L2 – 1.2 – Characteristics of Polynomial Functions Lesson
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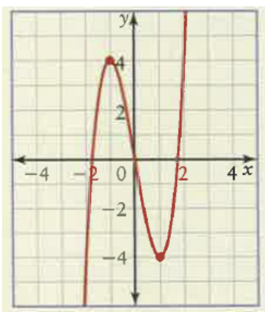
In section 1.1 we looked at power functions, which are single-term polynomial functions. Many polynomial functions are made up of two or more terms. In this section we will look at the characteristics of the graphs and equations of polynomial functions.

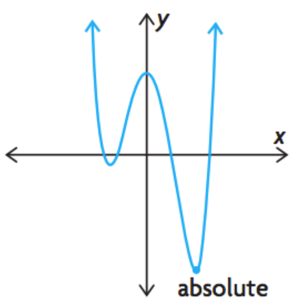
New Terminology – Local Min/Max vs. Absolute Min/Max

Local Min or Max Point – Points that are minimum or maximum points on some interval around that point.

Absolute Max or Min – The greatest/least value attained by a function for ALL values in its domain.



In this graph, (-1, 4) is a **local max** and (1, -4) is a **local min**. These are not absolute min and max points because there are other points on the graph of the function that are smaller and greater. Sometimes local min and max points are called **turning points**.



On the graph of this function…

There are **3** local min/max points. **2** are local min and **1** is a local max.

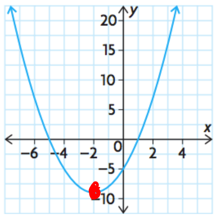
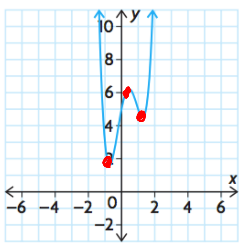
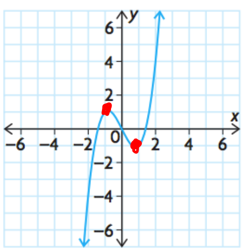
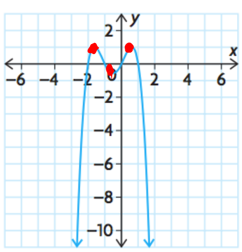
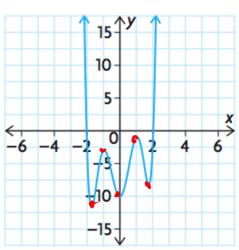
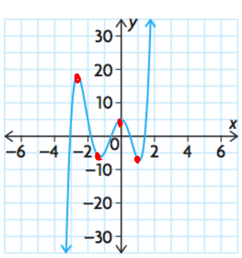
One of the local min points is also an absolute min (it is labeled) .

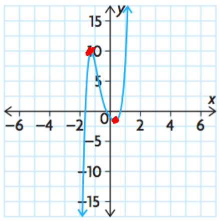
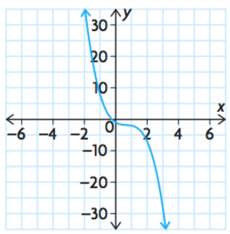
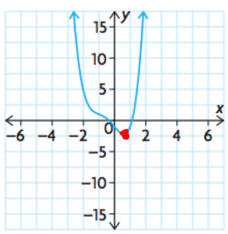
Investigation: Graphs of Polynomial Functions

The degree and the leading coefficient in the equation of a polynomial function indicate the end behaviours of the graph.

The degree of a polynomial function provides information about the shape, turning points (local min/max), and zeros (x-intercepts) of the graph.

Complete the following table using the equation and graphs given:

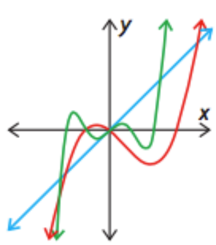
Equation and Graph	Degree	Even or Odd Degree?	Leading Coefficient	End Behaviour	Number of turning points	Number of x-intercepts
$f(x) = x^2 + 4x - 5$ 	2	Even	+1	as $x \to -\infty$, $y \to \infty$ as $x \to \infty$, $y \to \infty$ Q2 to Q1	1	2
$f(x) = 3x^4 - 4x^3 - 4x^2 + 5x + 5$ 	4	Even	+3	as $x \to -\infty$, $y \to \infty$ as $x \to \infty$, $y \to \infty$ Q2 to Q1	3	0
$f(x) = x^3 - 2x$ 	3	Odd	+1	as $x \to -\infty$, $y \to -\infty$ as $x \to \infty$, $y \to \infty$ Q3 to Q1	2	3
$f(x) = -x^4 - 2x^3 + x^2 + 2x$ 	4	Even	−1	as $x \to -\infty$, $y \to -\infty$ as $x \to \infty$, $y \to -\infty$ Q3 to Q4	3	4
$f(x) = 2x^6 - 12x^4 + 18x^2 + x - 10$ 	6	Even	+2	as $x \to -\infty$, $y \to \infty$ as $x \to \infty$, $y \to \infty$ Q2 to Q1	5	2
$f(x) = 2x^5 + 7x^4 - 3x^3 - 18x^2 + 5$ 	5	Odd	+2	as $x \to -\infty$, $y \to -\infty$ as $x \to \infty$, $y \to \infty$ Q3 to Q1	4	5

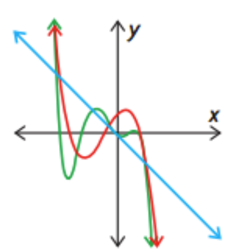
Equation and Graph	Degree	Even or Odd Degree?	Leading Coefficient	End Behaviour	Number of turning points	Number of x-intercepts
$f(x) = 5x^5 + 5x^4 - 2x^3 + 4x^2 - 3x$ 	5	Odd	+5	as $x \to -\infty$, $y \to -\infty$ as $x \to \infty$, $y \to \infty$ Q3 to Q1	2	3
$f(x) = -2x^3 + 4x^2 - 3x - 1$ 	3	Odd	-2	as $x \to -\infty$, $y \to \infty$ as $x \to \infty$, $y \to -\infty$ Q2 to Q4	0	1
$f(x) = x^4 + 2x^3 - 3x - 1$ 	4	Even	+ 1	as $x \to -\infty$, $y \to \infty$ as $x \to \infty$, $y \to \infty$ Q2 to Q1	1	2

Summary of Findings:

- A polynomial function of degree n has at most **$n - 1$** local max/min points (turning points)
- A polynomial function of degree n may have up to **n** distinct zeros (x-intercepts)
- If a polynomial function is **odd** degree, it must have at least one x-intercept, and an even number of turning points
- If a polynomial function is **even** degree, it may have no x-intercepts, and an odd number of turning points

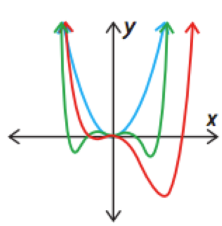
- An odd degree polynomial function extends from…
  - **3rd** quadrant to **1st** quadrant if it has a positive leading coefficient
  - **2nd** quadrant to **4th** quadrant if it has a negative leading coefficient

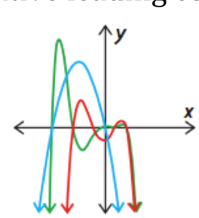




Note: *Odd degree polynomials have OPPOSITE end behaviours*

- An even degree polynomial function extends from…
  - **2nd** quadrant to **1st** quadrant if it has a positive leading coefficient
  - **3rd** quadrant to **4th** quadrant if is has a negative leading coefficient





Note: *Even degree polynomials have THE SAME end behaviour.*

Example 1: Describe the end behaviours of each function, the possible number of turning points, and the possible number of zeros. Use these characteristics to sketch possible graphs of the function

a) $f(x) = -3x^5 + 4x^3 - 8x^2 + 7x - 5$

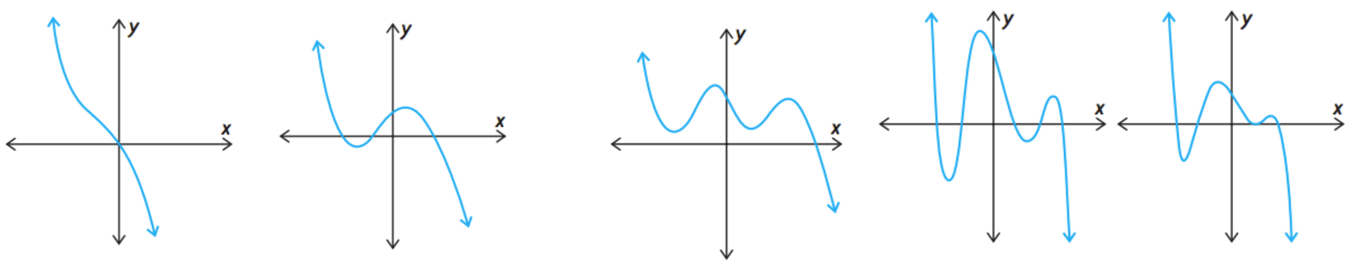
The degree is odd and the leading coefficient is negative so the function must extend from Q2 to Q4

As $x \to -\infty, y \to \infty$
As $x \to \infty, y \to -\infty$

The function can have at most 5 x-intercepts (1, 2, 3, 4, or 5)
The function can have at most 4 turning points (0, 2, or 4)

Note: *Odd degree functions must have an even number of turning points.*

Possible graphs of 5th degree polynomial functions with a negative leading coefficient:



b) $g(x) = 2x^4 + x^2 + 2$

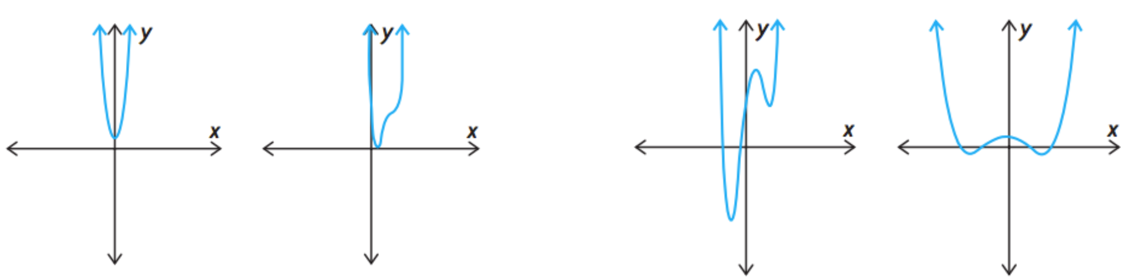
The degree is even and the leading coefficient is positive so the function must extend from the second quadrant to the first quadrant.

As $x \to -\infty, y \to \infty$
As $x \to \infty, y \to \infty$

The function can have at most 4 x-intercepts (0,1, 2, 3, or 4)
The function can have at most 3 turning points (1, or 3)

Note: *Even degree functions must have an odd number of turning points.*

Possible graphs of 4th degree polynomial functions with a positive leading coefficient:



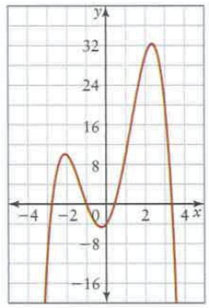
Example 2: Fill out the following chart

Degree	Possible # of x-intercepts	Possible # of turning points
1	1	0
2	0, 1, 2	1
3	1, 2, 3	0, 2
4	0, 1, 2, 3, 4	1, 3
5	1, 2, 3, 4, 5	0, 2, 4

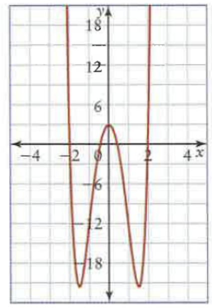
Example 3: Determine the key features of the graph of each polynomial function. Use these features to match each function with its graph. State the number of x-intercepts, the number of local max/min points, and the number of absolute max/min points for the graph of each function. How are these features related to the degree of each function?

a) $f(x) = 2x^3 - 4x^2 + x + 1$
b) $g(x) = -x^4 + 10x^2 + 5x - 4$
c) $h(x) = -2x^5 + 5x^3 - x$
d) $p(x) = x^6 - 16x^2 + 3$

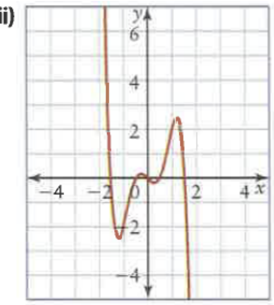
i)



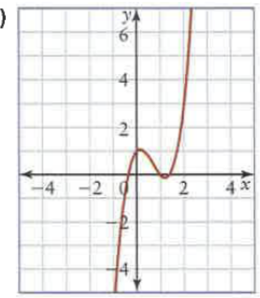
ii)



iii)



iv)



a) The function is cubic with a positive leading coefficient. The graph extends from Q3 to Q1. The y-intercept is 1. Graph iv) corresponds to this equation.

There are 3 x-intercepts and the degree is 3. The function has one local max and one local min, which is a total of two turning points, which is one less than the degree. There is no absolute max or min point.

b) The function is quartic with a negative leading coefficient. The graph extends from quadrant 3 to 4. The y-intercept is -4. Graph i) corresponds to this equation.

There are 4 x-intercepts and the degree is 4. The function has two local max and one local min, which is a total of 3 turning points, which is one less than the degree. The function has one absolute max point.

c) The function is quintic with a negative leading coefficient. The graph extends from quadrant 2 to 4. The y-intercept is 0. Graph iii) corresponds to this equation.

There are 5 x-intercepts and the degree is 5. The function has two local max and two local min, which is a total of 4, which is one less than the degree. The function has no absolute max or min points.

d) The function is degree 6 with a positive leading coefficient. The graph extends from quadrant 2 to 1. The y-intercept is 3. Graph ii) corresponds to this equation.

There are 4 x-intercepts and the degree is 6. The function has two local max and one local min, which is a total of 3, which is three less than the degree. The function has two absolute min points.

<u>Finite Differences</u>

For a polynomial function of degree n, where n is a positive integer, the n^{th} differences…

- are equal
- have the same sign as the leading coefficient
- are equal to $a \cdot n!$, where a is the leading coefficient

Note:

$n!$ is read as n factorial.

$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Example 4: The table of values represents a polynomial function. Use finite differences to determine

a) the degree of the polynomial function
b) the sign of the leading coefficient
c) the value of the leading coefficient

x	y	First differences	Second differences	Third differences
-3	-36			
-2	-12	$-12-(-36)=24$		
-1	-2	$-2-(-12)=10$	$10-24=-14$	
0	0	$0-(-2)=2$	$2-10=-8$	$-8-(-14)=6$
1	0	$0-0=0$	$0-2=-2$	$-2-(-8)=6$
2	4	$4-0=4$	$4-0=4$	$4-(-2)=6$
3	18	$18-4=14$	$14-4=10$	$10-4=6$
4	48	$48-18=30$	$30-14=16$	$16-10=6$

a) The third differences are constant. So, the table of values represents a cubic function. The degree of the function is 3.

b) The leading coefficient is positive since 6 is positive.

c)
$$6 = a \cdot n!$$
$$6 = a \cdot 3!$$
$$6 = a \cdot 6$$
$$\frac{6}{6} = a$$
$$a = 1$$

Therefore, the leading coefficient of the polynomial function is 1.

Example 5: For the function $2x^4 - 4x^2 + x + 1$ what is the value of the constant finite differences?

$$Finite\ differences = a \cdot n!$$
$$= 2 \cdot 4!$$
$$= 2 \cdot 24$$
$$= 48$$

Therefore, the fourth differences would all be 48 for this function.