5.2 Hypergeometric Probability Distributions

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Part 1: What is a Hypergeometric Probability Distribution

When a coach chooses a starting line-up for a game, a coach obviously has to choose a different player for each position. Similarly, when you deal cards from a standard deck, there can be no repetitions. In such situations, each selection reduces the number of items that could be selected in the next trial. Thus, the probabilities in these trials are dependent.

A hypergeometric distribution has a specified number of dependent trials having two possible outcomes, success or failure. The random variable is the number of successful outcomes in the specified number of trials. The individual outcomes cannot be repeated within these trials.

Part 2: Completing a Hypergeometric Probability Distribution

Example 1: Determine the probability distribution for the number of women on a three-person committee selected from a pool of 8 men and 10 women.

The selection process involves dependent events since each person who is already chosen for the jury cannot be selected again. The total number of ways the 3 people can be selected from the pool of 18 is:

$$n(S) = \binom{18}{3} = 816$$

There can be from 0 to 3 women on the committee. The men will fill the remaining positions. The probability distribution is as follows:

Number of Women, x	**Probability, $P(x)$**
0	$\frac{\binom{10}{0}\binom{8}{3}}{\binom{18}{3}} = 0.0686$
1	$\frac{\binom{10}{1}\binom{8}{2}}{\binom{18}{3}} = 0.3431$
2	$\frac{\binom{10}{2}\binom{8}{1}}{\binom{18}{3}} = 0.4412$
3	$\frac{\binom{10}{3}\binom{8}{0}}{\binom{18}{3}} = 0.1471$

The expected number of women is: $E(X) = \sum x \cdot P(x) = 0 \cdot P(0) + 1 \cdot P(1) + 2 \cdot P(2) + 3 \cdot P(3)$
$= 0(0.0686) + 1(0.3431) + 2(0.4412) + 3(0.1471)$
$= 1.67$

Part 3: Hypergeometric Probability Formula

To find the probability of x successes in r dependent trials, use the formula:

$$P(x) = \frac{\binom{a}{x}\binom{b}{r-x}}{\binom{n}{r}}$$

where a is the number of successful outcomes among a total of n possible outcomes, and b is the number of failures among all possible outcomes.

Looking back to example 1, we could have used the formula:

$P(x) = \frac{\binom{10}{x}\binom{8}{3-x}}{\binom{18}{3}}$ for each probability calculation.

$a = 10$
$b = 8$
$n = 18$
$r = 3$

Example 2: Suppose you are dealt a four-card hand from a standard deck of cards.

a) Create a table that shows the theoretical probability distribution of how many spades are in your hand.

There can be from 0 to 4 spades in your hand. The other suits can fill the remaining spots. We can use the following formula to calculate the probability of a three card hand with x spades in it:

$a = 13$
$b = 39$
$n = 52$
$r = 4$

$$P(x) = \frac{\binom{a}{x}\binom{n-a}{r-x}}{\binom{n}{r}} = \frac{\binom{13}{x}\binom{39}{4-x}}{\binom{52}{4}}$$

Number of Spades, x	Probability, $P(x)$
0	$\frac{\binom{13}{0}\binom{39}{4}}{\binom{52}{4}} = 0.3038$
1	$\frac{\binom{13}{1}\binom{39}{3}}{\binom{52}{4}} = 0.4388$
2	$\frac{\binom{13}{2}\binom{39}{2}}{\binom{52}{4}} = 0.2135$
3	$\frac{\binom{13}{3}\binom{39}{1}}{\binom{52}{4}} = 0.0412$
4	$\frac{\binom{13}{4}\binom{39}{0}}{\binom{52}{4}} = 0.0026$

b) What is the expected number of spades in a hand?

$$E(X) = \sum x \cdot P(x) = 0(0.3038) + 1(0.4388) + 2(0.2135) + 3(0.0412) + 4(0.0026) = 1$$

Example 3: A box contains eight yellow, four green, five purple, and three red candies jumbled together. You randomly pour five candies in to your hand. Create a table that shows the theoretical probability distribution of how many red candies are in your hand.

There can be from 0 to 3 red candies in your hand. The other candies will be one of the non-red colours. We can use the following formula to calculate the probability of pouring x red candies in to your hand:

$$P(x) = \frac{\binom{a}{x}\binom{b}{r-x}}{\binom{n}{r}} = \frac{\binom{3}{x}\binom{17}{5-x}}{\binom{20}{5}}$$

$a = 3$
$b = 17$
$n = 20$
$r = 5$

Number of Red Candies, x	**Probability, $P(x)$**
0	$\frac{\binom{3}{0}\binom{17}{5}}{\binom{20}{5}} = 0.3991$
1	$\frac{\binom{3}{1}\binom{17}{4}}{\binom{20}{5}} = 0.4605$
2	$\frac{\binom{3}{2}\binom{17}{3}}{\binom{20}{5}} = 0.1316$
3	$\frac{\binom{3}{3}\binom{17}{2}}{\binom{20}{5}} = 0.0088$