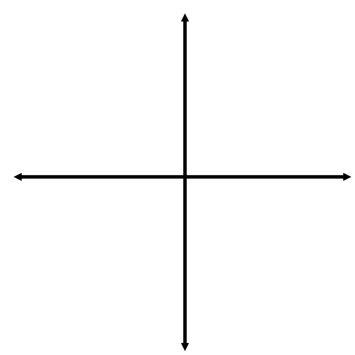
http://www.mathsisfun.com/geometry/unit-circle.html

Example 4: Find the EXACT value of each of the following

a) $\sin 270°$



b) $\cos 360°$



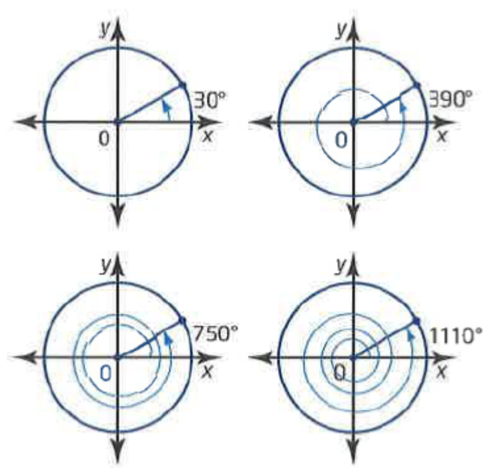
Part 4: Negative and Co-terminal Angles

Co-terminal angles are angles in standard position that have the __________________________.

Starting at 30° and rotating 360° counter clockwise will bring you back to the same terminal arm.

$$30° + 360° = 390°$$

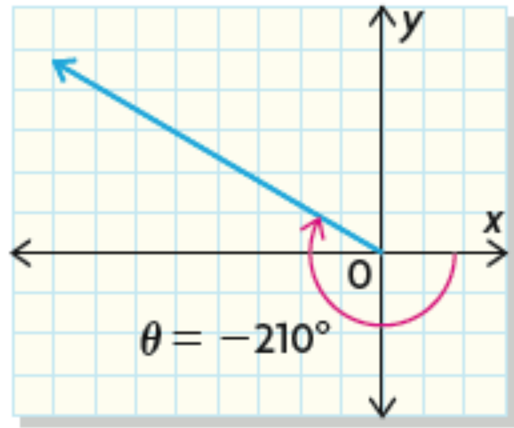
Therefore, 30° and 390° are co-terminal.



A negative angle is an angle measured ___________ from the positive x-axis.

You can find an equivalent (co-terminal) positive angle by adding $360°$ to the negative angle.

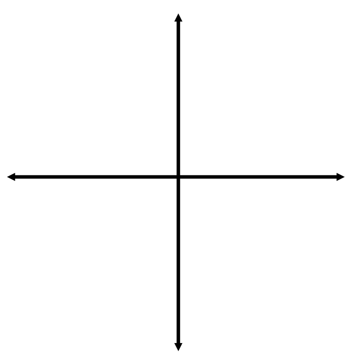
$-210°$ and $150°$ have the same terminal arm (co-terminal) and therefore have the same trigonometric ratios.



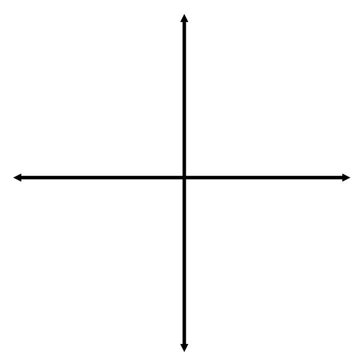
Example 5: Find three co-terminal angles of $60°$

Example 6: Find the EXACT value of each of the following

a) $\sin(-45°)$



b) $\cos(-60°)$



L3 – Solving Trigonometric Equations
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In this section, you will learn how to identify different angles that have the same trigonometric ratio, as well as learn how they are related.

To do this you will have to visualize the terminal arm rotating around a circle centred at the origin of a grid with a radius of r. This is done so that we can extend our understanding of trig functions for a broader class of angles and see how different angles are related.

http://www.mathsisfun.com/geometry/unit-circle.html

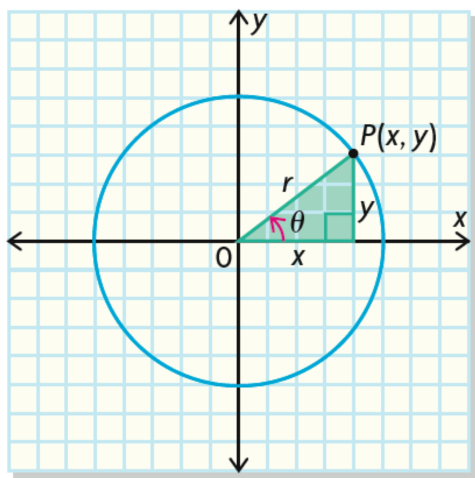
Some helpful reminders:

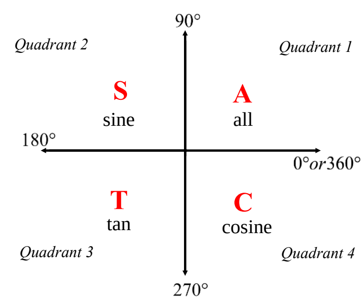
For any point $P(x, y)$ in the Cartesian plane, the trigonometric ratios for angles in standard position can be expressed in terms of x, y, and r.

$\sin\theta =$

$\cos\theta =$

$\tan\theta =$



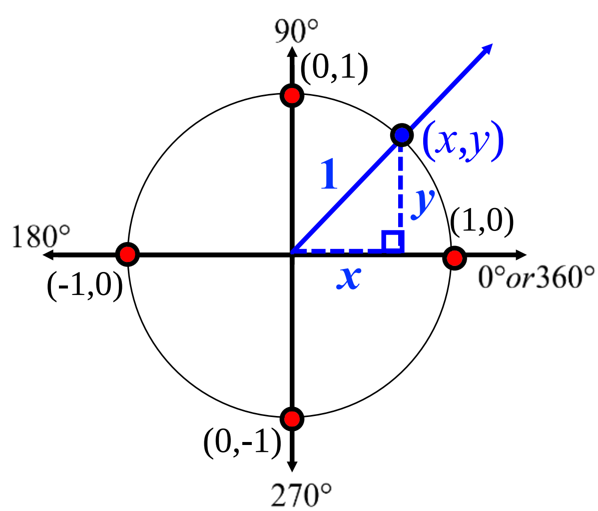


For any point $P(x, y)$ in the Cartesian plane that intersects the ***unit circle***, the trigonometric ratios for angles can be expressed in terms of x, y, and r.

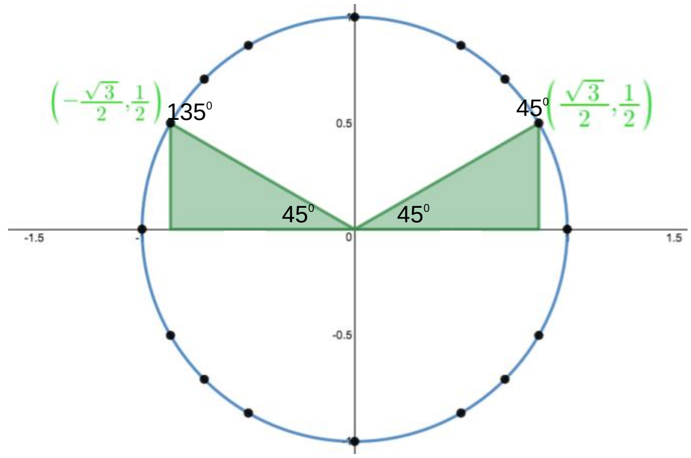
$\sin\theta =$

$\cos\theta =$

$\tan\theta =$



We know the x-coordinate of where the terminal arm Intersects the unit circle is equivalent to the cosine ratio and the y-coordinate is equivalent to the sine ratio.



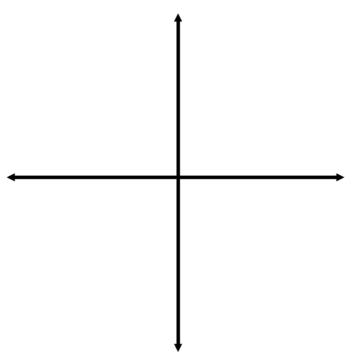
Notice that both $45°$ and $135°$ have the same ____________________. Since the angles fall in quadrants ____ and ____ respectively, they will have the exact same y-coordinate but the x-coordinates will have the same absolute value but will be opposite signs.

$\sin 135° =$

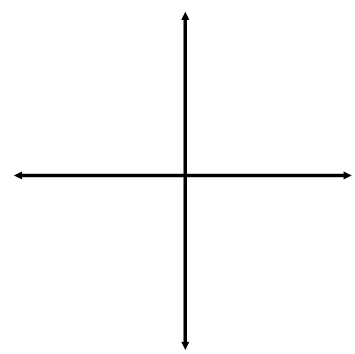
The important takeaway from this is that there are ______________ between $0°$ and $360°$ that have the exact same ratio. Using reference angles and the CAST rule, we can make sure to always find both possible angles between $0°$ and $360°$ that have the same trigonometric ratio.

Example 1: Solve the following equations for $0° \leq \theta \leq 360°$. Round answers to the nearest tenth of a degree.

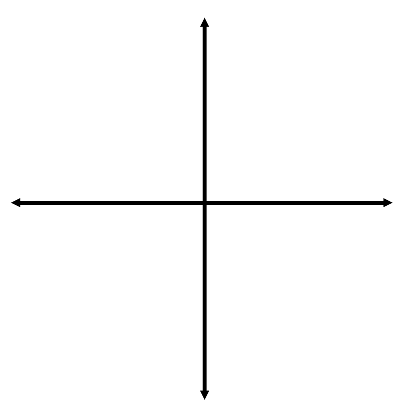
a) $\sin\theta = -0.7$



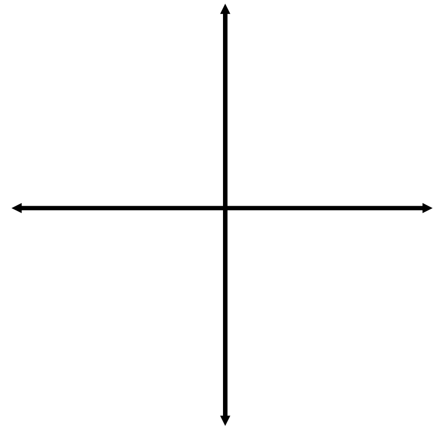
b) $\tan\theta = 2.1$



Example 2: The point $(-7, 19)$ lies on the terminal arm. Find the angle to the terminal arm (principal angle, θ) and find the related acute angle (reference angle, β).

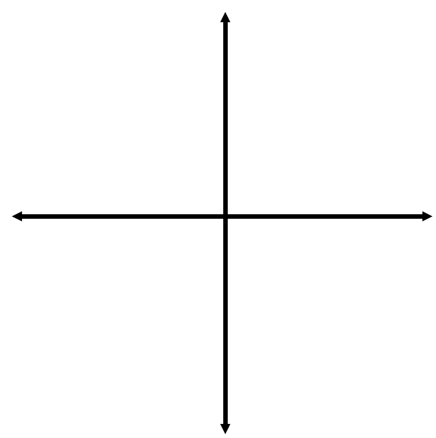


Example 3: The point $P(5, 11)$ lies on the terminal arm of angle θ in standard position. Draw a sketch of angle θ, determine the exact value of r, determine the primary trig ratios for angle θ, then calculate θ to the nearest tenth of a degree.



Example 4: Solve the following equation for $0° \leq \theta \leq 360°$.

$3\cos\theta + 1 = 0$



L4 – Reciprocal Trig Ratios
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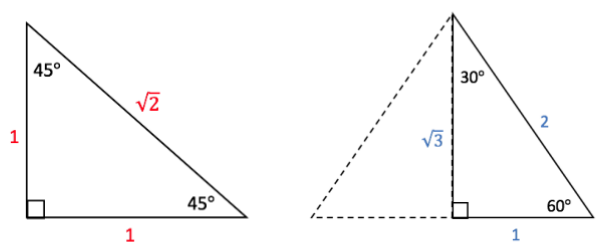
The reciprocal trigonometric ratios are reciprocals of the primary trigonometric ratios, and are defined as 1 divided by each of the primary trigonometric ratios:

Primary Trig Ratios	Reciprocal Trig Ratios
$sin\theta = \frac{opposite}{hypotenuse}$	$cosecant = \frac{1}{sin\theta} = \frac{hypotenuse}{opposite}$
$cos\theta = \frac{adjacent}{hypotenuse}$	$secant = \frac{1}{cos\theta} = \frac{hypotenuse}{adjacent}$
$tan\theta = \frac{opposite}{adjacent}$	$cotangent = \frac{1}{tan} = \frac{adjacent}{opposite}$

Don't forget your special triangles:



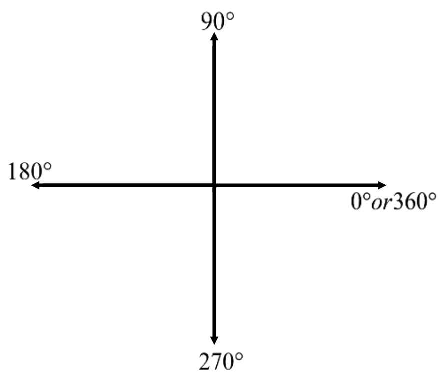
Example 1: Complete the following chart. Give exact values for each ratio.

	$sin\theta$	$csc\theta$	$cos\theta$	$sec\theta$	$tan\theta$	$cot\theta$
$0°$						
$30°$						
$45°$						
$60°$						
$90°$						

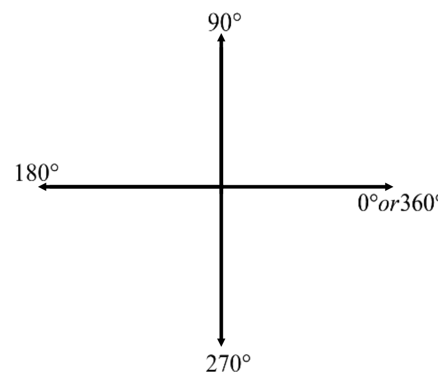
Example 2: The point (-9, 12) lies on the terminal arm of an angle in standard position. Determine exact expressions for the six trigonometric ratios for the angle.

Example 3: Solve the following equations for $0° \leq \theta \leq 90°$

a) $\csc\theta = 8$

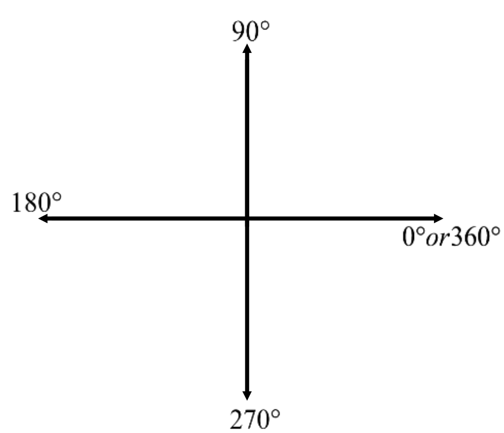


b) $\sec\theta = \frac{5}{2}$



Example 4: Solve the following equation for $0° \leq \theta \leq 360°$.

$\csc\theta + 2 = 0$



L5 – Problems in 2 and 3 Dimensions
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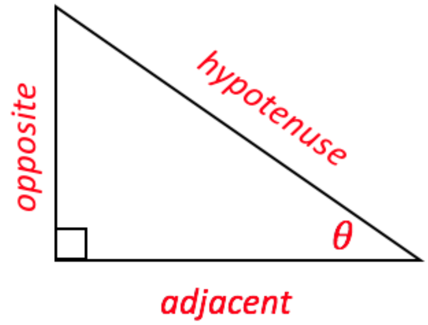
________________________ are used to solve triangles that contain a right angle.

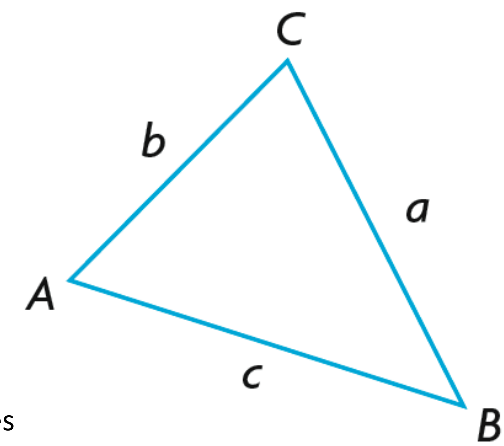
$\sin\theta = \frac{opposite}{hypotenuse}$ $\cos\theta = \frac{adjacent}{hypotenuse}$ $\tan\theta = \frac{opposite}{adjacent}$



The ____________________ and ______________________are used to solve oblique triangles. An oblique triangle is any triangle that is NOT a right triangle.

Sine Law can be used if you know:

i) 2 sides and one angle opposite a given side
ii) 2 angles and any side

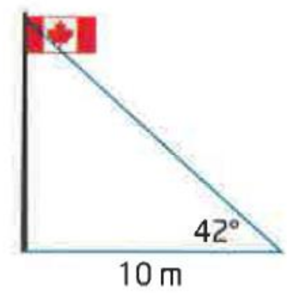


The Cosine Law can be used if you know:

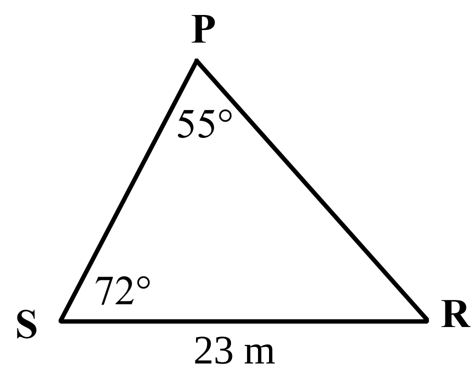
i) 2 sides and the angle contained by those 2 sides
ii) All 3 sides

Part 1: Problems in 2 Dimensions

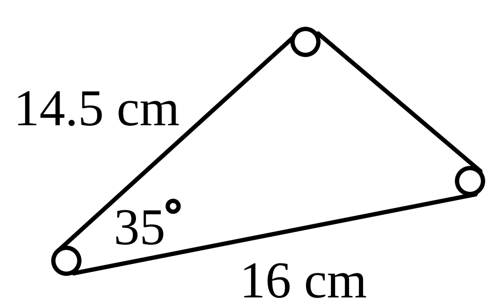
Example 1: Jonathan needs a new rope for his flagpole but is unsure of the length required. He measures a distance of 10m away from the base of the pole. From this point, the angle of elevation to the top of the pole is 42°. What is the height of the pole, to the nearest tenth of a meter?



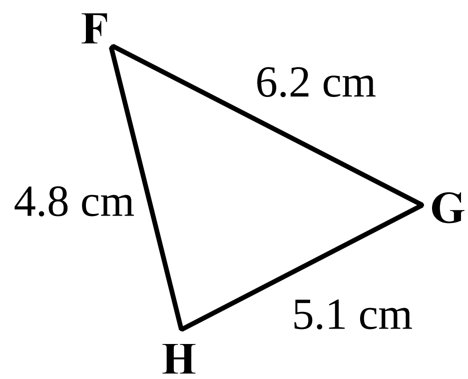
Example 2: Pam, Steven and Rachel are standing on a soccer field. Steven and Rachel are 23m apart. From Steven's point of view, the other two are separated by 72°. From Pam's point of view, the others are separated by an angle of 55°. Determine the distance from Pam to Rachel.



Example 3: A drive belt wraps around three pulleys as shown. Find the perimeter of the drive belt to the nearest tenth of a cm.

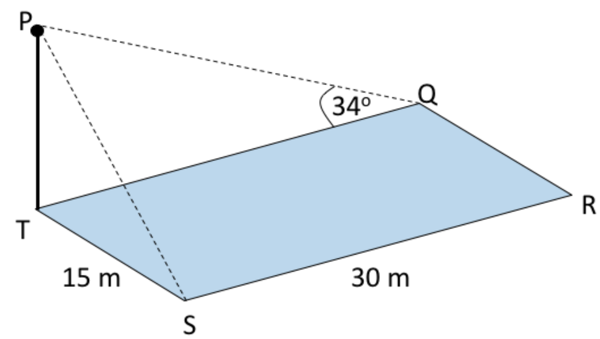


Example 4: Find the measure of angle G

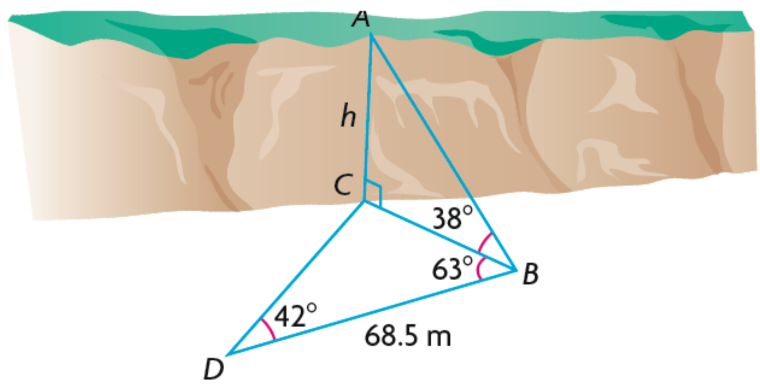


Part 2: Problems in 3 Dimensions

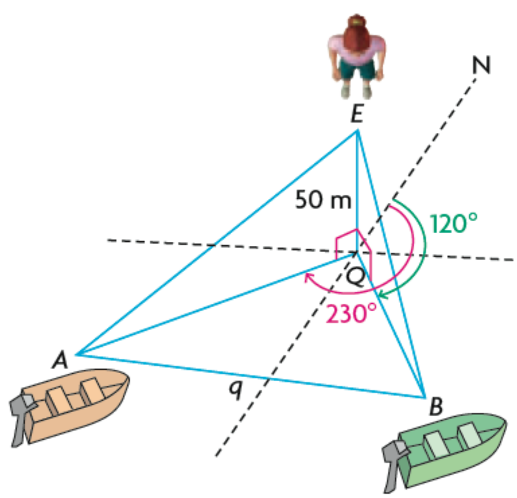
Example 4: A vertical flag pole TP stands in the corner of a rectangular field QRST. Using the information given in the diagram, calculate (a) The height of the flag pole and (b) The angle of elevation of P from S. Round answers to nearest tenth.



Example 5: From point B, Manny estimates the angle of elevation to the top of a cliff as $38°$. From point D, 68.5 meters away from Manny, Joe estimates the angle between the base of the cliff, himself, and Manny to be $42°$, while Manny estimates the angle between the base of the cliff, himself, and his friend Joe to be $63°$. What is the height of the cliff to the nearest tenth of a meter?



Example 6: Emma is on a 50 meter high bridge and sees two boats anchored below. From her position, boat A has a bearing of $230°$ and boat B has a bearing of $120°$. Emma estimates the angles of depression to be $38°$ for boat A and $35°$ for boat B. How far apart are the boats to the nearest meter?

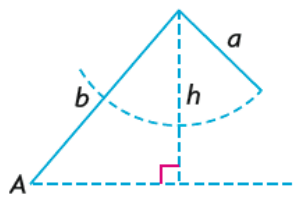


L6 – Ambiguous Case of Sine
MCR3U
Jensen

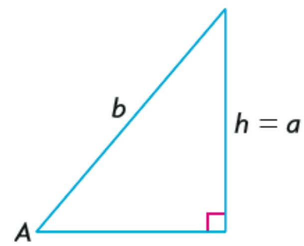
If you are given ________________________________ in a triangle, there are many scenarios to consider.

If $\angle A$, a, and b are given and $\angle A$ is acute, there are 4 scenarios to consider:

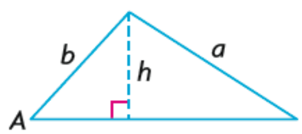
If $\angle A$ is acute and $a < h$, no triangle exists.



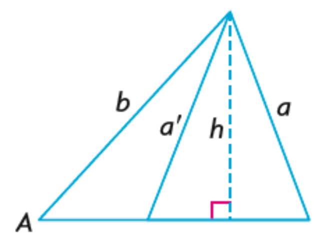
If $\angle A$ is acute and $a = h$, one right triangle exists.



If $\angle A$ is acute and $a > b$, one triangle exists.

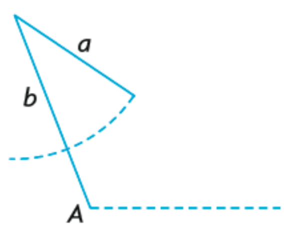


If $\angle A$ is acute and $h < a < b$, two triangles exist.

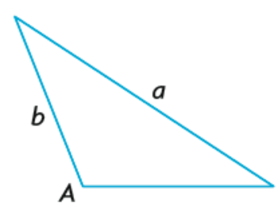


If $\angle A$, a, and b are given and $\angle A$ is obtuse, there are 2 scenarios to consider:

If $\angle A$ is obtuse and $a < b$ or $a = b$, no triangle exists.



If $\angle A$ is obtuse and $a > b$, one triangle exists.



The ambiguous case only occurs when two possible triangles exist for the same given information. This means, the ambiguous case must be considered if ______________________________________.

Example 1: In triangle ABC, side a = 12 cm, side b = 17 cm, and A = 21°. Find the measure of angle B.

Example 2: In triangle ABC, side a = 8 cm, side c = 10 cm, and A = 34°. Find angle C.

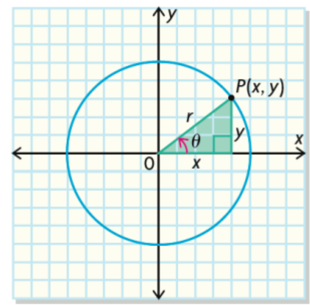
Example 3: In triangle ABC, side a = 14 cm, side b = 17 cm, and A = 54°. Find the measure of all missing sides and angles.

L7 – Trig Identities
MCR3U
Jensen

_______________: A mathematical equation that is true for ALL values of the given variables.

Part 1: Proving the Pythagorean and Quotient Identities

For this part you will need to remember that trig ratios can be written in terms of x and y



Example 1: Prove the quotient identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$

Example 2: Prove the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$

Fundamental Trigonometric Identities		
Reciprocal Identities	**Quotient Identities**	**Pythagorean Identities**
$\boldsymbol{csc\ \theta = \frac{1}{sin\ \theta}}$ $\boldsymbol{sec\ \theta = \frac{1}{cos\ \theta}}$ $\boldsymbol{cot\ \theta = \frac{1}{tan\ \theta}}$	$\boldsymbol{\frac{sin\ \theta}{cos\ \theta} = tan\ \theta}$ $\boldsymbol{\frac{\cos\theta}{\sin\theta} = \cot\theta}$	$\boldsymbol{sin^2\theta + cos^2\theta = 1}$

Tips and Tricks		
Reciprocal Identities	**Quotient Identities**	**Pythagorean Identities**
Square both sides $csc^2\ \theta = \frac{1}{sin^2\ \theta}$ $sec^2\ \theta = \frac{1}{cos^2\ \theta}$ $cot^2\ \theta = \frac{1}{tan^2\ \theta}$	Square both sides $\frac{sin^2\ \theta}{cos^2\ \theta} = tan^2\ \theta$ $\frac{\cos^2\theta}{\sin^2\theta} = \cot^2\theta$	Rearrange the identity $sin^2\theta = 1 - cos^2\theta$ $cos^2\theta = 1 - sin^2\theta$

General tips for proving identities:

- **i)** Try to change everything to $sin\ \theta$ or $cos\ \theta$
- **ii)** If you have to fractions being added or subtracted, find a common denominator and combine the fractions
- **iii)** Use difference of squares → $1 - sin^2\ \theta = (1 - sin\ \theta)(1 + sin\ \theta)$
- **iv)** Use the power rule → $sin\ ^6\ \theta = (sin^2\ \theta)^3$

We will use the preceding identities to help us prove more complex identities in the following examples.

Example 3: Prove each of the following identities

a) $\frac{\cos\theta\tan\theta}{\sin\theta} = 1$

b) $\tan^2\theta + 1 = \sec^2\theta$

c) $\cos^2 x = (1 - \sin x)(1 + \sin x)$

d) $\frac{\sin^2 x}{1-\cos x} = 1 + \cos x$

e) $\sin\theta \sec\theta \cot\theta = 1$

f) $\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = \frac{2\tan x}{\cos x}$

g) $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$

h) $\tan x + \frac{\cos x}{1+\sin x} = \sec x$