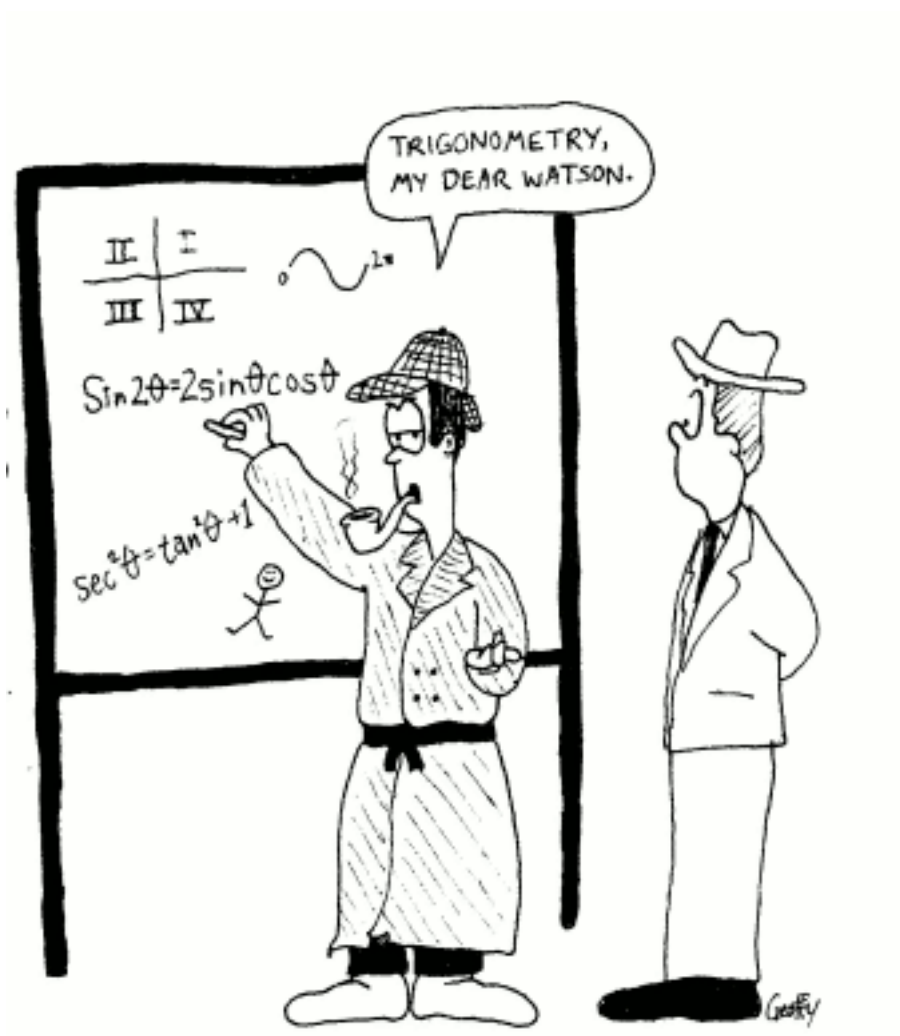
UNIT 5

Chapter 4/5 Part 2- Trig Identities and Equations

Lesson Package

MHF4U

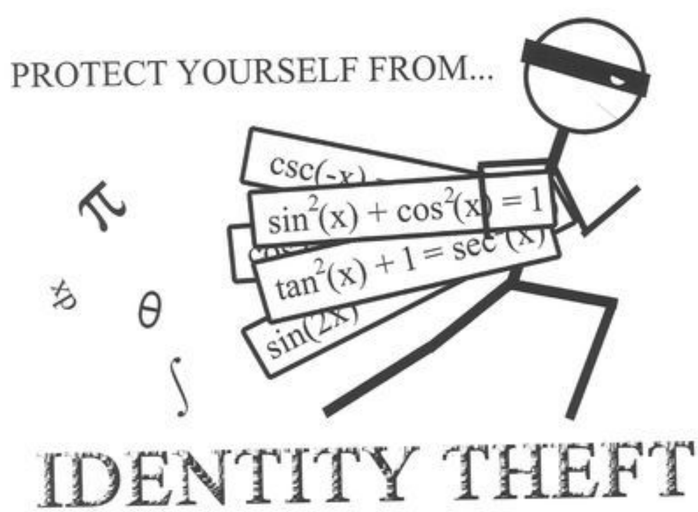


Chapter 4/5 Part 2 Outline

Unit Goal: By the end of this unit, you will be able to solve trig equations and prove trig identities.

Section	Subject	Learning Goals	Curriculum Expectations
L1	Transformation Identities	- recognize equivalent trig expressions by using angles in a right triangle and by performing transformations	B3.1
L2	Compound Angles	- understand development of compound angle formulas and use them to find exact expressions for non-special angles	B3.2
L3	Double Angle	- use compound angle formulas to derive double angle formulas - use formulas to simplify expressions	B3.3
L4	Proving Trig Identities	- Be able to prove identities using identities learned throughout the unit	B3.3
L5	Solve Linear Trig Equations	- Find all solutions to a linear trig equation	B3.4
L5	Solve Trig Equations with Double Angles	- Find all solutions to a trig equation involving a double angle	B3.4
L5	Solve Quadratic Trig Equations	- Find all solutions to a quadratic trig equation	B3.4
L5	Applications of Trig Equations	- Solve problems arising from real world applications involving trig equations	B2.7

Assessments	F/A/O	Ministry Code	P/O/C	KTAC
Note Completion	A		P	
Practice Worksheet Completion	F/A		P	
Quiz – Solving Trig Equations	F		P	
PreTest Review	F/A		P	
Test – Trig Identities and Equations	O	B3.1, 3.2, 3.3, 3.4 B2.7	P	K(21%), T(34%), A(10%), C(34%)



L1 – 4.3 Co-function Identities
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Part 1: Remembering How to Prove Trig Identities

Fundamental Trigonometric Identities		
Reciprocal Identities	**Quotient Identities**	**Pythagorean Identities**
$csc\ \theta = \frac{1}{sin\ \theta}$ $sec\ \theta = \frac{1}{cos\ \theta}$ $cot\ \theta = \frac{1}{tan\ \theta}$	$\frac{sin\ \theta}{cos\ \theta} = tan\ \theta$ $\frac{\cos\theta}{\sin\theta} = \cot\theta$	$\sin^2\theta + \cos^2\theta = 1$

Tips and Tricks		
Reciprocal Identities	**Quotient Identities**	**Pythagorean Identities**
Square both sides $csc^2\ \theta = \frac{1}{sin^2\ \theta}$ $sec^2\ \theta = \frac{1}{cos^2\ \theta}$ $cot^2\ \theta = \frac{1}{tan^2\ \theta}$	Square both sides $\frac{sin^2\ \theta}{cos^2\ \theta} = tan^2\ \theta$ $\frac{\cos^2\theta}{\sin^2\theta} = \cot^2\theta$	Rearrange the identity $sin^2\theta = 1 - cos^2\theta$ $cos^2\theta = 1 - sin^2\theta$ Divide by either $\sin^2\theta$ or $\sin^2\theta$ $1 + cot^2\theta = \csc^2\theta$ $tan^2\theta + 1 = sec^2\theta$

General tips for proving identities:

i) Separate into LS and RS. Terms may NOT cross between sides.
ii) Try to change everything to $sin\ \theta$ or $cos\ \theta$
iii) If you have two fractions being added or subtracted, find a common denominator and combine the fractions.
iv) Use difference of squares → $1 - sin^2\ \theta = (1 - sin\ \theta)(1 + sin\ \theta)$
v) Use the power rule → $sin^6\ \theta = (sin^2\ \theta)^3$

Example 1: Prove each of the following identities

a) $\tan^2 x + 1 = \sec^2 x$

LS	RS
$= \tan^2 x + 1$	$= \sec^2 x$
QI: $= \dfrac{\sin^2 x}{\cos^2 x} + \dfrac{\cos^2 x}{\cos^2 x}$ (common denom)	R.I.: $= \dfrac{1}{\cos^2 x}$
$= \dfrac{\sin^2 x + \cos^2 x}{\cos^2 x}$	
P.I.: $= \dfrac{1}{\cos^2 x}$	

LS=RS

b) $\cos^2 x = (1 - \sin x)(1 + \sin x)$

LS	RS
$= \cos^2 x$	$= (1 - \sin x)(1 + \sin x)$
	DOS: $= 1 - \sin^2 x$
	PI: $= \cos^2 x$

LS=RS

c) $\dfrac{\sin^2 x}{1 - \cos x} = 1 + \cos x$

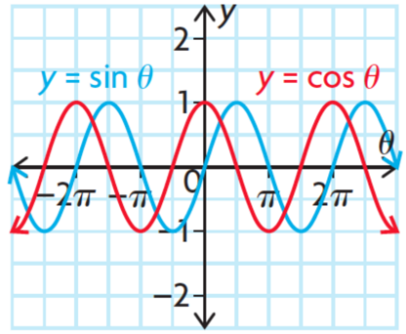
LS	RS
$= \dfrac{\sin^2 x}{1 - \cos x}$	$= 1 + \cos x$
PI: $= \dfrac{1 - \cos^2 x}{1 - \cos x}$	
DOS: $= \dfrac{(\cancel{1 - \cos x})(1 + \cos x)}{\cancel{1 - \cos x}}$	
$= 1 + \cos x$	

LS=RS

Part 2: Transformation Identities

Because of their periodic nature, there are many equivalent trigonometric expressions .

Horizontal translations of $\frac{\pi}{2}$ that involve both a sine function and a cosine function can be used to obtain two equivalent functions with the same graph.



Translating the cosine function $\frac{\pi}{2}$ to the **right**, $f(x) = \cos\left(x - \frac{\pi}{2}\right)$ results in the graph of the sine function, $f(x) = \sin x$.

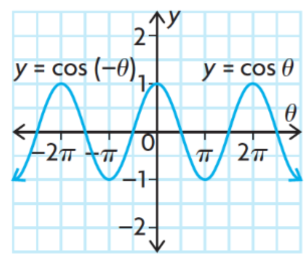
Similarly, translating the sine function $\frac{\pi}{2}$ to the **left**, $f(x) = \sin\left(x + \frac{\pi}{2}\right)$ results in the graph of the cosine function, $f(x) = \cos x$.

Transformation Identities

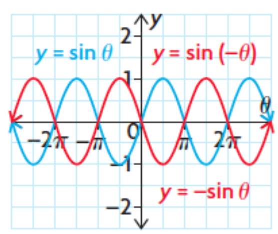
$$\cos\left(x - \frac{\pi}{2}\right) = \sin x \qquad \sin\left(x + \frac{\pi}{2}\right) = \cos x$$

Part 3: Even/Odd Function Identities

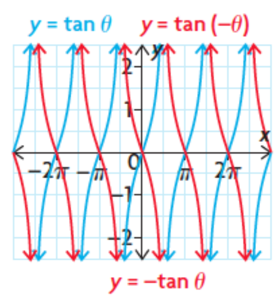
Remember that $\mathbf{cos}\ \boldsymbol{x}$ is an **even** function. Reflecting its graph across the y-axis results in two equivalent functions with the same graph.



$\sin x$ and $\tan x$ are both **odd** functions. They have rotational symmetry about the origin.



$\sin(-\theta) = -\sin\theta$



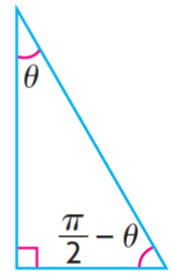
$\tan(-\theta) = -\tan\theta$

Even/Odd Identities

$$\cos(-x) = \cos x \qquad \sin(-x) = -\sin x \qquad \tan(-x) = -\tan x$$

Part 4: Co-function Identities

The co-function identities describe trigonometric relationships between complementary angles in a right triangle.



Co-Function Identities	
$\cos\left(\frac{\pi}{2} - x\right) = \sin x$	$\sin\left(\frac{\pi}{2} - x\right) = \cos x$

We could identify other equivalent trigonometric expressions by comparing principle angles drawn in standard position in quadrants II, III, and IV with their related acute (reference) angle in quadrant I.

Principle in Quadrant II	Principle in Quadrant III	Principle in Quadrant IV
$\sin(\pi - x) = \sin x$	$\sin(\pi + x) = -\sin x$	$\sin(2\pi - x) = -\sin x$

Example 2: Prove both co-function identities using transformation identities

a) $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

b) $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

a)

LS	RS
$= \cos(\pi/2 - x)$	$= \sin x$
$= \cos(-x + \pi/2)$ factor -1	
$= \cos[-(x - \pi/2)]$ even	
$= \cos(x - \pi/2)$ T.I.	
$= \sin x$	

LS=RS

b)

LS	RS
$= \sin(\pi/2 - x)$	$= \cos x$ even
	$= \cos(-x)$ TI
	$= \sin(-x + \pi/2)$
	$= \sin(\pi/2 - x)$

LS=RS

Part 5: Apply the Identities

Example 3: Given that $\sin\frac{\pi}{5} \cong 0.5878$, use equivalent trigonometric expressions to evaluate the following:

a) $\cos\frac{3\pi}{10}$

$$= \sin\left(\frac{\pi}{2} - \frac{3\pi}{10}\right)$$

$$= \sin\left(\frac{5\pi}{10} - \frac{3\pi}{10}\right)$$

$$= \sin\left(\frac{2\pi}{10}\right)$$

$$= \sin\left(\frac{\pi}{5}\right)$$

$$\cong 0.5878$$

b) $\cos\frac{7\pi}{10}$

$$= \sin\left(\frac{\pi}{2} - \frac{7\pi}{10}\right)$$

$$= \sin\left(\frac{5\pi}{10} - \frac{7\pi}{10}\right)$$

$$= \sin\left(-\frac{2\pi}{10}\right)$$

$$= -\sin\left(\frac{\pi}{5}\right)$$

$$\cong -0.5878$$

L2 – 4.4 Compound Angle Formulas
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Compound angle: *an angle that is created by adding or subtracting two or more angles.*

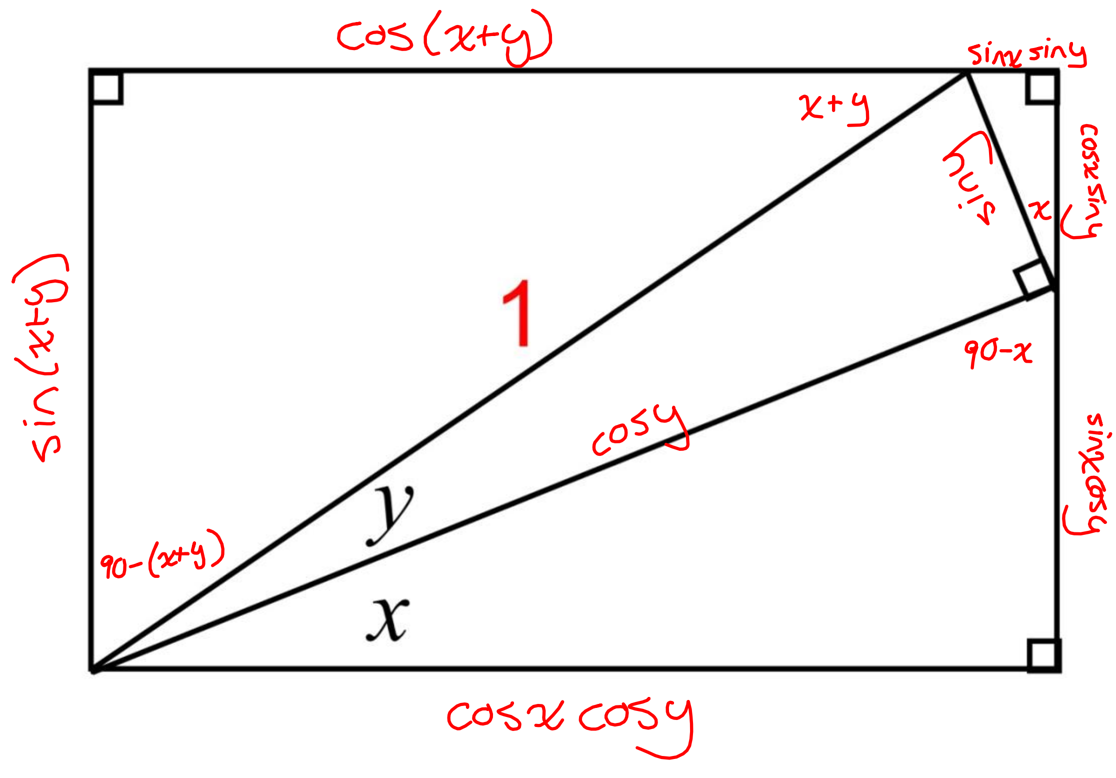
Part 1: Proof of $\cos(x - y)$

Normal algebra rules do not apply:

$$\cos(x - y) \neq \cos x - \cos y$$

So what does $\cos(x - y) =?$

Using the diagram below, label all angles and sides:



$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

Part 2: Proofs of other compound angle formulas

Even/Odd Properties	
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$

Example 1: Prove $\cos(x - y) = \cos x \cos y + \sin x \sin y$

LS	**RS**
$= \cos(x - y)$	$= \cos x \cos y + \sin x \sin y$
$= \cos[x + (-y)]$	
$= \cos x \cos(-y) - \sin x \sin(-y)$	
$= \cos x \cos y - \sin x\,(-\sin y)$	
$= \cos x \cos y + \sin x \sin y$	

LS = RS

Example 2:

a) Prove $\sin(x - y) = \sin x \cos y - \cos x \sin y$

LS	**RS**
$= \sin(x - y)$	$= \sin x \cos y - \cos x \sin y$
$= \sin x \cos(-y) + \cos x \sin(-y)$	
$= \sin x \cos y + \cos x\,(-\sin y)$	
$= \sin x \cos y - \cos x \sin y$	

LS = RS

Compound Angle Formulas

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

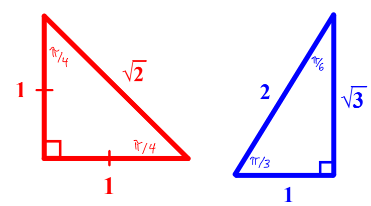
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Part 3: Determine Exact Trig Ratios for Angles other than Special Angles



By expressing an angle as a sum or difference of angles in the special triangles, exact values of other angles can be determined.

Example 3: Use compound angle formulas to determine exact values for

a) $\sin\frac{\pi}{12}$

$$\sin\frac{\pi}{12} = \sin\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right)$$

$$= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

b) $\tan\left(-\frac{5\pi}{12}\right)$

$$\tan\left(-\frac{5\pi}{12}\right) = -\tan\left(\frac{5\pi}{12}\right)$$

$$= -\tan\left(\frac{2\pi}{12} + \frac{3\pi}{12}\right)$$

$$= -\frac{\tan\left(\frac{2\pi}{12}\right) + \tan\left(\frac{3\pi}{12}\right)}{1 - \tan\left(\frac{2\pi}{12}\right)\tan\left(\frac{3\pi}{12}\right)}$$

$$= -\frac{\tan\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{4}\right)}{1 - \tan\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{4}\right)}$$

$$= -\frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}}$$

$$= -\frac{\frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}}{\sqrt{3}} - \frac{1}{\sqrt{3}}}$$

$$= -\frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$$

$$= -\frac{1+\sqrt{3}}{\sqrt{3}-1}$$

Part 4: Use Compound Angle Formulas to Simplify Trig Expressions

Example 4: Simplify the following expression

$$\cos\frac{7\pi}{12}\cos\frac{5\pi}{12}+\sin\frac{7\pi}{12}\sin\frac{5\pi}{12}$$

$$=\cos\left(\frac{7\pi}{12}-\frac{5\pi}{12}\right)$$

$$=\cos\frac{2\pi}{12}$$

$$=\cos\frac{\pi}{6}$$

$$=\frac{\sqrt{3}}{2}$$

Part 5: Application

Example 5: Evaluate $\sin(a+b)$, where a and b are both angles in the second quadrant; given $\sin a=\frac{3}{5}$ and $\sin b=\frac{5}{13}$

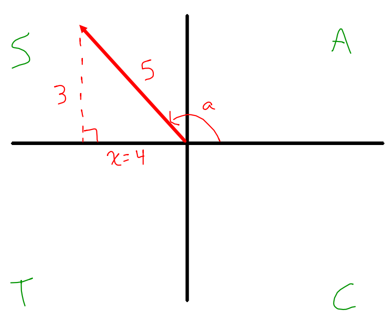
Start by drawing both terminal arms in the second quadrant and solving for the third side.

$$\sin(a+b)=\sin a\cos b+\cos a\sin b$$

$$=\left(\frac{3}{5}\right)\left(-\frac{12}{13}\right)+\left(-\frac{4}{5}\right)\left(\frac{5}{13}\right)$$

$$=-\frac{36}{65}-\frac{20}{65}$$

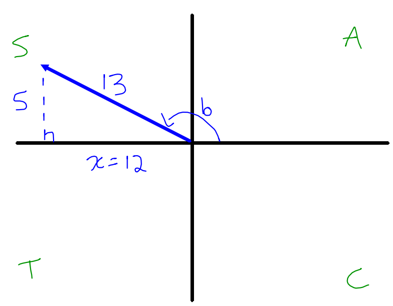
$$=-\frac{56}{65}$$



$x^2+3^2=5^2$

$x^2=16$

$x=4$



$x^2+5^2=13^2$

$x^2=144$

$x=12$

L3 – 4.5 Double Angle Formulas
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Part 1: Proofs of Double Angle Formulas

Example 1: Prove $\sin(2x) = 2\sin x \cos x$

LS	RS
$= \sin(2x)$	$= 2\sin x \cos x$
$= \sin(x + x)$	
$= \sin x \cos x + \cos x \sin x$	
$= 2\sin x \cos x$	

LS = RS

Example 2: Prove $\cos(2x) = \cos^2 x - \sin^2 x$

LS	RS
$= \cos(2x)$	$= \cos^2 x - \sin^2 x$
$= \cos(x + x)$	
$= \cos x \cos x - \sin x \sin x$	
$= \cos^2 x - \sin^2 x$	

LS = RS

Note: *There are alternate versions of* $\cos 2x$ *where either* $cos^2\, x$ *OR* $sin^2\, x$ *are changed using the Pythagorean Identity.*

Double Angle Formulas

$$\sin(2x) = 2\sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$\cos(2x) = 2\cos^2 x - 1$$

$$\cos(2x) = 1 - 2\sin^2 x$$

$$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$$

Part 2: Use Double Angle Formulas to Simplify Expressions

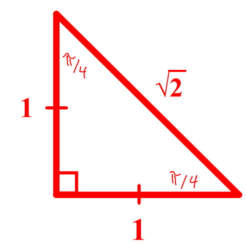
Example 1: Simplify each of the following expressions and then evaluate

a) $2\sin\frac{\pi}{8}\cos\frac{\pi}{8}$

$= \sin\left[2\left(\frac{\pi}{8}\right)\right]$

$= \sin\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}}$

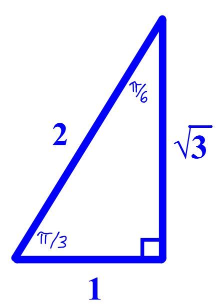


b) $\frac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}}$

$= \tan\left[2\left(\frac{\pi}{6}\right)\right]$

$= \tan\frac{\pi}{3}$

$= \sqrt{3}$



Part 3: Determine the Value of Trig Ratios for a Double Angle

If you know one of the primary trig ratios for any angle, then you can determine the other two. You can then determine the primary trig ratios for this angle doubled.

Example 2: If $\cos\theta = -\frac{2}{3}$ and $\frac{\pi}{2} \le \theta \le \pi$, determine the value of $\cos(2\theta)$ and $\sin(2\theta)$

We can solve for $\cos(2\theta)$ without finding the sine ratio if we use the following version of the double angle formula:

$$\cos(2\theta) = 2\cos^2\theta - 1$$

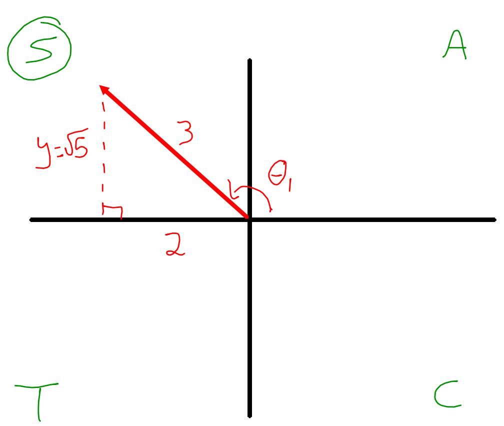
$$\cos(2\theta) = 2\left(-\frac{2}{3}\right)^2 - 1$$

$$\cos(2\theta) = 2\left(\frac{4}{9}\right) - 1$$

$$\cos(2\theta) = \frac{8}{9} - \frac{9}{9}$$

$$\cos(2\theta) = -\frac{1}{9}$$

Now to find $\sin(2\theta)$:



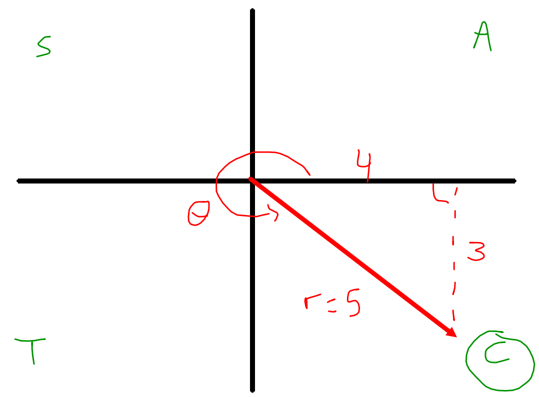
$$2^2 + y^2 = 3^2$$
$$y^2 = 5$$
$$y = \sqrt{5}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$= 2\left(\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right)$$
$$= \frac{-4\sqrt{5}}{9}$$

$\cos(2\theta) = -\frac{1}{9}$ and $\sin(2\theta) = -\frac{4\sqrt{5}}{9}$

Example 3: If $\tan\theta = -\frac{3}{4}$ and $\frac{3\pi}{2} \le \theta \le 2\pi$, determine the value of $\cos(2\theta)$.

We are given that the terminal arm of the angle lies in quadrant **4**:



$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$= \left(\frac{4}{5}\right)^2 - \left(\frac{-3}{5}\right)^2$$
$$= \frac{16}{25} - \frac{9}{25}$$
$$= \frac{7}{25}$$

$$3^2 + 4^2 = r^2$$
$$25 = r^2$$
$$r = 5$$

$$\boxed{\cos(2\theta) = \frac{7}{25}}$$

L4 – 4.5 Prove Trig Identities
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Using your sheet of all identities learned this unit, prove each of the following:

Example 1: Prove $\frac{\sin(2x)}{1+\cos(2x)} = \tan x$

LS	RS
$= \frac{\sin(2x)}{1+\cos(2x)}$	$= \tan x$
$= \frac{\sin(2x)}{1+2\cos^2 x - 1}$	$= \frac{\sin x}{\cos x}$
$= \frac{\cancel{2}\sin x \cancel{\cos x}}{\cancel{2}\cos^{\cancel{2}} x}$	
$= \frac{\sin x}{\cos x}$	

LS=RS

Example 2: Prove $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$

LS	RS
$= \cos(\pi/2 + x)$	$= -\sin x$
$= \cos(\pi/2)\cos x - \sin(\pi/2)\sin x$	
$= 0(\cos x) - 1\sin x$	
$= -\sin x$	

LS=RS

Example 3: Prove $\csc(2x) = \frac{\csc x}{2\cos x}$

LS

$= \csc(2x)$

$= \frac{1}{\sin(2x)}$

$= \frac{1}{2\sin x\cos x}$

RS

$= \frac{\csc x}{2\cos x}$

$= \csc x \cdot \frac{1}{2\cos x}$

$= \frac{1}{\sin x} \cdot \frac{1}{2\cos x}$

$= \frac{1}{2\sin x\cos x}$

LS=RS

Example 4: Prove $\cos x = \frac{1}{\cos x} - \sin x\tan x$

LS

$= \cos x$

RS

$= \frac{1}{\cos x} - \sin x\tan x$

QI

$= \frac{1}{\cos x} - \sin x\left(\frac{\sin x}{\cos x}\right)$

$= \frac{1}{\cos x} - \frac{\sin^2 x}{\cos x}$

PI

$= \frac{\cos^2 x}{\cos x}$

$= \cos x$

LS=RS

Example 5: Prove $\tan(2x) - 2\tan(2x)\sin^2 x = \sin 2x$

LS

$= \tan(2x) - 2\tan(2x)\sin^2 x$

factor

$= \tan(2x)\left[1 - 2\sin^2 x\right]$

DA

$= \tan(2x)\cos(2x)$

QI

$= \dfrac{\sin(2x)}{\cos(2x)} \cdot \cos(2x)$

$= \sin(2x)$

RS

$= \sin(2x)$

LS=RS

Example 6: Prove $\frac{\cos(x-y)}{\cos(x+y)} = \frac{1+\tan x \tan y}{1-\tan x \tan y}$

LS

$= \dfrac{\cos(x-y)}{\cos(x+y)}$

$= \dfrac{\cos x \cos y + \sin x \sin y}{\cos x \cos y - \sin x \sin y}$

RS

$= \dfrac{1+\tan x \tan y}{1-\tan x \tan y}$

$= \dfrac{1+\left(\frac{\sin x}{\cos x}\right)\left(\frac{\sin y}{\cos y}\right)}{1-\left(\frac{\sin x}{\cos x}\right)\left(\frac{\sin y}{\cos y}\right)} \times \dfrac{\cos x \cos y}{\cos x \cos y}$

$= \dfrac{\cos x \cos y + \sin x \sin y}{\cos x \cos y - \sin x \sin y}$

L5 – 5.4 Solve Linear Trigonometric Equations
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In the previous lesson we have been working with identities. Identities are equations that are true for ANY value of x. In this lesson, we will be working with equations that are not identities. We will have to solve for the value(s) that make the equation true.

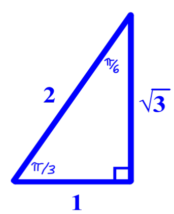
Remember that 2 solutions are possible for an angle between 0 and 2π with a given ratio. Use the reference angle and CAST rule to determine the angles.

When solving a trigonometric equation, consider all 3 tools that can be useful:

1. Special Triangles
2. Graphs of Trig Functions
3. Calculator

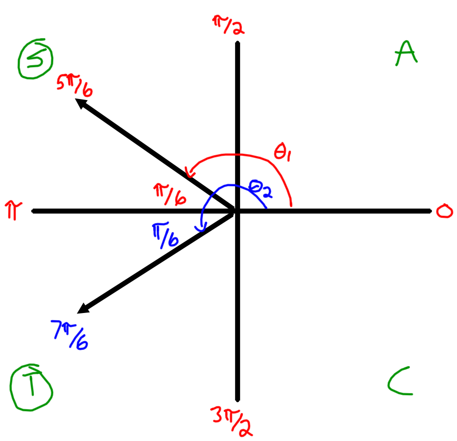
Example 1: Find all solutions for $\cos\theta = -\frac{\sqrt{3}}{2}$ in the interval $0 \le x \le 2\pi$

$-\frac{\sqrt{3}}{2}$ → Graph NO → $\triangle$ Yes



$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Put ~~reference~~ angle into quadrants 2 and 3 to make the ratio negative



$\theta_1 = \pi - \pi/6$
$= 5\pi/6$

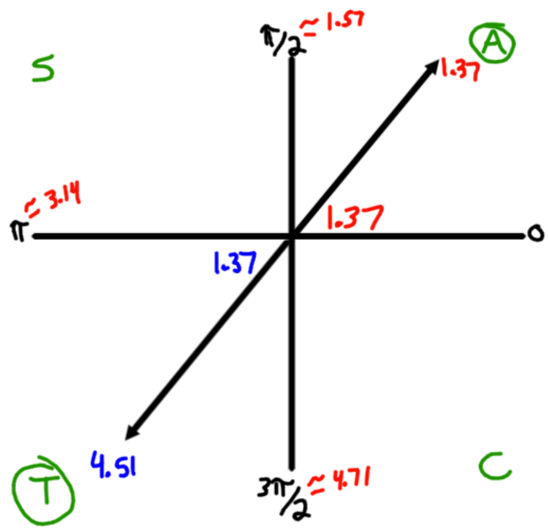
$\theta_2 = \pi + \pi/6$
$= 7\pi/6$

$$\boxed{\cos\frac{5\pi}{6} = \cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}}$$

Example 2: Find all solutions for $\tan\theta = 5$ in the interval $0 \le x \le 2\pi$

$\tan\theta = 5 \rightarrow$ Graph NO $\rightarrow \Delta$ NO $\rightarrow$ CALCULATOR

NOTE: Tan is $\oplus$ in Q1 and Q3



$\tan\theta = 5$

$\theta_1 = \tan^{-1}(5)$

$\theta_1 \approx 1.37$ radians

$\theta_2 = \pi + 1.37$

$\theta_2 \approx 4.51$ radians

$$\boxed{\tan 1.37 = \tan 4.51 = 5}$$

Example 3: Find all solutions for $2\sin x + 1 = 0$ in the interval $0 \le x \le 2\pi$

$2\sin x + 1 = 0$

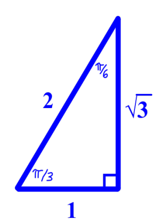
$2\sin x = -1$

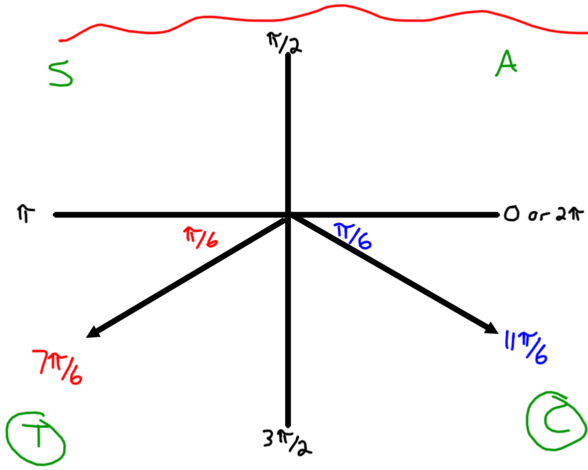
$\sin x = -1/2$

Graph NO $\rightarrow \Delta$ Yes

$\sin \pi/6 = 1/2$

use as reference angle in Q3 + Q4 to make ratio $\ominus$





$\theta_1 = \pi + \pi/6$

$= 7\pi/6$

$\theta_2 = 2\pi - \pi/6$

$= 11\pi/6$

$$\boxed{\sin\frac{7\pi}{6} = \sin\frac{11\pi}{6} = -\frac{1}{2}}$$

Example 4: Solve $3(\tan x + 1) = 2$, where $0 \le x \le 2\pi$

$3(\tan x + 1) = 2$

$\tan x + 1 = \frac{2}{3}$

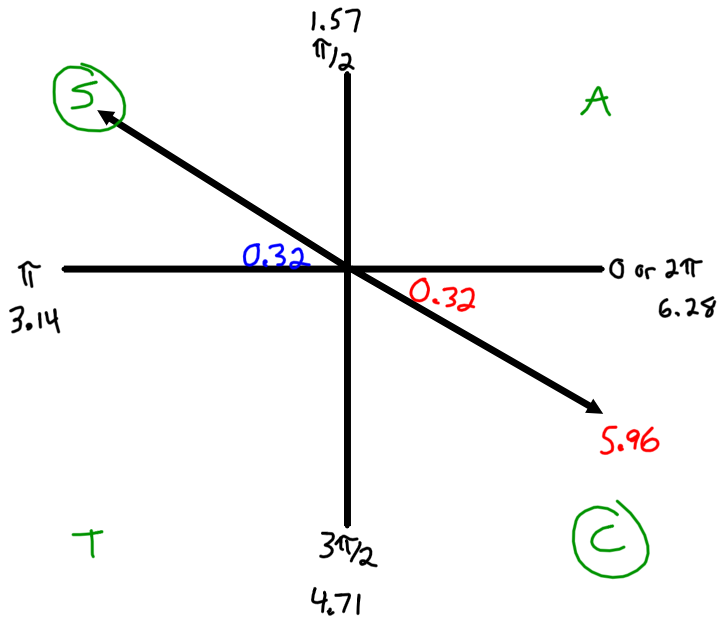
$\tan x = \frac{2}{3} - 1$

$\tan x = -\frac{1}{3}$

Graph → Δ → CALCULATOR

NO   NO

NOTE: tan is ⊖ in Q2 + Q4



$\tan x = \frac{-1}{3}$

$x_1 = \tan^{-1}\left(\frac{-1}{3}\right)$

$x_1 = -0.32175$

$x_1 = 2\pi - 0.32175$

$x_1 \simeq 5.96$ radians

$x_2 = \pi - 0.32$

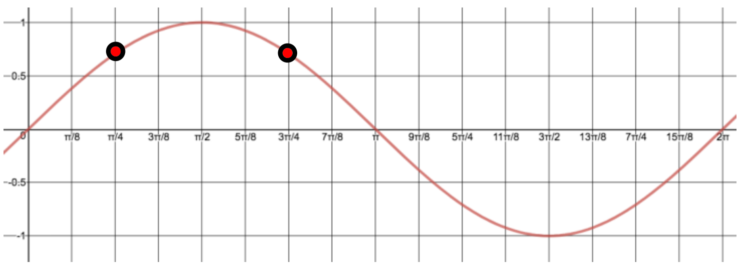
$x_2 \simeq 2.82$ radians

$$\boxed{\tan 5.96 = \tan 2.82 = -\frac{1}{3}}$$

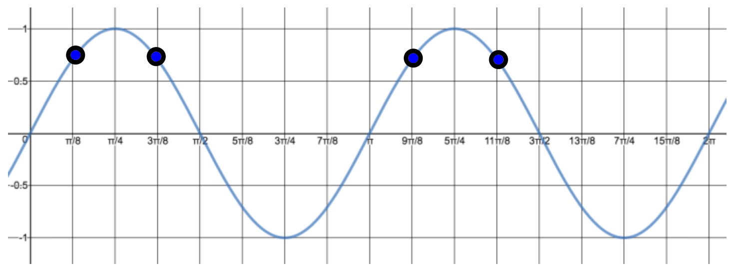
L6 – 5.4 Solve Double Angle Trigonometric Equations
MHF4U
Jensen

Part 1: Investigation

$\mathbf{y = \sin x}$



$\mathbf{y = \sin(2x)}$



a) What is the period of both of the functions above? How many cycles between 0 and 2π radians?

For $y = \sin x$ → $period = 2\pi$

For $y = \sin(2x)$ → $period = \frac{2\pi}{2} = \pi$

b) Looking at the graph of $y = \sin x$, how many solutions are there for $\sin x = \frac{1}{\sqrt{2}} \approx 0.71$?

2 solutions

$$\sin\frac{\pi}{4} = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$$

c) Looking at the graph of $y = \sin(2x)$, how many solutions are there for $\sin(2x) = \frac{1}{\sqrt{2}} \approx 0.71$?

4 solutions

$$\sin\left[2\left(\frac{\pi}{8}\right)\right] = \sin\left[2\left(\frac{3\pi}{8}\right)\right] = \sin\left[2\left(\frac{9\pi}{8}\right)\right] = \sin\left[2\left(\frac{11\pi}{8}\right)\right] = \frac{1}{\sqrt{2}}$$

d) When the period of a function is cut in half, what does that do to the number of solutions between 0 and 2π radians?

Doubles the number of solutions

Part 2: Solve Linear Trigonometric Equations that Involve Double Angles

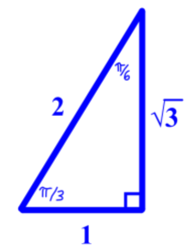
Example 1: $\sin(2\theta) = \frac{\sqrt{3}}{2}$ where $0 \le \theta \le 2\pi$

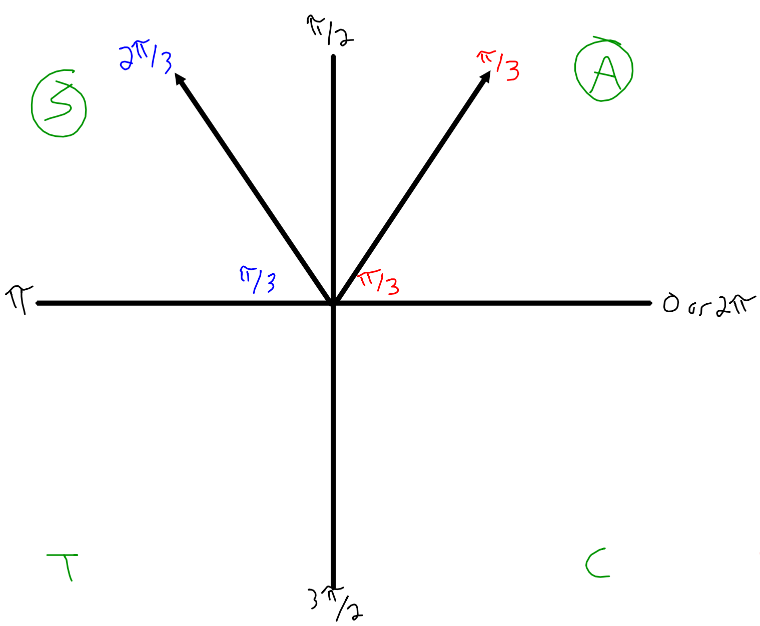
Let $2\theta = x$

$\sin x = \frac{\sqrt{3}}{2}$ → Graph → NO; △ → yes

$\sin \pi/3 = \frac{\sqrt{3}}{2}$

Put reference angle in Q1 + Q2 where sine is (+)





$x_1 = \pi/3$

$x_2 = \pi - \pi/3$

$x_2 = 2\pi/3$

$2\theta = x$

$2\theta = \pi/3$ $\qquad$ $2\theta = 2\pi/3$

$\theta_1 = \pi/6$ $\qquad$ $\theta_2 = 2\pi/6$

$\theta_2 = \pi/3$

$y = \sin(2\theta)$ has a period of π; add π to θ_1 and θ_2 to find other angles $0 \le \theta \le 2\pi$ that have equivalent ratios

$\theta_3 = \theta_1 + \pi$ $\qquad$ $\theta_4 = \theta_2 + \pi$

$= \pi/6 + \pi$ $\qquad$ $\theta_4 = \pi/3 + \pi$

$= \frac{7\pi}{6}$ $\qquad$ $\theta_4 = \frac{4\pi}{3}$

$$\boxed{\sin\left[2\left(\frac{\pi}{6}\right)\right] = \sin\left[2\left(\frac{\pi}{3}\right)\right] = \sin\left[2\left(\frac{7\pi}{6}\right)\right] = \sin\left[2\left(\frac{4\pi}{3}\right)\right] = \frac{\sqrt{3}}{2}}$$

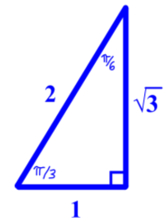
Example 2: $\cos(2\theta) = -\frac{1}{2}$ where $0 \le \theta \le 2\pi$

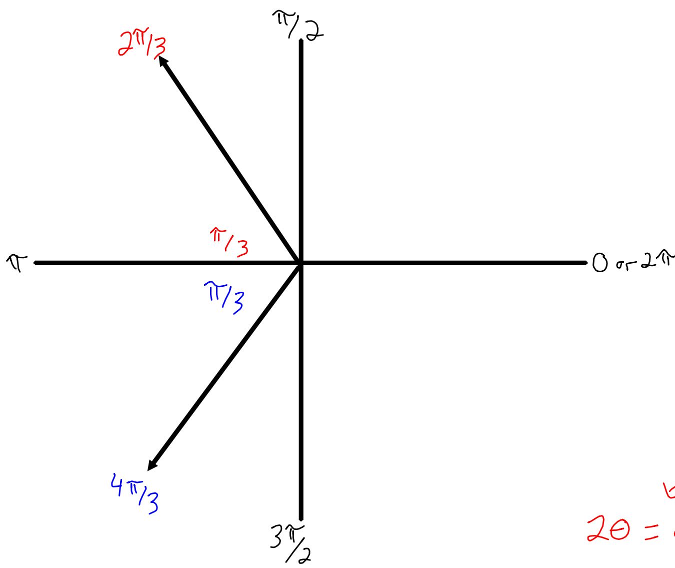
Let $2\theta = x$

$\cos x = -\frac{1}{2}$ → Graph NO → Δ yes

$\cos \frac{\pi}{3} = \frac{1}{2}$

Put reference angle in Q2 + Q3 where cosine is ⊖





$x_1 = \pi - \frac{\pi}{3}$

$x_1 = \frac{2\pi}{3}$

$x_2 = \pi + \frac{\pi}{3}$

$x_2 = \frac{4\pi}{3}$

$2\theta = x$

$2\theta = \frac{2\pi}{3}$

$\theta_1 = \frac{2\pi}{6}$

$\theta_1 = \frac{\pi}{3}$

$2\theta = \frac{4\pi}{3}$

$\theta_2 = \frac{4\pi}{6}$

$\theta_2 = \frac{2\pi}{3}$

Remember that $\cos(2\theta)$ has a period of π; add π to θ_1 and θ_2 to find other solutions $0 \le \theta \le 2\pi$

$\theta_3 = \theta_1 + \pi$

$\theta_3 = \frac{\pi}{3} + \pi$

$\theta_3 = \frac{4\pi}{3}$

$\theta_4 = \theta_2 + \pi$

$\theta_4 = \frac{2\pi}{3} + \pi$

$\theta_4 = \frac{5\pi}{3}$

$$\boxed{\cos\left[2\left(\frac{\pi}{3}\right)\right] = \cos\left[2\left(\frac{2\pi}{3}\right)\right] = \cos\left[2\left(\frac{4\pi}{3}\right)\right] = \cos\left[2\left(\frac{5\pi}{3}\right)\right] = -\frac{1}{2}}$$

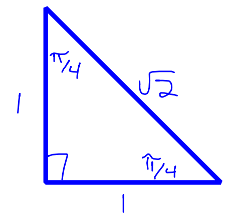
Example 3: $\tan(2\theta) = 1$ where $0 \le \theta \le 2\pi$

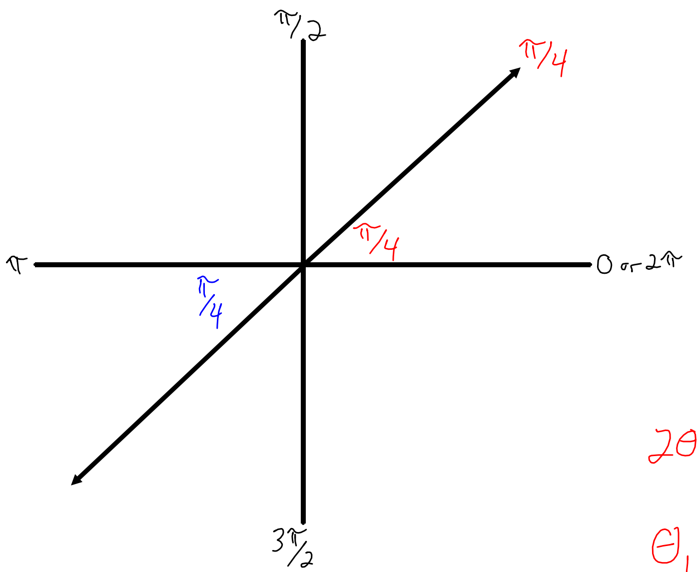
Let $2\theta = x$

$\tan x = 1$ → Graph → Δ
NO yes

$\tan\frac{\pi}{4} = 1$

Put reference angle in Q1 + Q3 where tangent is (+)





$x_1 = \frac{\pi}{4}$

$x_2 = \pi + \frac{\pi}{4}$

$x_2 = \frac{5\pi}{4}$

$2\theta = x$

$2\theta = \frac{\pi}{4}$ $\qquad$ $2\theta = \frac{5\pi}{4}$

$\theta_1 = \frac{\pi}{8}$ $\qquad$ $\theta_2 = \frac{5\pi}{8}$

Remember that $\tan(2\theta)$ has a period of $\frac{\pi}{2}$; add $\frac{\pi}{2}$ to θ_1 and θ_2 to find other solutions $0 \le \theta \le 2\pi$

$$\theta_3 = \theta_2 + \frac{\pi}{2} = \frac{5\pi}{8} + \frac{4\pi}{8} = \frac{9\pi}{8}$$

$$\theta_4 = \theta_3 + \frac{\pi}{2} = \frac{9\pi}{8} + \frac{4\pi}{8} = \frac{13\pi}{8}$$

$$\boxed{\tan\left[2\left(\frac{\pi}{8}\right)\right] + \tan\left[2\left(\frac{5\pi}{8}\right)\right] + \tan\left[2\left(\frac{9\pi}{8}\right)\right] + \tan\left[2\left(\frac{13\pi}{8}\right)\right] = 1}$$

L7 – 5.4 Solve Quadratic Trigonometric Equations
MHF4U
Jensen

A quadratic trigonometric equation may have multiple solutions in the interval $0 \le x \le 2\pi$.

You can often **factor** a quadratic trigonometric equation and then solve the resulting two linear trigonometric equations. In cases where the equation cannot be factored, use the **quadratic formula** and then solve the resulting linear trigonometric equations.

You may need to use a Pythagorean identity, compound angle formula, or double angle formula to create a quadratic equation that contains only a single trigonometric function whose arguments all match.

Remember that when solving a linear trigonometric equation, consider all 3 tools that can be useful:

1. Special Triangles
2. Graphs of Trig Functions
3. Calculator

Part 1: Solving Quadratic Trigonometric Equations

Example 1: Solve each of the following equations for $0 \le x \le 2\pi$

a) $(\sin x + 1)\left(\sin x - \frac{1}{2}\right) = 0$

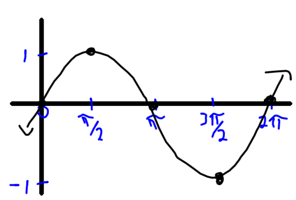
$(\sin x+1)(\sin x - \frac{1}{2}) = 0$

* set both factors equal to zero and solve *

$\sin x + 1 = 0$

$\sin x = -1$ → Graph Yes!

$\boxed{x_1 = \frac{3\pi}{2}}$

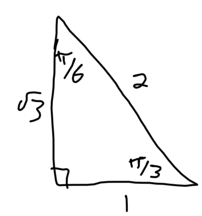


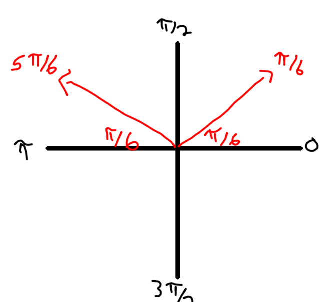
$\sin x - \frac{1}{2} = 0$

$\sin x = \frac{1}{2}$ → Graph No → Δ Yes

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

Put in Q1 + Q2





$\boxed{x_2 = \pi/6}$

$x_3 = \pi - \frac{\pi}{6}$

$\boxed{x_3 = \frac{5\pi}{6}}$

$\boxed{\text{Solutions are } x = \frac{3\pi}{2}, \frac{\pi}{6}, \text{ or } \frac{5\pi}{6} \text{ radians}}$

b) $\sin^2 x - \sin x = 2$

$\sin^2 x - \sin x = 2$

$\sin^2 x - \sin x - 2 = 0$

Let $\sin x = x$

$x^2 - x - 2 = 0$ P: -2 S: -1 -2 and 1

$(x-2)(x+1) = 0$

$(\sin x - 2)(\sin x + 1) = 0$

$\sin x - 2 = 0$

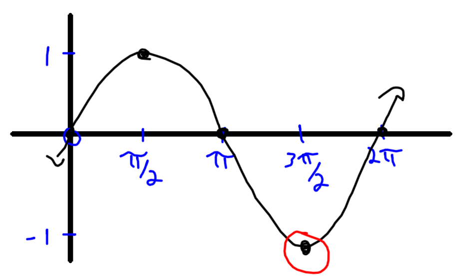
$\sin x = 2$

No solutions

$\sin x + 1 = 0$

$\sin x = -1$ → Graph Yes

$\boxed{x = \frac{3\pi}{2}}$



$\boxed{\text{The only solution is } x = \frac{3\pi}{2}}$

c) $2\sin^2 x - 3\sin x + 1 = 0$

$2\sin^2 x - 3\sin x + 1 = 0$

Let $x = \sin x$

$2x^2 - 3x + 1 = 0$ P: 2 S: -3 -2 and -1

$2x^2 - 2x - 1x + 1 = 0$

$(2x^2 - 2x) + (-1x + 1) = 0$

$2x(x-1) - 1(x-1) = 0$

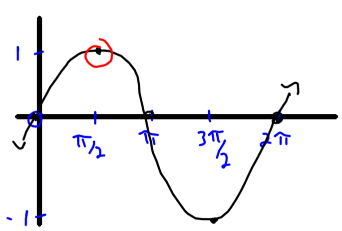
$(x-1)(2x-1) = 0$

$(\sin x - 1)(2\sin x - 1) = 0$

$\sin x - 1 = 0$

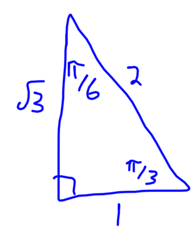
$\sin x = 1$ ← Graph

$\boxed{x_1 = \frac{\pi}{2}}$



$2\sin x - 1 = 0$

$\sin x = \frac{1}{2}$ →



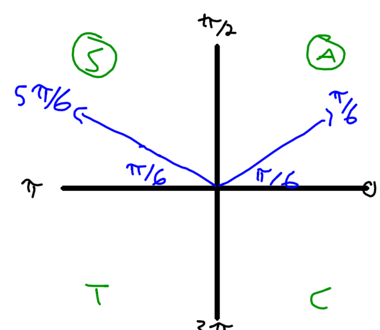
$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

Put in Q1 + Q2

$x_2 = \frac{\pi}{6}$

$x_3 = \pi - \frac{\pi}{6}$

$\boxed{x_3 = \frac{5\pi}{6}}$



$\boxed{\text{The solutions are } x = \frac{\pi}{2}, \frac{\pi}{6}, \text{ or } \frac{5\pi}{6}}$

Part 2: Use Identities to Help Solve Quadratic Trigonometric Equations

Example 2: Solve each of the following equations for $0 \leq x \leq 2\pi$

a) $2\sec^2 x - 3 + \tan x = 0$

$2\sec^2 x - 3 + \tan x = 0$

P.I. $\sec^2 x = \tan^2 x + 1$

$2(\tan^2 x + 1) - 3 + \tan x = 0$

$2\tan^2 x + 2 - 3 + \tan x = 0$

$2\tan^2 x + \tan x - 1 = 0$

Let $x = \tan x$

$2x^2 + x - 1 = 0$ P: -2 S: 1 2 and -1

$2x^2 + 2x - 1x - 1 = 0$

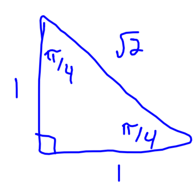
$2x(x+1) - 1(x+1) = 0$

$(x+1)(2x-1) = 0$

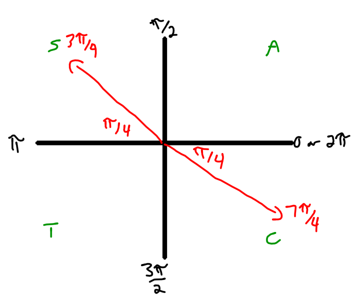
$(\tan x + 1)(2\tan x - 1) = 0$

$\tan x + 1 = 0$

$\tan x = -1$ ← Δ



Put $\frac{\pi}{4}$ in Q2 + Q4



$x_1 = \pi - \frac{\pi}{4}$

$\boxed{x_1 = \frac{3\pi}{4}}$

$x_2 = 2\pi - \frac{\pi}{4}$

$\boxed{x_2 = \frac{7\pi}{4}}$

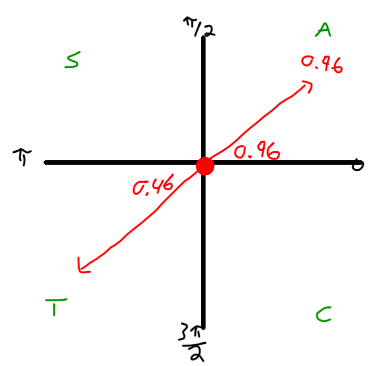
$2\tan x - 1 = 0$

$\tan x = \frac{1}{2}$

$x = \tan^{-1}\left(\frac{1}{2}\right)$

$x \simeq 0.46$ radians

Put in Q1 + Q3



$\boxed{x_3 = 0.46}$

$x_4 = \pi + 0.46$

$\boxed{x_4 = 3.60}$

The solutions are $x = \frac{3\pi}{4}, \frac{7\pi}{4}, 0.46$, OR 3.60 RADIANS

b) $3\sin x + 3\cos(2x) = 2$

$$3\sin x + 3\cos(2x) = 2$$
$$3\sin x + 3(1 - 2\sin^2 x) = 2$$
$$3\sin x + 3 - 6\sin^2 x - 2 = 0$$
$$-6\sin^2 x + 3\sin x + 1 = 0$$

Let $x = \sin x$

$$-6x^2 + 3x + 1 = 0$$

P: -6
S: 3

NOT FACTORABLE
USE Q.F.

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(-6)(1)}}{2(-6)}$$

$$x = \frac{-3 \pm \sqrt{33}}{-12}$$

$x \doteq -0.23$

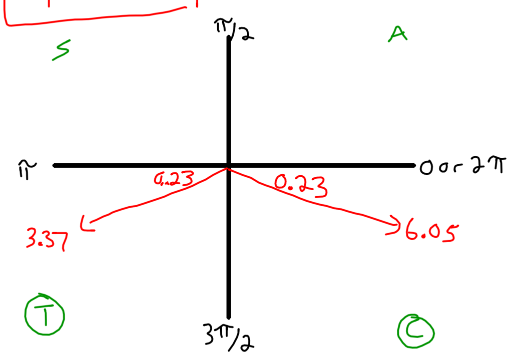
$\sin x = -0.23$ ← CALC

$x_1 = \sin^{-1}(-0.23)$

$x_1 = -0.23$

$x_1 = 2\pi - 0.23$

$\boxed{x_1 = 6.05}$



$x_2 = \pi + 0.23$

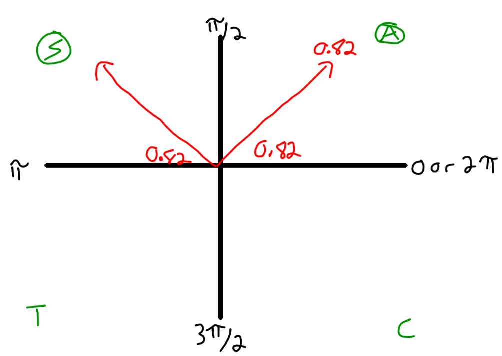
$\boxed{x_2 = 3.37}$

$x \doteq 0.73$

$\sin x = 0.73$ ← CALC

$x = \sin^{-1}(0.73)$

$\boxed{x_1 = 0.82}$



$x_4 = \pi - 0.82$

$\boxed{x_4 = 2.32}$

The solutions are $x = 0.82, 2.32, 3.37$, OR 6.05 RADIANS

L8 – 5.4 Applications of Trigonometric Equations
MHF4U
Jensen

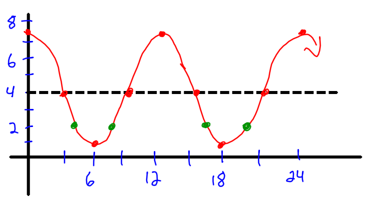
Part 1: Application Questions

Example 1: Today, the high tide in Matthews Cove, New Brunswick, is at midnight. The water level at high tide is 7.5 m. The depth, *d* meters, of the water in the cove at time *t* hours is modelled by the equation

$$d(t) = 3.5\cos\left(\frac{\pi}{6}t\right) + 4$$

Jenny is planning a day trip to the cove tomorrow, but the water needs to be at least 2 m deep for her to maneuver her sailboat safely. Determine the best time when it will be safe for her to sail into Matthews Cove?

$d(t) = 3.5\cos\left(\frac{\pi}{6}t\right) + 4$



amp = 3.5

period = $\frac{2\pi}{\left(\frac{\pi}{6}\right)}$ = 12 hours

max = 4+3.5 = 7.5 m

min = 4−3.5 = 0.5 m

set d(t) = 2 and solve for t

$2 = 3.5\cos\left(\frac{\pi}{6}t\right) + 4$

$\frac{-2}{3.5} = \cos\left(\frac{\pi}{6}t\right)$

let $x = \frac{\pi}{6}t$

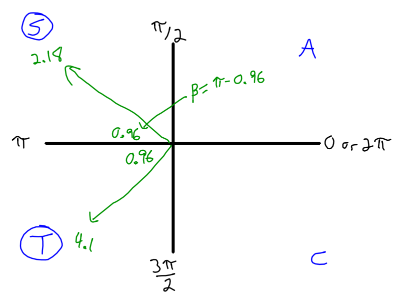
$\cos x = \frac{-2}{3.5}$ ← CALCULATOR solutions in Q2 + Q3

$x_1 = \cos^{-1}\left(\frac{-2}{3.5}\right)$

$\boxed{x_1 = 2.18 \text{ radians}}$

$x_2 = \pi + 0.96$

$\boxed{x_2 = 4.1 \text{ radians}}$



$x = \frac{\pi}{6}t$

$2.18 = \frac{\pi}{6}t$

$\frac{6(2.18)}{\pi} = t$

$t_1 = 4.16$ hours

$t_3 = t_1 + 12$

$t_3 = 16.16$ hours

$4.1 = \frac{\pi}{6}t$

$\frac{6(4.1)}{\pi} = t$

$t_2 = 7.83$ hours

$t_4 = t_2 + 12$

$t_4 = 19.83$ hours

* Add period = 12 to find other solutions

Longest interval above 2 m is between 7.83 hours and 16.16 hours.

∴ she can safely sail between 7:50 am and 4:10 pm

Example 2: A city's daily temperature, in degrees Celsius, can be modelled by the function

$$t(d) = -28\cos\left(\frac{2\pi}{365}d\right) + 10$$

where d is the day of the year and 1 = January 1. On days where the temperature is approximately 32°C or above, the air conditioners at city hall are turned on. During what days of the year are the air conditioners running at city hall?

$$t(d) = -28\cos\left(\frac{2\pi}{365}d\right) + 10$$

$$32 = -28\cos\left(\frac{2\pi}{365}d\right) + 10$$

$$-\frac{22}{28} = \cos\left(\frac{2\pi}{365}d\right) \qquad * \text{ let } x = \frac{2\pi}{365}d$$

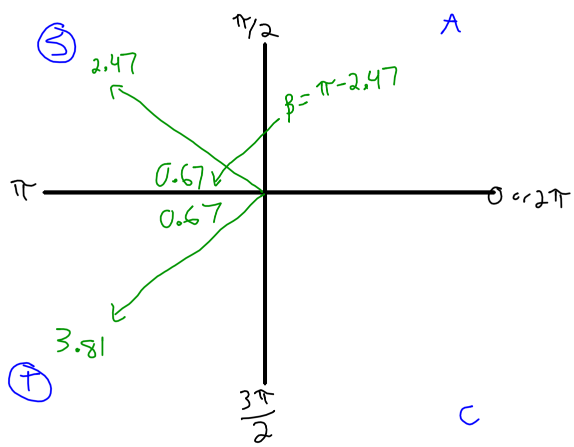
$$\cos x = -\frac{22}{28} \rightarrow$$ CALCULATOR solutions in Q2 + Q3

$$x_1 = \cos^{-1}\left(-\frac{22}{28}\right)$$

$$\boxed{x_1 = 2.47 \text{ radians}}$$

$$x_2 = \pi + 0.67$$

$$\boxed{x_2 = 3.81 \text{ radians}}$$



$$x = \frac{2\pi}{365}d$$

$$2.47 = \frac{2\pi}{365}d \qquad 3.81 = \frac{2\pi}{365}d$$

$$\frac{365(2.47)}{2\pi} = d \qquad \frac{365(3.81)}{2\pi} = d$$

$$d_1 \doteq 143 \qquad d_2 \doteq 221$$

∴ They will use the air conditioning between day 143 to day 221.

Example 3: A Ferris wheel with a 20 meter diameter turns once every minute. Riders must climb up 1 meter to get on the ride.

a) Write a cosine equation to model the height of the rider, h meters, t seconds after the ride has begun. Assume they start at the min height.

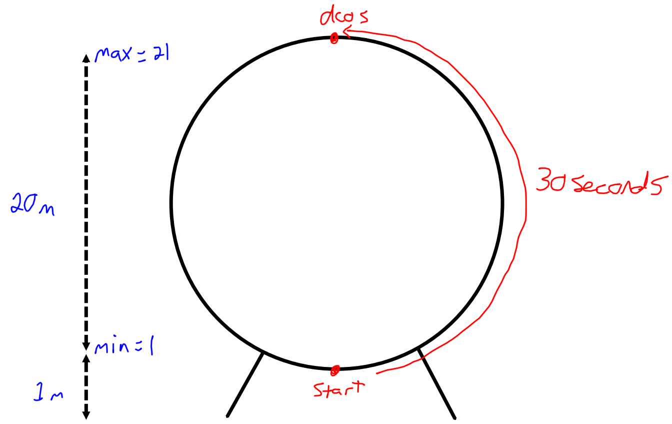
$$a = \frac{max - min}{2} = \frac{21 - 1}{2} = 10$$

$$k = \frac{2\pi}{period} = \frac{2\pi}{60} = \frac{\pi}{30}$$

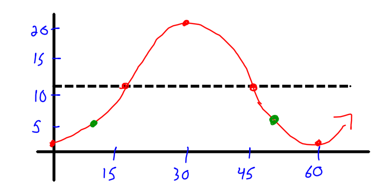
$$c = max - a = 21 - 10 = 11$$

$$d_{cos} = 30$$

$$h(t) = 10\cos\left[\frac{\pi}{30}(t - 30)\right] + 11$$



b) What will be the first 2 times that the rider is at a height of 5 meters?



Set $h(t) = 5$ and solve for t

$$5 = 10\cos\left[\frac{\pi}{30}(t-30)\right] + 11$$

$$\frac{-6}{10} = \cos\left[\frac{\pi}{30}(t-30)\right]$$

Let $x = \frac{\pi}{30}(t-30)$

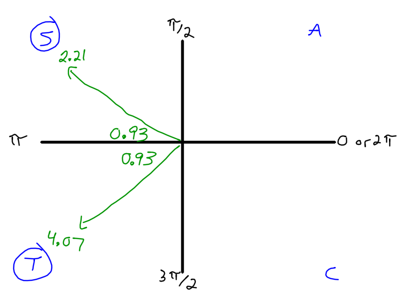
$\cos x = \frac{-6}{10}$ → CALCULATOR solutions in Q2 + Q3

$x_1 = \cos^{-1}\left(\frac{-6}{10}\right)$

$\boxed{x_1 = 2.21 \text{ radians}}$

$x_2 = \pi + 0.93$

$\boxed{x_2 = 4.07 \text{ radians}}$



$x = \frac{\pi}{30}(t-30)$

$2.21 = \frac{\pi}{30}(t-30)$

$\frac{30(2.21)}{\pi} + 30 = t$

$\boxed{t_1 = 51.1 \text{ seconds}}$

$4.07 = \frac{\pi}{30}(t-30)$

$\frac{30(4.07)}{\pi} + 30 = t$

$t_2 = 68.87$ seconds

subtract period = 60 to find an earlier solution

$t_3 = t_2 - 60$

$\boxed{t_3 = 8.87 \text{ seconds}}$

∴ The first two times the rider is at a height of 5m are 8.87 seconds and 51.1 seconds