Section 4.7 – Combinations

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Part 1: Handshake Problem

Problem: There are 26 students in this class. If every student shook every other students hand once, how many different handshakes would take place?

Guess: _____

Ideas on how to solve the problem:

Each of the 26 students will have 25 handshakes (one with each other student in the class). Therefore 26 x 25 seems to make sense….

However, this would count the handshake between each pair twice, and in this instance the order does NOT matter. Therefore we must divide our answer by 2 (because there are 2! arrangements within each pair that include the same people)

Solution: $n(handshakes) = C(26, 2) = \frac{26!}{(26-2)!2!} = \frac{26!}{24!2!} = \frac{26 \times 25}{2} = 325$

Part 2: Combinations

PERMUTATIONS are ordered sets where: ABC/BCA/CAB etc. are all different possibilities, because with permutations ORDER MATTERS.

HOWEVER, for many counting problems, order is not important!

COMBINATIONS are unordered sets where: ABC/BCA/CAB etc. are considered the same, because with combinations order DOES NOT matter.

Permutations	Combinations
When we consider permutations, we are considering groupings *and* order. $P(n, r) = \frac{n!}{(n-r)!}$	When we consider combinations, we are considering only the number of different groupings. Order within the groupings is not considered. $C(n, r) = \frac{n!}{(n-r)!\,r!}$

Example 1:

You won four free tickets for the 2016 Stanley Cup Final (Pittsburgh vs. Edmonton….hopefully). Determine how many ways you can select 3 of your 10 best friends to come to the game with you.

> before you start you must ask yourself, does the order you pick your friends really matter?
>
> **If yes --> permutation** **If no --> combination**

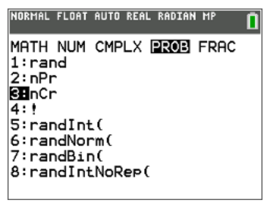
$$n(groups\ of\ 3\ friends) = C(10,3) = \frac{10!}{(10-3)!\,3!} = \frac{10!}{7!\,3!} = 120$$

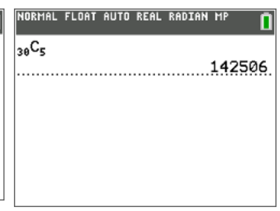
Example 3:

From a class of 30 students, determine how many ways a five-person committee can be selected to organize a class party:

a) With no restrictions

$$n(committees) = C(30,5) = 142\ 506$$





30 → MATH → PROB → nCr → 5 → ENTER

b) With Marnie on the committee

$$n(committees) = C(29,4) = 23\ 751$$

Example 4:

How many 5-card hands can be dealt out of a deck of 52 cards?

$$n(hands) = \binom{52}{5} = 2\ 598\ 960$$

Note: $\binom{52}{5}$ *means the same as* $C(52,5)$

Example 5:

A coach of a co-ed basketball team must select five players to start the game from a team that consists of six females and five males. How many ways can this be achieved if Tanya must choose three females and two males to start the game. (assume order does not matter)

> Remember the **Multiplication Rule for Counting**:
>
> To find the number of outcomes for a series of events, find the product of the possible outcomes at each step in the sequence.
>
> n(a,b) = n(A) x n(B)

$n(3\ females\ \cap 2\ males) = n(3\ females) \times n(2\ males)$

$= \binom{6}{3} \times \binom{5}{2}$

$= 20 \times 10$

$= 200$

Example 6:

In how many ways can 6 people be selected from a group that consists of four adults and eight children if the group must contain at least two adults?

Solution 1: Direct Reasoning

In this situation, the condition that the group must have at least two adults must be satisfied. This can happen three ways:

2 adults and 4 children **or** 3 adults and 3 children **or** 4 adults and 2 children

> Remember the **Additive Principle for the Union of Sets**:
>
> n(a ∪ b) = n(A) + n(B)
>
> (for disjoint sets)

$n(atleast\ 2\ adults) = n(2A, 4C) + n(3A, 3C) + n(4A, 2C)$

$= \binom{4}{2}\binom{8}{4} + \binom{4}{3}\binom{8}{3} + \binom{4}{4}\binom{8}{2}$

$= 6(70) + 4(56) + 1(28)$

$= 420 + 224 + 28$

$= 672$

Solution 2: Indirect Reasoning

In this situation, the solution can also be found by subtracting the number of ways the condition is not satisfied (0 or 1 adult in the group) from the total number of combinations.

$n(atleast\ 2\ adults) = n(groups) - [n(0A, 6C) + n(1A, 5C)]$

$= \binom{12}{6} - \left[\binom{4}{0}\binom{8}{6} + \binom{4}{1}\binom{8}{5}\right]$

$= 924 - [1(28) + 4(56)]$

$= 924 - 252$

$= 672$

Part 3: Using Combinations to Find Probabilities

Example 7:

Five cards are dealt at random from a deck of 52 playing cards. Determine the probability that you will have:

a) the 10-J-Q-K-A of the same suit (a royal flush)?

$$\boldsymbol{P(royal\ flush) = \frac{n(royal\ flush)}{n(hands)} = \frac{4}{C(52,5)} = \frac{4}{2\ 598\ 960} = \frac{1}{649\ 740}}$$

b) four of a kind?

$$P(4\ of\ a\ kind) = \frac{n(4\ of\ a\ kind\ hands)}{n(hands)} = \frac{13\binom{48}{1}}{\binom{52}{5}} = \frac{624}{2\ 598\ 960} = \frac{1}{4165}$$

Note: There are 13 cards in each suit and the hand must contain four of the same card. The remaining card in the hand can be any card of the remaining 48.

Example 8:

A company that has 40 employees chooses a committee of 7 to represent employee retirement issues. When the committee was formed, none of the 18 minority employees were selected. Do you think the committee selection was biased? Give mathematical evidence for your decision.

Possible solution:

The number of ways 7 employees can be chosen from 40:

$$n(committees) = \binom{40}{7} = 18\,643\,560$$

The number of ways 7 employees can be chosen from 22 non-minorities:

$$n(committees\ with\ no\ minorities) = \binom{22}{7} = 170\,544$$

The probability that it contained no minorities if it was chosen randomly is:

$$P(no\ minorities) = \frac{n(committees\ with\ no\ minorities)}{n(committees)} = \frac{170\,544}{18\,643\,560} = 0.00915$$

Since there is only about a 0.9% chance the committee would contain no minorities if it was chosen at random, it seems reasonable to conclude that the committee selection was biased.