Chapter 5- Trig Functions

Lesson Package

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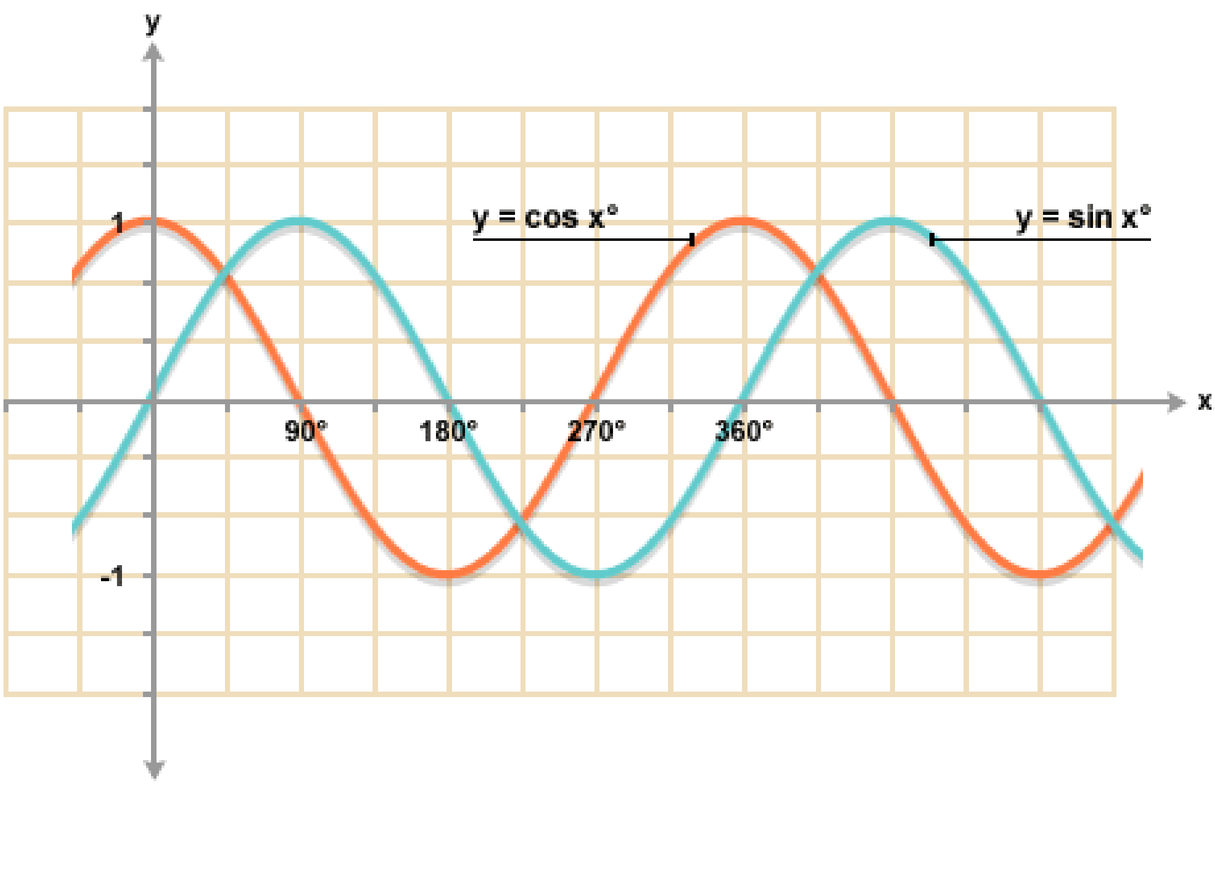

Chapter 5 Outline

Unit Goal: Be able to identify and represent sinusoidal functions, and solve problems involving sinusoidal functions, including problems arising from real-world applications.

Section	Subject	Learning Goals	Curriculum Expectations
L1	Modeling Periodic Behaviour	- describe key properties of periodic functions and predict future values by extrapolating	D2.1, D2.2
L2	Graphing Sine and Cosine Functions	- graph sin x and cos x for angles given in degrees	D2.3, D2.4
L3	Transformations of Sine and Cosine Part 1	- given the equation of the a sinusoidal function, use transformations to graph it	D2.5, D2.6, D2.7, D2.8
L4	Transformations of Sine and Cosine Part 2	- given the graph of a sinusoidal function, determine an equation that defines it	D2.5, D2.6, D2.7, D2.8
L5	Trig Applications Part 1	- solve problems that arise from real world applications involving periodic phenomena	D3.2, D3.3, D3.4
L6	Trig Applications Part 2	- solve problems that arise from real world applications involving periodic phenomena	D3.2, D3.3, D3.4

Assessments	F/A/O	Ministry Code	P/O/C	KTAC
Note Completion	A		P	
Practice Worksheet Completion	F/A		P	
PreTest Review	F/A		P	
Test – Trig Geometry	O	D2.1, D2.2, D2.3, D2.4, D2.5, D2.6, D2.7, D2.8, D3.4	P	K(21%), T(34%), A(10%), C(34%)

L1 - Modeling Periodic Behaviour
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Section 1: Definitions

***PERIODIC FUNCTION*:** *a function that has a pattern of y-values that repeats at regular intervals.*

CYCLE: *one complete repetition of a pattern.*

PERIOD: *the horizontal length of one cycle on a graph.*

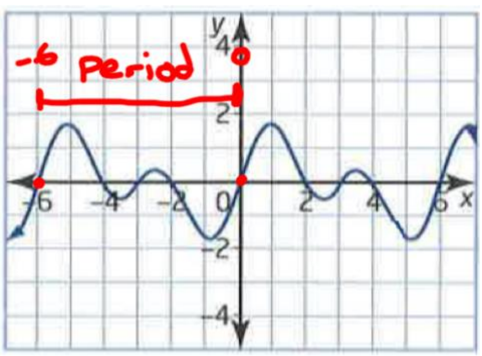
AMPLITUDE: *half the distance between the maximum and minimum values of a periodic function.*

Section 2: Recognizing Properties of Periodic Functions

How to find the PERIOD of a function: *choose a convenient x-coordinate to start at and then move to the right and estimate the x-coordinate of the where the next cycle begins. Find the difference of these x-coordinates to calculate the period of the function.*

Example 1: Determine whether the functions are periodic or not. If it is, state the period of the function.

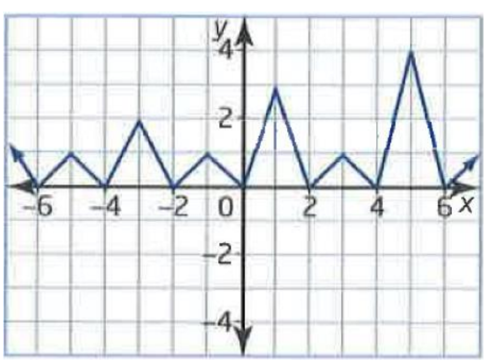
i)



The pattern of $y - values$ in one section of the graph repeats in the next section. Therefore, the function IS periodic.

$$period = 0 - (-6)$$
$$= 6$$

ii)

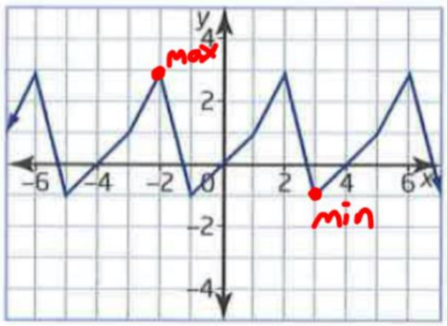


The pattern of $y - values$ in one section of the graph does NOT repeat in the next section. Therefore, the function is NOT periodic.

Example 2: Is the function periodic? If so, what is the amplitude?

How to find the AMPLITUDE of a function: *the amplitude is half the difference between the max and min values. Use the formula:*

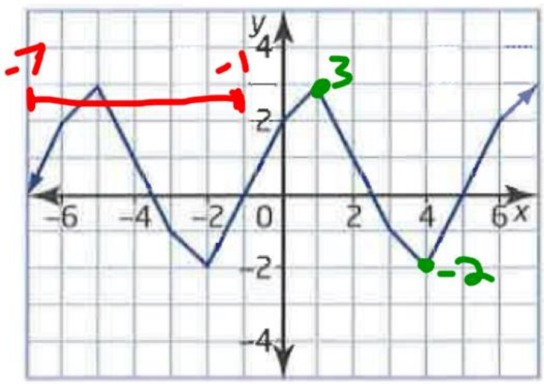
$$amplitude = \frac{y_{max} - y_{min}}{2}$$



Yes, the function is periodic.

$$amplitude = \frac{3-(-1)}{2} = \frac{4}{2} = 2 \text{ units}$$

Example 3: In the following periodic function, determine the period and amplitude.

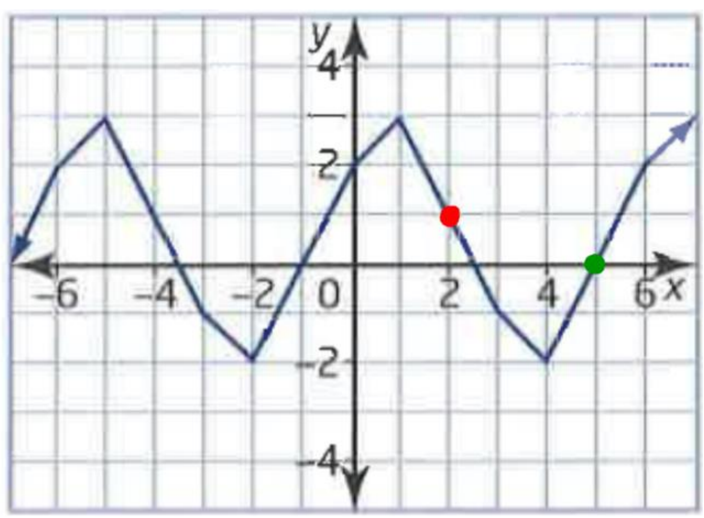


$$period = -1 - (-7) = 6\ units$$

$$amplitude = \frac{3-(-2)}{2} = \frac{5}{2}\ units$$

Section 3: Predicting Values of a Periodic Function

Example 4: For the following function…



a) determine $f(2)$ and $f(5)$

$f(2) = 1$

$f(5) = 0$

b) determine $f(8), f(-10), and\ f(14)$

Hint:

i) figure out the period of the function

ii) add or subtract the period of the function until you get back to a value on the graph that you know.

$period = 6\ units$

$f(8) = f(8-6)$
$= f(2)$
$= 1$

$f(-10) = f(-10+6)$
$= f(-4)$
$= 1$

$f(14) = f(14-6)$
$= f(8)$
$= 1$

c) determine 4 values of x so that $f(x) = 2$

Hint:

Keep adding/subtracting the period value to the x-value where y = 2.

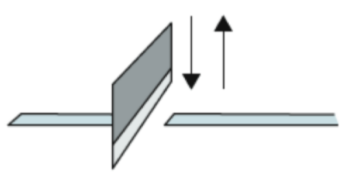
From graph: $f(0) = 2$

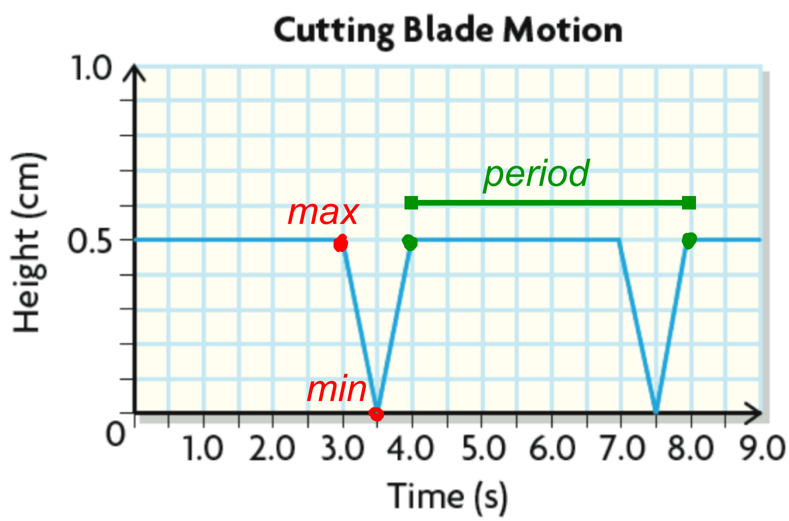
$f(0+6) = 2$
$f(6) = 2$

$f(6+6) = 2$
$f(12) = 2$

$f(0+6) = 2$
$f(6) = 2$

Example 5: A cutting machine chops strips of plastic into their appropriate lengths. The following graph shows the motion of the cutting blade on the machine in terms of time.





a) State the max height of the blade, the minimum height, and the amplitude of the function.

$\max height = 0.5\ cm$
$\min height = 0\ cm$

$amplitude = \frac{y_{max} - y_{min}}{2} = \frac{0.5-0}{2} = 0.25 \text{ cm}$

b) What is the period of this function?

$period = 8 - 4 = 4\ seconds$

c) State the next two times that the blade will strike the cutting surface?

Last strike was at 7.5 seconds → $f(7.5) = 0$

$f(7.5 + 4) = 0$
$f(11.5) = 0$

$f(11.5 + 4) = 0$
$f(15.5) = 0$

Therefore, the next strikes will be at 11.5 seconds and 15.5 seconds.

L2 - Graphing Sine and Cosine Functions

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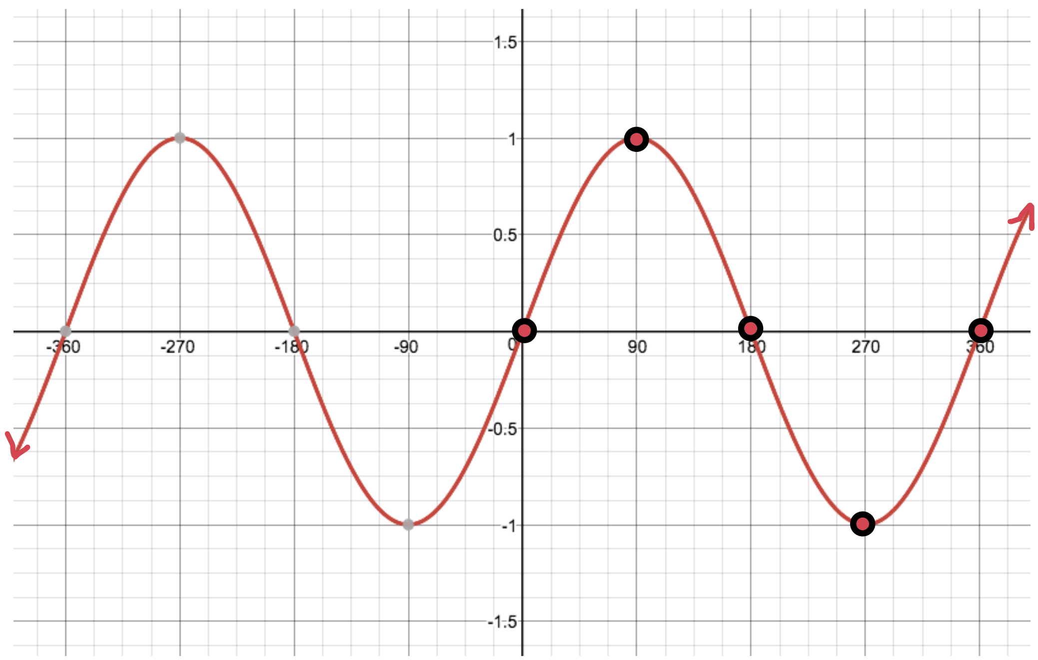
Section 1: Graphing Sine and Cosine **DESMOS demonstration**

To graph sine and cosine, we will be using a Cartesian plane that has angles for x values.

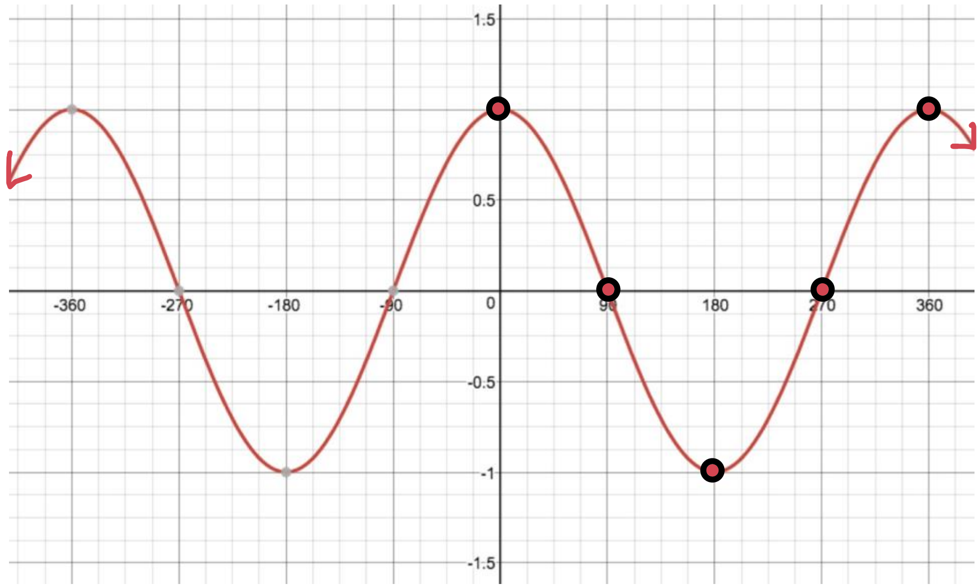
Example 1: Complete the following table of values for the function $f(x) = \sin(x)$. Use special triangles, the unit circle, or a calculator to find values for the function at 30° intervals. Use the table to graph the function.

x	$f(x)$
0	**0**
30	0.5
60	$\frac{\sqrt{3}}{2} \sim 0.87$
90	**1**
120	$\frac{\sqrt{3}}{2} \sim 0.87$
150	0.5
180	**0**
210	-0.5
240	$-\frac{\sqrt{3}}{2} \sim -0.87$
270	**-1**
300	$-\frac{\sqrt{3}}{2} \sim -0.87$
330	-0.5
360	**0**



Example 2: Complete the following table of values for the function $f(x) = \cos(x)$. Use special triangles, the unit circle, or a calculator to find values for the function at 30° intervals. Use the table to graph the function.

x	$f(x)$
0	**1**
30	$\frac{\sqrt{3}}{2} \sim 0.87$
60	0.5
90	**0**
120	-0.5
150	$-\frac{\sqrt{3}}{2} \sim -0.87$
180	**-1**
210	$-\frac{\sqrt{3}}{2} \sim -0.87$
240	-0.5
270	**0**
300	0.5
330	$\frac{\sqrt{3}}{2} \sim 0.87$
360	**1**



Section 2: Properties of Sine and Cosine Functions

Domain: $\{X \in \mathbb{R}\}$

Range: $\{Y \in \mathbb{R}| -1 \leq y \leq 1\}$

Period: 360°

Amplitude: $\frac{max-min}{2} = \frac{1-(-1)}{2} = 1\ unit$

Section 3: Transformations of the Sine and Cosine Functions

$y = a\sin[k(x-d)] + c$

Desmos Demonstration

a	k	d	c
Vertical stretch or compression by a factor of a. Vertical reflection if $a < 0$ $\|a\| = amplitude$	Horizontal stretch or compression by a factor of $\frac{1}{k}$. Horizontal reflection if $k < 0$. $\frac{360}{\|k\|} = period$	Phase shift $d > 0; shift\ right$ $d < 0; shift\ left$	Vertical shift $c > 0; shift\ up$ $c < 0; shift\ down$

Example 3: For the function $y = 3\sin[2(\theta + 60°)] - 1$, state the...

Amplitude: $a = 3$	Period: $period = \frac{360}{\|k\|} = \frac{360}{2} = 180°$
Phase shift: $d = -60°$; Shift left 60°	Vertical shift: $c = -1$; Shift down 1 unit
Max: $max = c + \|a\| = -1 + 3 = 2$ units	Min: $min = c - \|a\| = -1 - 3 = -4$ units

L3 – Transformations of Sine and Cosine Part 1

Equation → Graph

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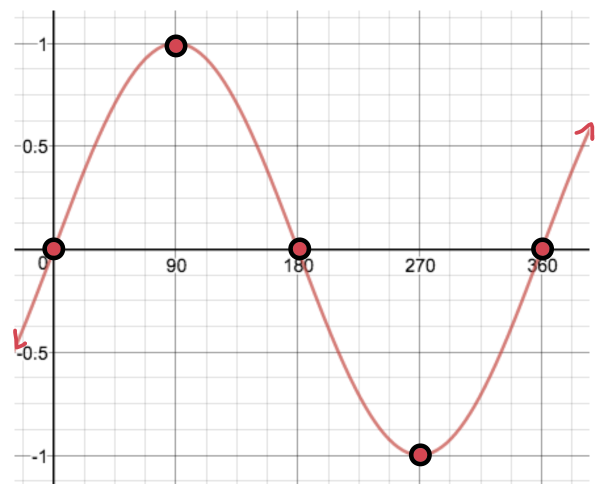
Section 1: Review of Sine and Cosine Functions

$$y = a\sin[k(x-d)] + c \quad \text{OR} \quad y = a\cos[k(x-d)] + c$$

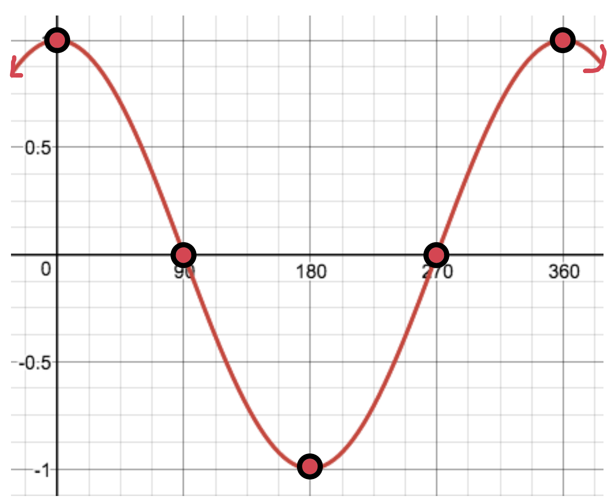
a	k	d	c
Vertical stretch or compression by a factor of a. Vertical reflection if $a < 0$ $\lvert a\rvert = amplitude$	Horizontal stretch or compression by a factor of $\frac{1}{k}$. Horizontal reflection if $k < 0$. $\frac{360}{\lvert k\rvert} = period$	Phase shift $d > 0;\, shift\ right$ $d < 0;\, shift\ left$	Vertical shift $c > 0;\, shift\ up$ $c < 0;\, shift\ down$

Graphs of parent functions $y = \sin x$ and $y = \cos x$ using key points:

x	y
0	0
90	1
180	0
270	-1
360	0



x	y
0	1
90	0
180	-1
270	0
360	1



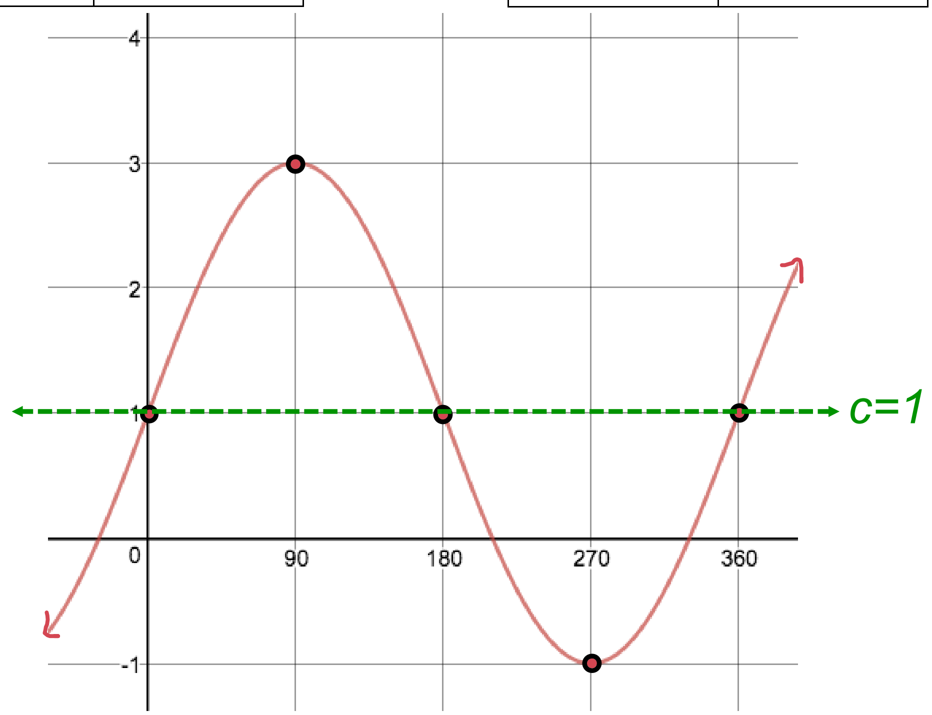
Section 2: Graphing Transformed Sinusoidal Functions

Example 1: Graph $y = 2\sin x + 1$ using transformations. Then state the amplitude, period, and number of cycles between 0° and 360°.

$a = 2$; vertical stretch by a factor of 2 $(2y)$
$c = 1$; vertical shift up 1 unit $(y + 1)$

$y = \sin x$	
x	y
0	0
90	1
180	0
270	-1
360	0

$y = 2\sin x + 1$	
x	$2y + 1$
0	1
90	3
180	1
270	-1
360	1



Amplitude: $amplitude = |a| = 2$

Period: $period = \frac{360}{|k|} = \frac{360}{1} = 360°$

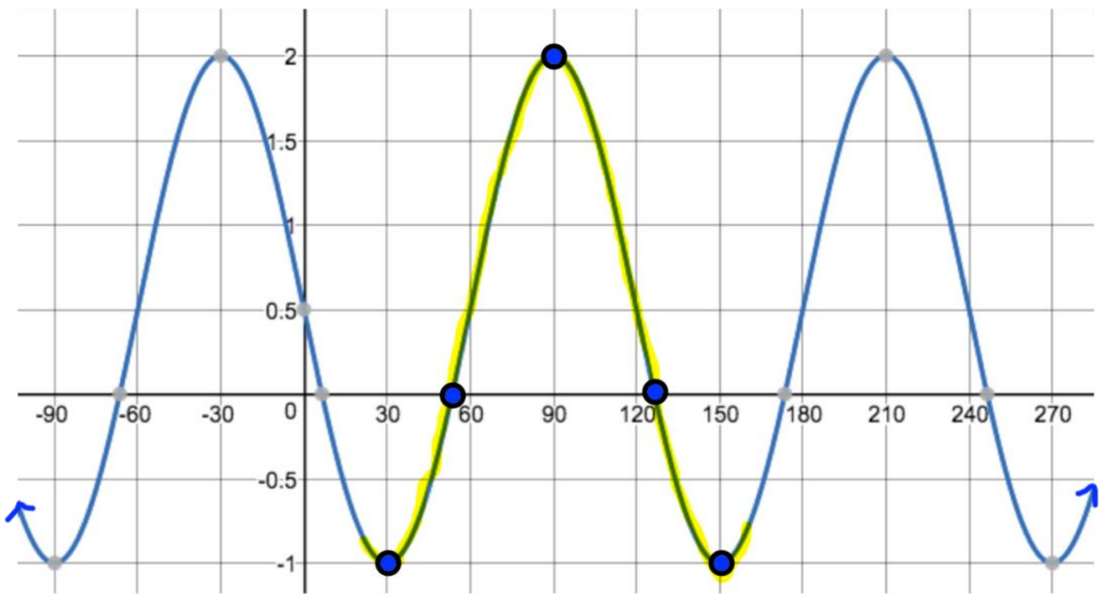
Number of cycles between 0° and 360°: $\#of\ cycles = |k| = 1$

Example 2: Graph $y = -1.5\cos[3(x - 30°)] + 0.5$ using transformations. Then state the amplitude, period, and number of cycles between 0° and 360°.

$a = -1.5$; vertical stretch by a factor of 1.5 and a vertical reflection $(-1.5y)$
$k = 3$; horizontal compression by a factor of $\frac{1}{3}$ $\left(\frac{x}{3}\right)$
$d = 30$; phase shift 30° to the right $(x + 30)$
$c = 0.5$; vertical shift 0.5 units up $(y + 0.5)$

$y = \cos x$	
x	y
0	1
90	0
180	-1
270	0
360	1

$y = -1.5\cos[3(x - 30°)] + 0.5$	
$\frac{x}{3} + 30$	$-1.5y + 0.5$
30	-1
60	0.5
90	2
120	0.5
150	-1



Amplitude: $amplitude = |a| = 1.5$

Period: $period = \frac{360}{|k|} = \frac{360}{3} = 120°$

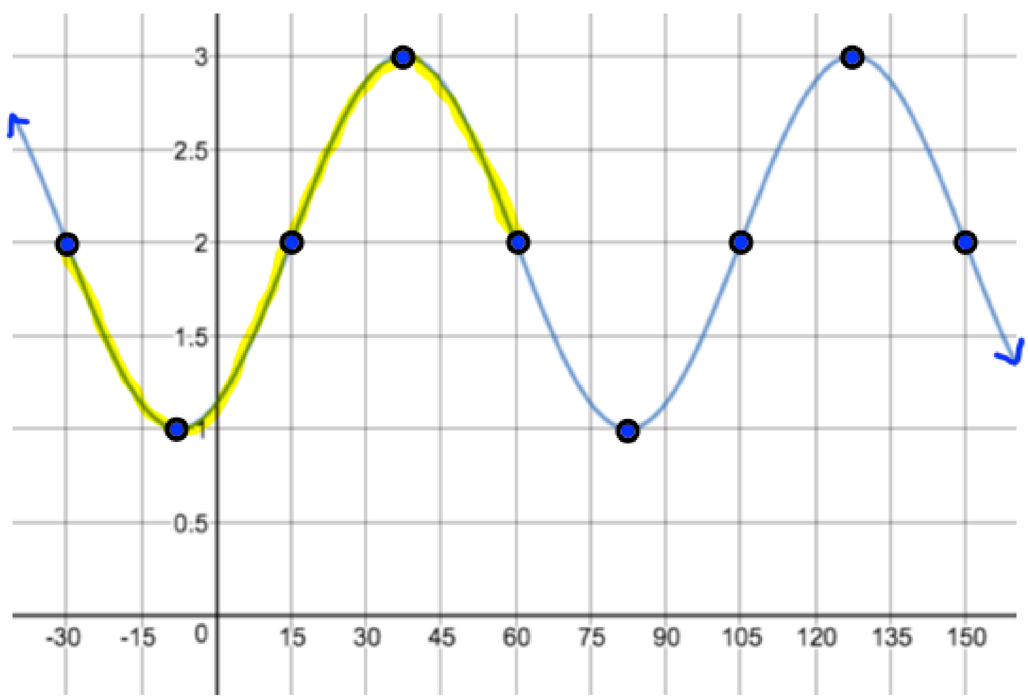
Number of cycles between 0° and 360°: $\#of\ cycles = |k| = 3$

Example 3: Graph $y = \sin[-4(x - 60°)] + 2$ using transformations. Then state the amplitude, period, and number of cycles between 0° and 360°.

$k = -4$; horizontal compression by a factor of $\frac{1}{4}$, and horizontal reflection $\left(\frac{-x}{4}\right)$
$d = 60$; phase shift 60° to the right $(x + 60)$
$c = 2$; vertical shift 2 units up $(y + 2)$

$y = \sin x$	
x	y
0	0
90	1
180	0
270	-1
360	0

$y = \sin[-4(x - 60°)] + 2$	
$\frac{-x}{4} + 60$	$y + 2$
60	2
37.5	3
15	2
-7.5	1
-30	2



Amplitude: $amplitude = |a| = 1$

Period: $period = \frac{360}{|k|} = \frac{360}{4} = 90°$

Number of cycles between 0° and 360°: $\#of\ cycles = |k| = 4$

L4 – Transformations of Sine and Cosine Part 2

Graph → Equation

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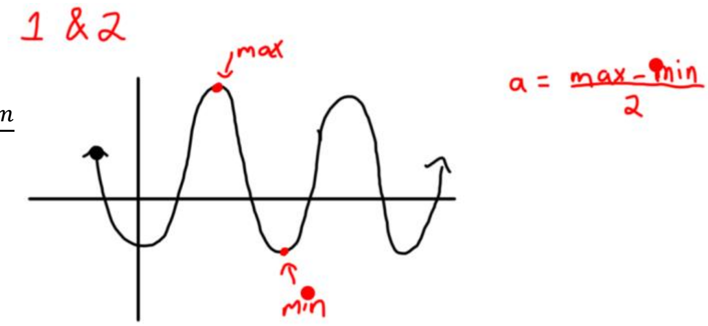
Section 1: How to Determine the Equation of a Sine or Cosine Function Given its Graph

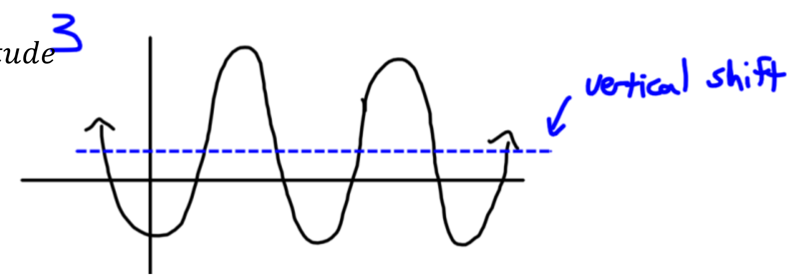
1) Find the max and min of the function

2) Find the amplitude of the function (a-value): $a = \frac{max - min}{2}$



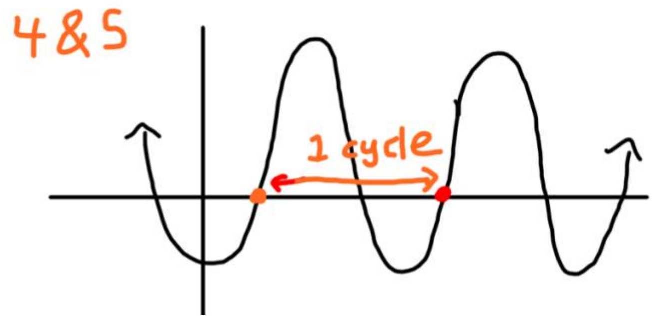
3) Find the vertical shift (c-value): $c = max - amplitude$

(this finds the 'middle' of the function)



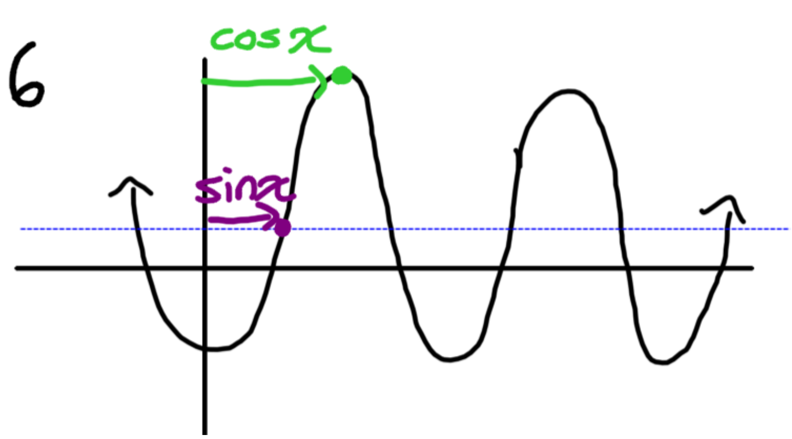
4) Find the period (in degrees) of the function using a starting point and ending point of a full cycle

5) Calculate the k-value. $k = \frac{360}{period}$ → $period = \frac{360}{|k|}$



6) Determine the phase shift (d-value)

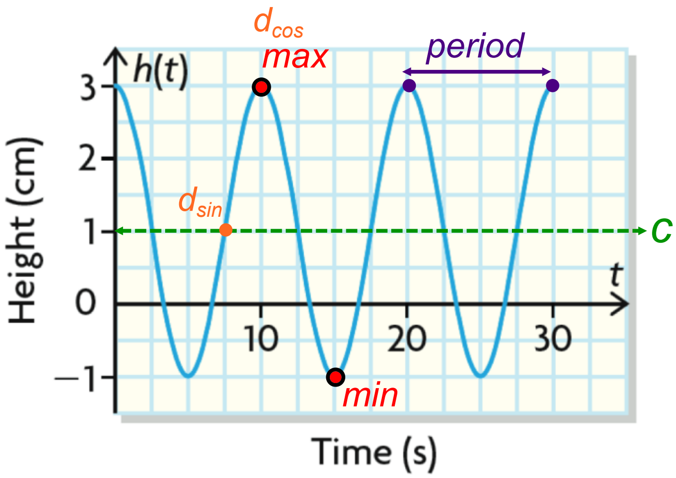
- for $\sin x$: trace along the center line and find the distance between the y-axis and the bottom left of the closest rising midline.
- for $\cos x$: the distance between the y-axis and the closest maximum point



Section 2: Determining the Equation of a Sinusoidal Function Given its Graph

Example 1: For each of the following graphs, determine the equation of a sine and cosine function that represents each graph:

a)



$$a = \frac{max - min}{2} = \frac{3 - (-1)}{2} = 2$$

$$k = \frac{360}{period} = \frac{360}{30 - 20} = \frac{360}{10} = 36$$

$$c = max - |a| = 3 - 2 = 1$$

$$d_{sin} = 7.5$$

$$d_{cos} = 10$$

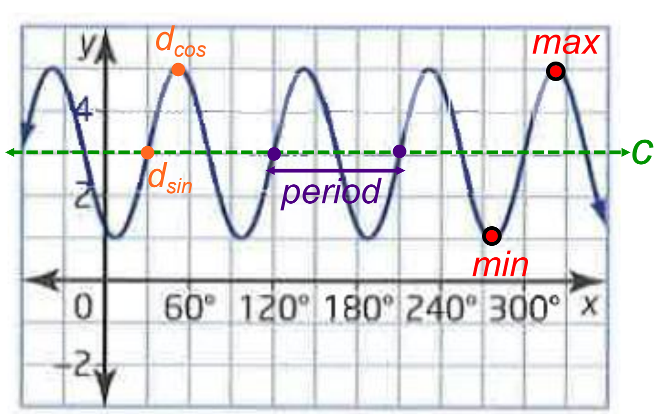
$d_{sin} \rightarrow$ look for x-value of closest rising midline

$d_{cos} \rightarrow$ look for x-value of closest maximum

$$y = 2\cos[36(x - 10)] + 1$$

$$y = 2\sin[36(x - 7.5)] + 1$$

b)



$$a = \frac{max - min}{2} = \frac{5 - 1}{2} = 2$$

$$k = \frac{360}{period} = \frac{360}{210 - 120} = \frac{360}{90} = 4$$

$$c = max - |a| = 5 - 2 = 3$$

$$d_{sin} = 30$$

$$d_{cos} = d_{sin} + \frac{90}{|k|} = 30 + \frac{90}{4} = 52.5$$

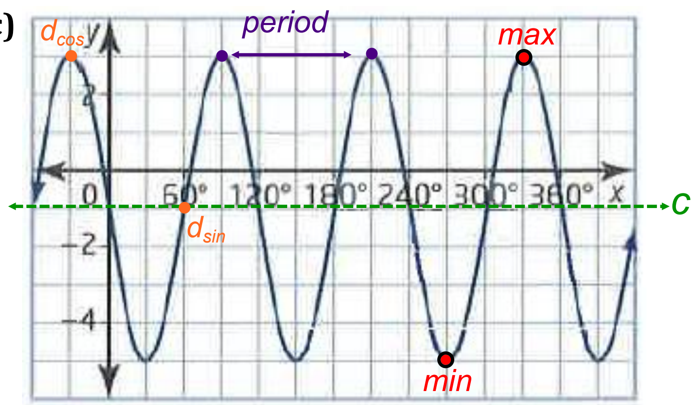
$$y = 2\cos[4(x - 52.5)] + 3$$

$$y = 2\sin[4(x - 30)] + 3$$

Note: The $x - value$ of the maximum point was not obvious from the graph. You need to know that maximum points are always $\frac{90}{|k|}$ to the right of the rising midline point. Also, if you knew where the maximum point was, the rising midline point would be $\frac{90}{|k|}$ to the left of the max.

$$\boldsymbol{d_{cos} = d_{sin} + \frac{90}{|k|}} \quad \textbf{OR} \quad \boldsymbol{d_{sin} = d_{cos} - \frac{90}{|k|}}$$

c)



$$a = \frac{max - min}{2} = \frac{3-(-5)}{2} = 4$$

$$k = \frac{360}{period} = \frac{360}{210-90} = \frac{360}{120} = 3$$

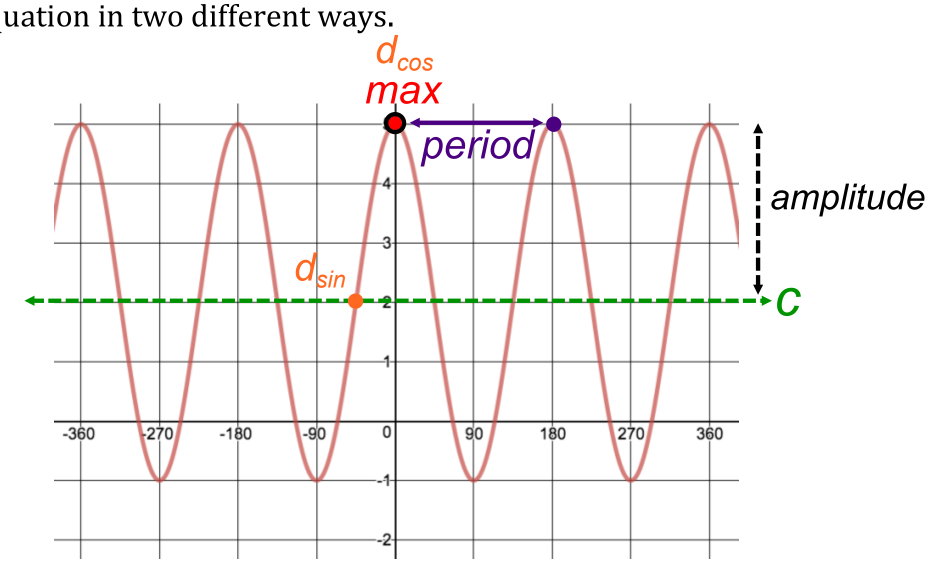
$$c = max - |a| = 3 - 4 = -1$$

$$d_{sin} = 60$$

$$d_{cos} = -30$$

$$y = 4\cos[3(x+30)] - 1 \qquad y = 4\sin[3(x-60)] - 1$$

Example 2: A sinusoidal function has an amplitude of 3 units, a period of 180 degrees and a max point at (0, 5). Represent the function with an equation in two different ways.



$$a = 3$$

$$k = \frac{360}{period} = \frac{360}{180} = 2$$

$$c = max - |a| = 5 - 3 = 2$$

$$d_{cos} = 0$$

$$d_{sin} = d_{cos} - \frac{90}{|k|} = 0 - \frac{90}{2} = -45$$

$$y = 3\cos(2x) + 2 \qquad y = 3\sin[2(x+45)] + 2$$

Example 3: A sinusoidal function has an amplitude of 5 units, a period of 120 degrees and a maximum at (0, 3). Represent the function with an equation in two different ways.

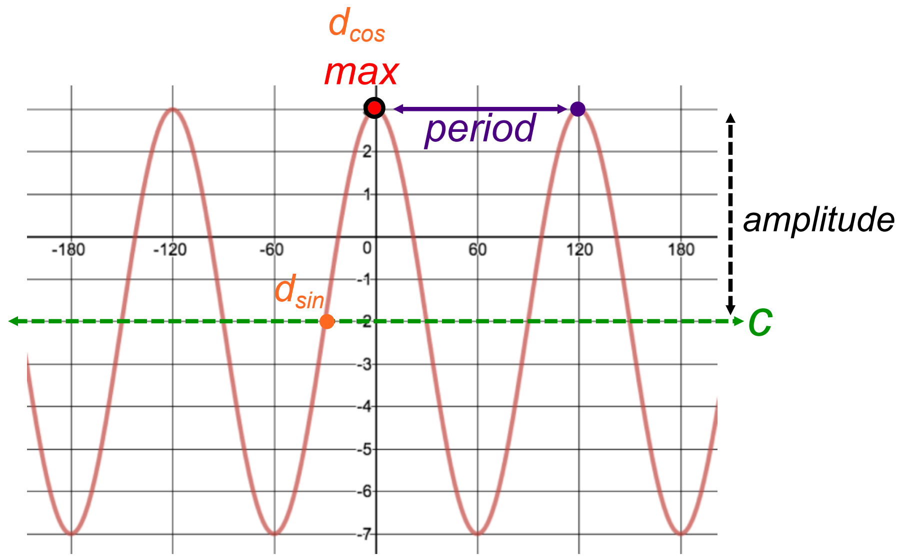
$a = 5$

$k = \frac{360}{period} = \frac{360}{120} = 3$

$c = max - |a| = 3 - 5 = -2$

$d_{cos} = 0$

$d_{sin} = d_{cos} - \frac{90}{|k|} = 0 - \frac{90}{3} = -30$



$y = 5\cos(3x) - 2$

$y = 5\sin[3(x + 30)] - 2$

L5 – Trig Applications Part 1

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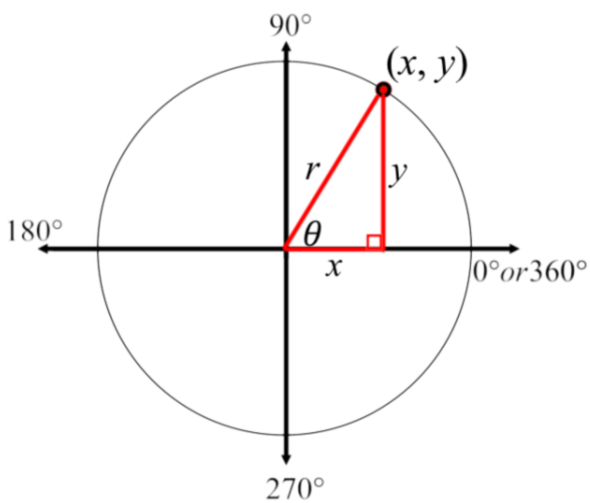
Before we do application questions, it will be good to know the connection between what we learned last chapter and the functions from this chapter:

Desmos - Sine Graph   Desmos - Cosine Graph

Section 1: Remembering the Unit Circle

The circle being used has radius r. The radius and the coordinates of a point on the circle (x, y) are related to the primary trig ratios. Study the circle and write expressions for $\sin\theta$, $\cos\theta$, and $\tan\theta$ in terms of x, y, and r.

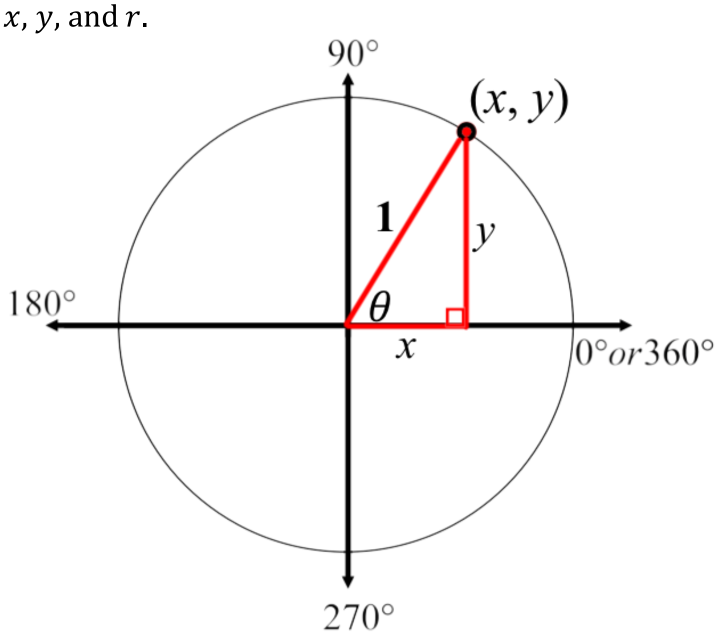


$\sin\theta = \dfrac{y}{r}$

$\cos\theta = \dfrac{x}{r}$

$\tan\theta = \dfrac{y}{x}$

A UNIT CIRLCE has a radius of 1. Use the unit circle to write expressions for $\sin\theta$, $\cos\theta$, and $\tan\theta$ in terms of x, y, and r.



$\sin\theta = y$

$\cos\theta = x$

$\tan\theta = \dfrac{y}{x}$

Summary of findings for trig ratios using the unit circle:

The sine function:

graphs the relationship between the angle and the **VERTICAL** displacement from the *x*-axis.

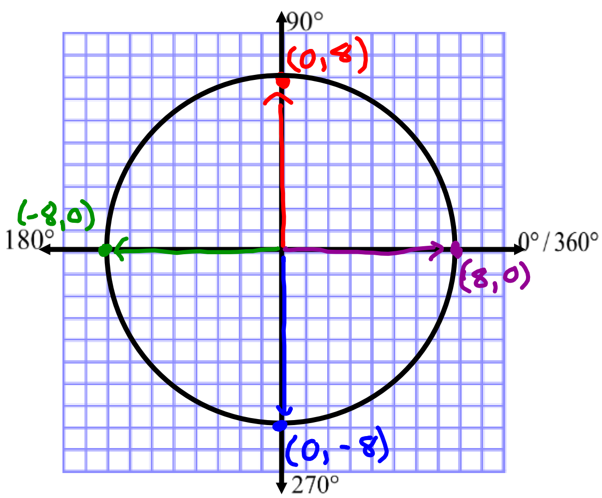
The cosine function:

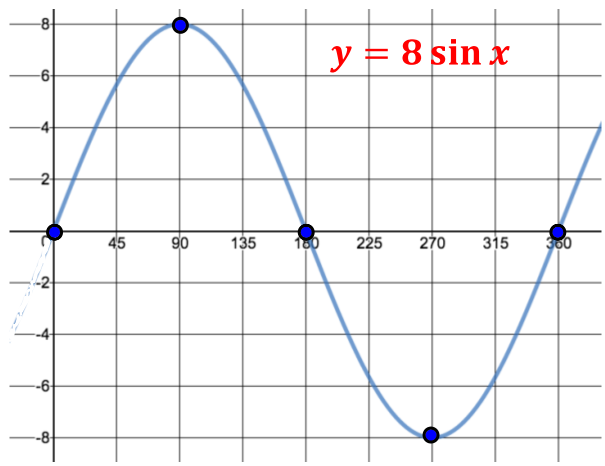
graphs the relationship between the angle and the **HORIZONTAL** displacement from the *y*-axis.

Section 2: Modeling with Graphs

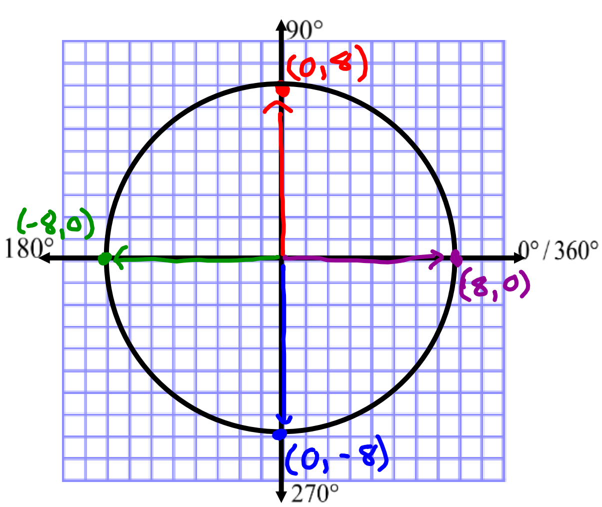
Example 1: You are in a car of a Ferris wheel. The wheel has a radius of 8m and turns counterclockwise. Let the origin be at the center of the wheel. Begin your sketch when the radius from the center of the wheel to your car is along the positive x-axis.

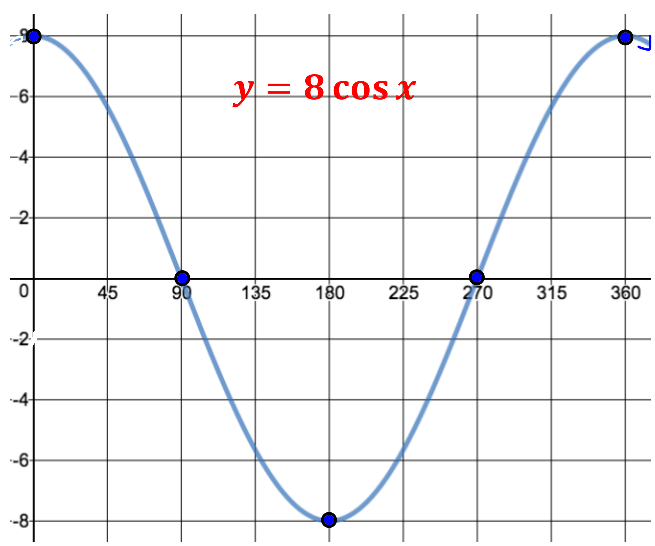
a) Sketch the graph of **vertical displacement** versus the angle of rotation for 1 complete rotation.



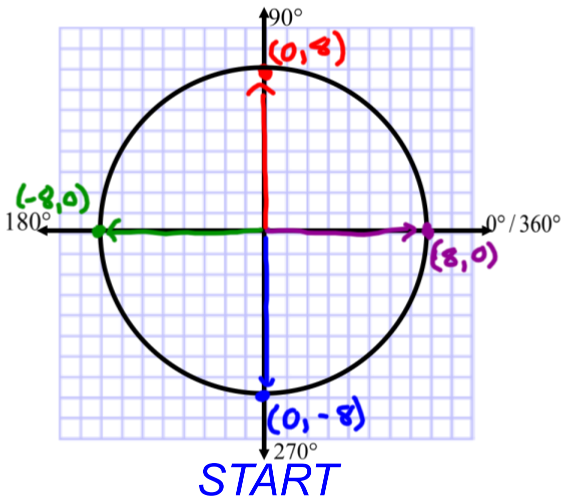


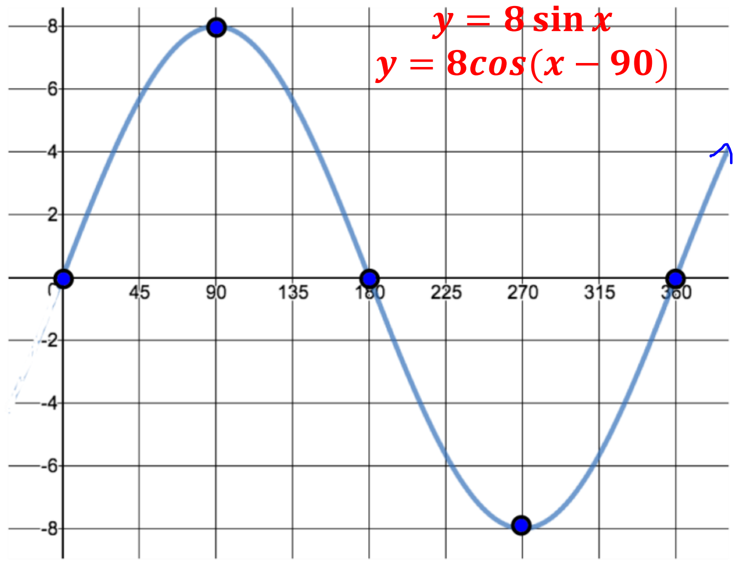
b) Sketch the graph of **horizontal displacement** versus the angle of rotation for 1 complete rotation starting along the positive x-axis.





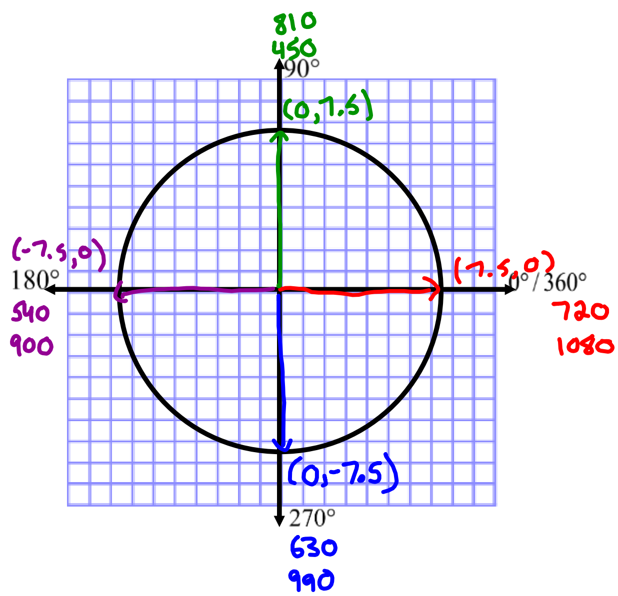
c) Sketch the graph of **horizontal displacement** versus the angle of rotation for 1 complete rotation if your car starts at the **bottom** of the Ferris Wheel.

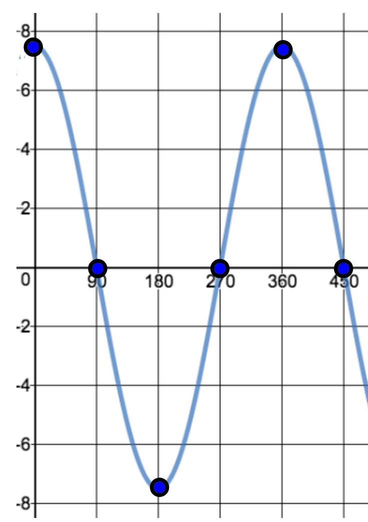




Example 2: A carousel rotates at a constant speed. It has a diameter of 15m. A horse that is directly in line with the center, horizontally, rotates around 3 full times. Create a graph that models the **horizontal distance** from the center as the horse rotates around.

Note: $radius = \frac{15}{2} = 7.5$

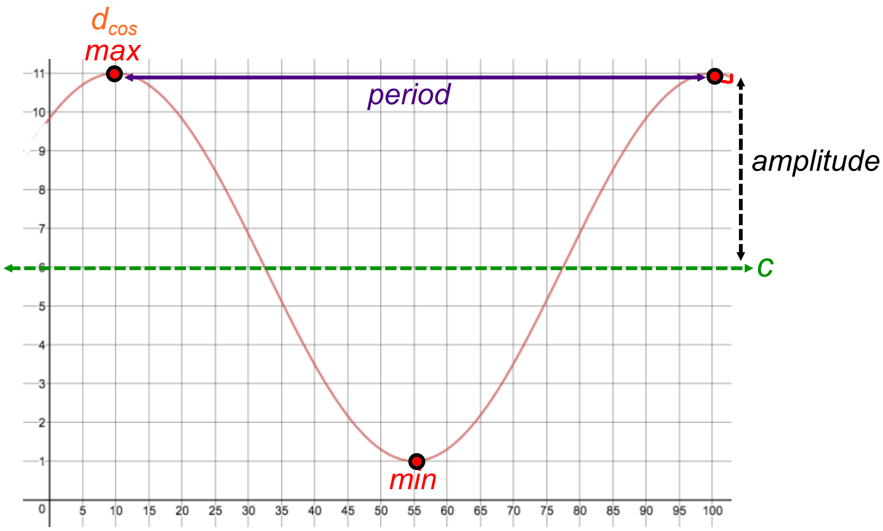




Section 3: Modeling with Equations

Example 3: A group of students is tracking a friend, John, who is riding a Ferris wheel. They know that John reaches the maximum height of 11m at 10 seconds, and then reaches the minimum height of 1m at 55 seconds.

a) Develop an equation of a sine and cosine function that models John's height above the ground.



$$a = \frac{max - min}{2} = \frac{11 - 1}{2} = 5$$

$$k = \frac{360}{period} = \frac{360}{100 - 10} = \frac{360}{90} = 4$$

$$c = max - |a| = 11 - 5 = 6$$

$$d_{cos} = 10$$

$$d_{sin} = d_{cos} - \frac{90}{|k|} = 10 - \frac{90}{4} = -12.5$$

$$y = 5\cos[4(x - 10)] + 6 \qquad y = 5\sin[4(x + 12.5)] + 6$$

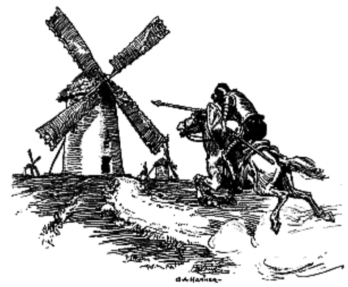
b) What is John's height above the ground after 78 seconds?

$$y = 5\cos[4(78 - 10)] + 6$$

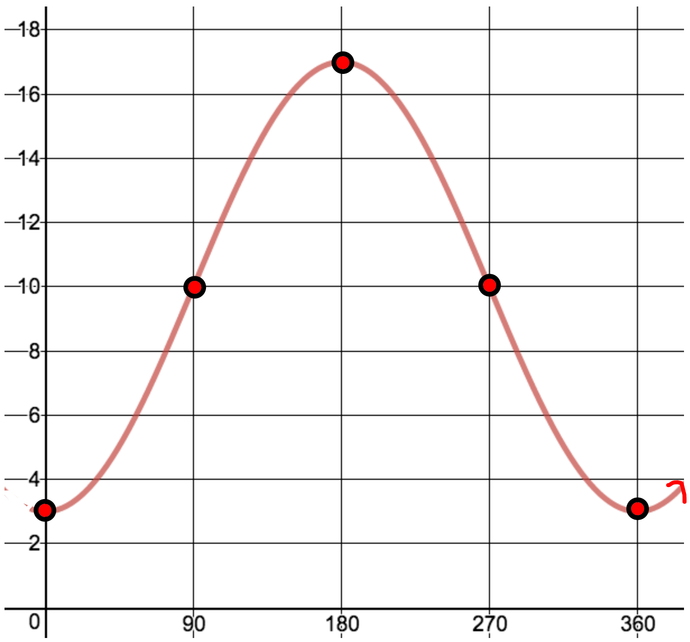
$$y = 5\cos[272] + 6$$

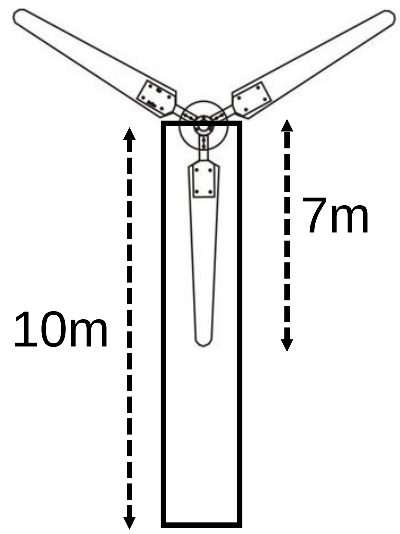
$$y = 6.2\text{m}$$

Example 4: Don Quixote, a fictional character in a Spanish novel, attacked windmills because he thought they were giants. At one point, he got snagged by one of the blades and was hoisted into the air. The center of the windmill is 10 meters off the ground and each blade is 7 meters long. The blade picked him up when it was at its lowest point.



a) Graph Don's height above the ground during one full rotation around the windmill





b) Determine an equation for a sine and cosine function that represents his height above the ground in relation to the angle of rotation.

$$a = \frac{max - min}{2} = \frac{17 - 3}{2} = 7$$

$$k = \frac{360}{period} = \frac{360}{360} = 1$$

$$c = max - |a| = 17 - 7 = 10$$

$$d_{cos} = 180$$

$$d_{sin} = 90$$

$$y = 7\cos(x - 180) + 10 \qquad y = 7\sin(x - 90) + 10$$

L6 – Trig Applications Part 2
MCR3U
Jensen

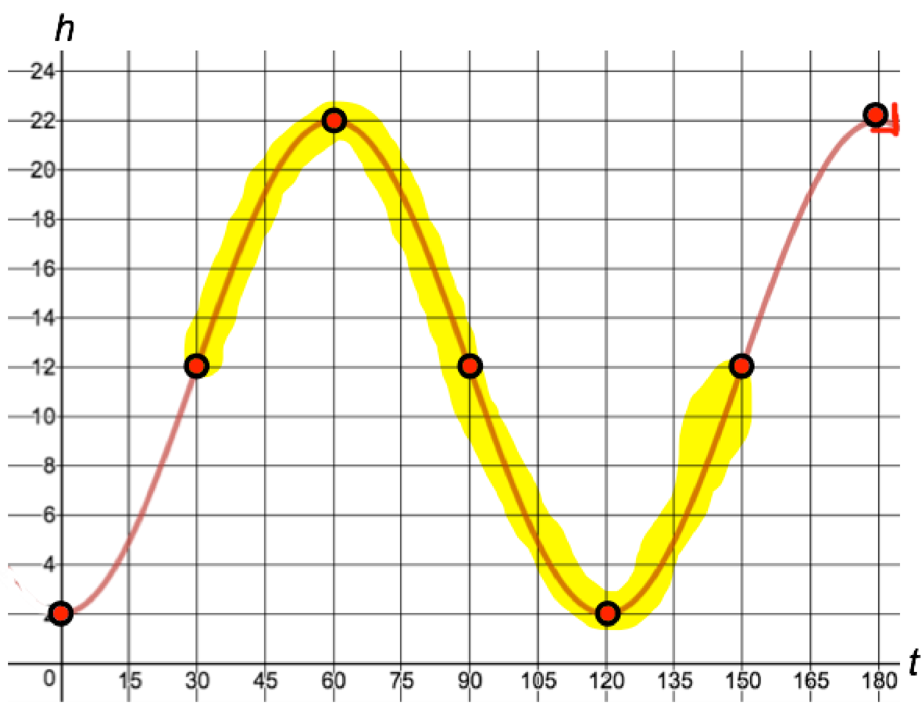
Example 1: The height, h, in meters, above the ground of a rider on a Ferris wheel after t seconds can be modelled by the sine function:

$$h(t) = 10\sin[3(t - 30)] + 12$$

a) Graph the function using transformations

$y = \sin x$	
x	y
0	0
90	1
180	0
270	-1
360	0

$h(t) = 10\sin[3(t - 30)] + 12$	
$\frac{x}{3} + 30$	$10y + 12$
30	12
60	22
90	12
120	2
150	12



b) Determine the max height, min height, and time for one revolution.

$max = 22$ m
$min = 2$ m
$period = 150 - 30 = 120$ seconds

c) Represent the function using the equation of a cosine function

$a = 10$

$k = 3$

$c = 12$

$$d_{cos} = d_{sin} + \frac{90}{|k|} = 30 + \frac{90}{3} = 60$$

$$\boldsymbol{h(t) = 10\cos[3(t - 60)] + 12}$$

d) What is the height of the rider after 35 seconds? Use both equations to verify your answer.

$$\boldsymbol{h(35) = 10\cos[3(35 - 60)] + 12}$$

$$\boldsymbol{h(35) = 10\cos[-75] + 12}$$

$$\boldsymbol{h(35) = 14.6\ \text{m}}$$

$$\boldsymbol{h(35) = 10\sin[3(35 - 30)] + 12}$$

$$\boldsymbol{h(35) = 10\sin[15] + 12}$$

$$\boldsymbol{h(35) = 14.6\ \text{m}}$$

Example 2: Skyscrapers sway in high-wind conditions. In one case, at $t = 2$ seconds, the top floor of a building swayed 30 cm to the left (-30 cm) and at $t = 12$ seconds, the top floor swayed 30 cm to the right (+30 cm) of its starting position.

a) What is the equation of a cosine function that describes the motion of the building in terms of time?

$$a = \frac{max - min}{2} = \frac{30 - (-30)}{2} = 30$$

$$k = \frac{360}{period} = \frac{360}{20} = 18$$

$$c = max - |a| = 30 - 30 = 0$$

$$d_{cos} = 12$$

$$y = 30\cos[18(t - 12)]$$

b) What is the equation of a sine function that describes the motion of the building in terms of time?

$d_{sin} = d_{cos} - \frac{90}{|k|} = 12 - \frac{90}{18} = 7$

$y = 30\cos[18(t-7)]$

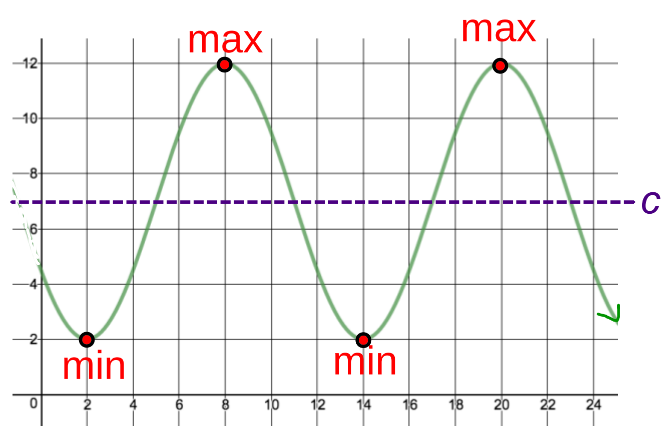
Example 3: The height of the tide on a given day at $'t'$ hours after midnight is modelled by:

$$h(t) = 5\sin[30(t-5)] + 7$$

a) Find the max and min values for the height of the depth of the water.

$max = c + |a| = 7 + 5 = 12$

$min = c - |a| = 7 - 5 = 2$



b) What time is high tide? What time is low tide?

Note: $period = \frac{360}{k} = \frac{360}{30} = 12$; therefore there are 2 cycles in a 24 hour period.

The first rising midline is at $t = 5$. A max will occur $\frac{90}{k}$ to the right of the rising midline.

Therefore, there is a max at $5 + \frac{90}{k} = 5 + \frac{90}{30} = 8$. There will be another high tide in 12 hours (since this is the period of the function).

High tide = 8am AND 8 pm

The first rising midline is at $t = 5$. A min will occur $\frac{90}{k}$ to the left of the rising midline.

Therefore, there is a min at $5 - \frac{90}{k} = 5 - \frac{90}{30} = 2$. There will be another high tide in 12 hours (since this is the period of the function).

Low tide = 2am AND 2 pm

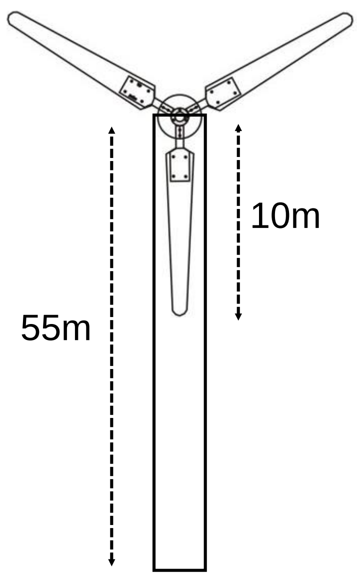
c) What is the depth of the water at 9 am?

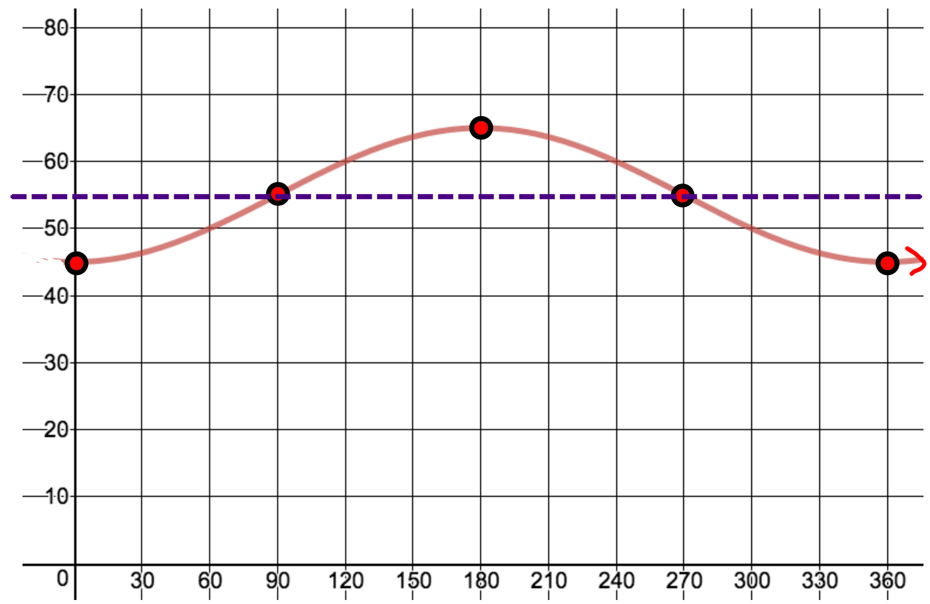
$h(9) = 5\sin[30(9-5)] + 7$

$h(9) = 5\sin[120] + 7$

$h(9) = 11.3 \text{ m}$

Example 4a: A wind turbine has a height of 55m from the ground to the center of the turbine. Graph one cycle of the vertical displacement of a 10m blade turning counterclockwise. Assume the blade starts pointing straight down.





Example 4b: Model the rider's height above the ground versus angle using a transformed sine and cosine function.

$$a = \frac{max - min}{2} = \frac{65-45}{2} = 10$$

$$k = \frac{360}{period} = \frac{360}{360} = 1$$

$c = max - |a| = 65 - 10 = 55$

$d_{cos} = 180$

$d_{sin} = 90$

$h = 10\cos(x - 180) + 55$ y$= 10\sin(x - 90) + 55$