Unit 6

Rates of Change

Workbook

MHF4U

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

W1 – 1.5 Average Rates of Change
MHF4U
Jensen

SOLUTIONS

1) Calculate the average rate of change for the function $g(x) = 4x^2 - 5x + 1$ over each interval.

a) $2 \leq x \leq 4$

$m = \frac{g(4) - g(2)}{4 - 2}$

$= \frac{45 - 7}{2}$

$= 19$

b) $2 \leq x \leq 3$

$m = \frac{g(3) - g(2)}{3 - 2}$

$= \frac{22 - 7}{1}$

$= 15$

c) $2 \leq x \leq 2.5$

$m = \frac{g(2.5) - g(2)}{2.5 - 2}$

$= \frac{13.5 - 7}{0.5}$

$= 13$

d) $2 \leq x \leq 2.25$

$m = \frac{g(2.25) - g(2)}{2.25 - 2}$

$= \frac{10 - 7}{0.25}$

$= 12$

e) $2 \leq x \leq 2.1$

$m = \frac{g(2.1) - g(2)}{2.1 - 2}$

$= \frac{8.14 - 7}{0.1}$

$= 11.4$

f) $2 \leq x \leq 2.01$

$m = \frac{g(2.01) - g(2)}{2.01 - 2}$

$= \frac{7.1104 - 7}{0.01}$

$= 11.04$

2) An emergency flare is shot into the air. Its height, in meters, above the ground at various times in its flight is given in the following table:

Time (s)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Height (m)	2.00	15.75	27.00	35.75	42.00	45.75	47.00	45.75	42.00

Determine the average rate of change in the height of the flare during each interval

a) $1.0 \leq t \leq 2.0$

$m = \frac{h(2) - h(1)}{2 - 1}$

$m = \frac{42 - 27}{1}$

$m = 15$ m/s

b) $3.0 \leq t \leq 4.0$

$m = \frac{h(4) - h(3)}{4 - 3}$

$= \frac{42 - 47}{1}$

$= -5$ m/s

3) What is the average rate of change in the values of the function $f(x) = 4x$ from $x = 2$ to $x = 6$? What about from $x = 2$ to $x = 26$? What do your results indicate about $f(x)$?

$2 \le x \le 6$

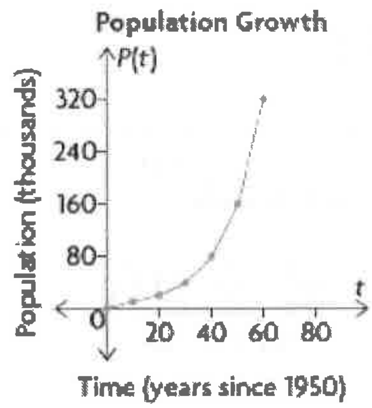
$$m = \frac{f(6) - f(2)}{6 - 2}$$
$$= \frac{24 - 8}{4}$$
$$= 4$$

$2 \le x \le 26$

$$m = \frac{f(26) - f(2)}{26 - 2}$$
$$= \frac{104 - 8}{24}$$
$$= 4$$

Average rate of change will always be 4 because it is a linear function with a slope of 4.

4) The population of a city has continued to grow since 1950. The population, P, in thousands, and the time t, in years, since 1950 are given in the table below and in the graph.



Time (years)	0	10	20	30	40	50	60
Population (thousands)	5	10	20	40	80	160	320

a) Calculate the average rate of change in the population for the following intervals of time.

i) $0 \le t \le 20$

$$m = \frac{P(20) - P(0)}{20 - 0}$$
$$= \frac{20000 - 5000}{20}$$
$$= 750 \text{ ppl/year}$$

ii) $20 \le t \le 40$

$$m = \frac{P(40) - P(20)}{40 - 20}$$
$$= \frac{80000 - 20000}{20}$$
$$= 3000 \text{ ppl/year}$$

iii) $40 \le t \le 60$

$$m = \frac{P(60) - P(40)}{60 - 40}$$
$$= \frac{320000 - 80000}{20}$$
$$= 12000 \text{ ppl/year}$$

iv) $0 \le t \le 60$

$$m = \frac{P(60) - P(0)}{60 - 0}$$
$$= \frac{320000 - 5000}{60}$$
$$= 5250 \text{ ppl/year}$$

b) Is the population growth constant?

NO.

5) A company that sells sweatshirts finds that the profit can be modelled by $P(s) = -0.30s^2 + 3.5s + 11.15$, where $P(s)$ is the profit, in thousands of dollars, and s is the number of sweatshirts sold (expressed in thousands).

Calculate the average rate of change in the profit for the following intervals.

i) $1 \leq s \leq 2$

$$m = \frac{P(2) - P(1)}{2-1}$$

$$= \frac{16.95 - 14.35}{1}$$

$= \$2.6$ / sweatshirt

ii) $2 \leq s \leq 3$

$$m = \frac{P(3) - P(2)}{3-2}$$

$$= \frac{18.95 - 16.95}{1}$$

$= \$2$ / sweatshirt

iii) $3 \leq s \leq 4$

$$m = \frac{P(4) - P(3)}{4-3}$$

$$= \frac{20.35 - 18.95}{1}$$

$= \$1.40$/ sweatshirt

iv) $4 \leq s \leq 5$

$$m = \frac{P(5) - P(4)}{5-4}$$

$$= \frac{21.15 - 20.35}{1}$$

$= \$0.80$/ sweatshirt

b) As the number of sweatshirts sold increases, what do you notice about the average rate of change in profit on each sweatshirt? What does this mean?

Rate of change is positive but decreasing. Profits are going up but at a decreasing rate.

c) Predict if the rate of change in profit will stay positive. Explain what this means.

$$x\text{-vertex} = \frac{-b}{2a} = \frac{-3.5}{2(-0.3)} \doteq 5.83$$

~~&~~ at around 6000 sweatshirts sold, profits will start to decrease.

Answer Key

1)a) 19 **b)** 15 **c)** 13 **d)** 12 **e)** 11.4 **f)** 11.04
2)a)i) 15 m/s **ii)** -5 m/s
3) 4; 4; the average rate of change is always 4 because the function is linear, with a slope of 4.
4)a)i) 750 ppl/year **ii)** 3000 ppl/year **iii)** 12 000 ppl/year **iv)** 5250 ppl/year
b) no, the rate of growth increases as the time increases.
)i) \$2.60/sweatshirt **ii)** \$2.00/sweatshirt **iii)** \$1.40/sweatshirt **iv)** \$0.80/sweatshirt
, The rate of change is still positive, but it is decreasing. This means that the profit is still increasing, but at a decreasing rate.
c) No; after 6000 sweatshirts are sold, the rate of change becomes negative. This means that the profit begins to decrease after 6000 sweatshirts are sold.

W2 – 1.6 Instantaneous Rates of Change

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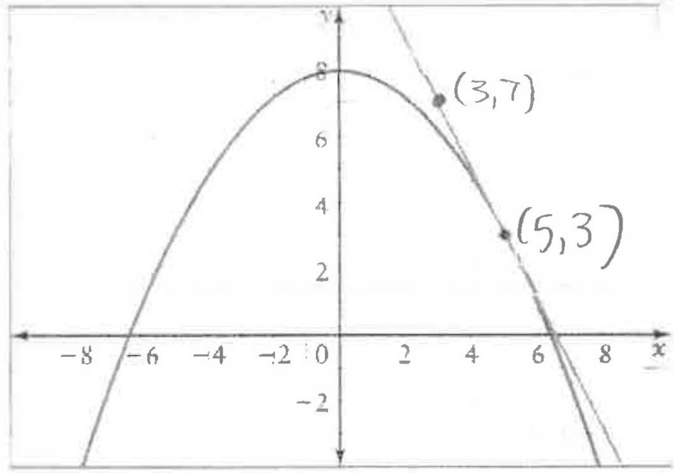
Solutions

1) Consider the graph shown.



a) State the coordinates of the tangent point

$(5,3)$

b) State the coordinates of another point on the tangent line

$(3,7)$

c) Use the points you found to find the slope of the tangent line

$m = \frac{3-7}{5-3} = \frac{-4}{2} = -2$

d) What does the slope of the tangent line represent?

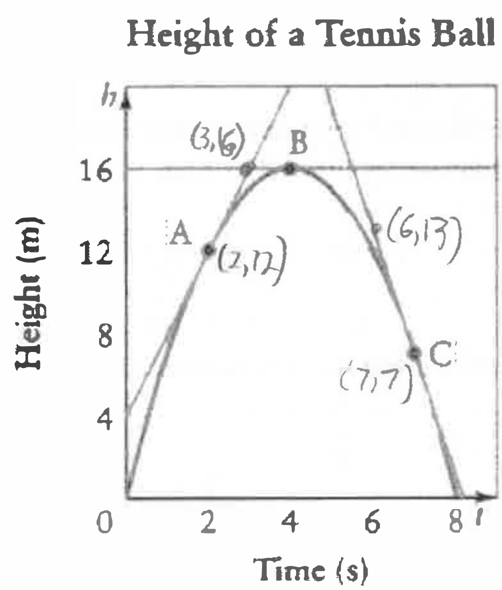
instantaneous rate of change at $x = 5$

2)a) At each of the indicated points on the graph, is the instantaneous rate of change positive, negative, or zero?

A: positive

B: zero

C: Negative



b) Estimate the instantaneous rate of change at points A and C.

A

$\left.\frac{dH}{dt}\right|_{t=2} \approx \frac{16-12}{3-2} = 4\,m/s$

C

$\left.\frac{dH}{dt}\right|_{t=7} \simeq \frac{13-7}{6-7} = -6\,m/s$

c) Interpret the values in part b) for the situation represented by the graph.

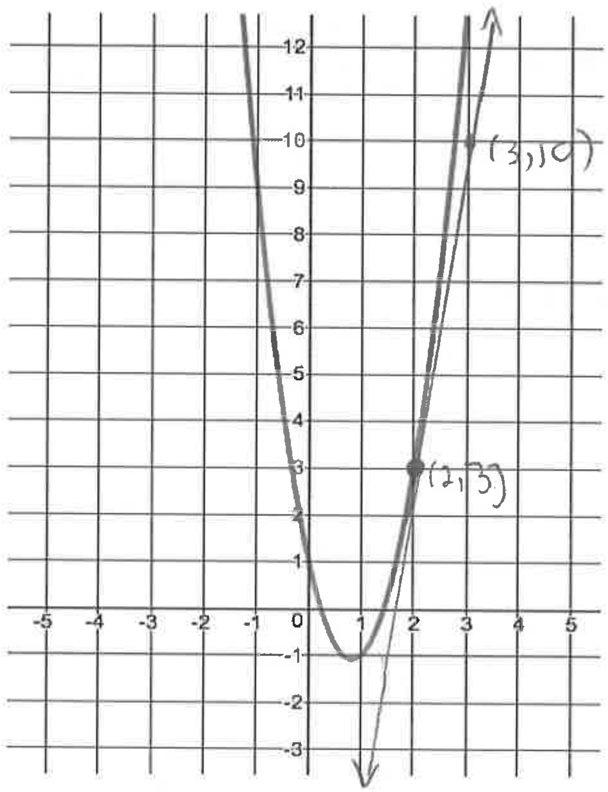
change in distance with respect to time gives a velocity.

3) Use the graph of each function to estimate the instantaneous rate of change at $x = 2$ by drawing a tangent line and calculating it's slope.

$3x^2 - 5x + 1$

$m = \frac{\Delta y}{\Delta x} = \frac{10-3}{3-2} = 7$

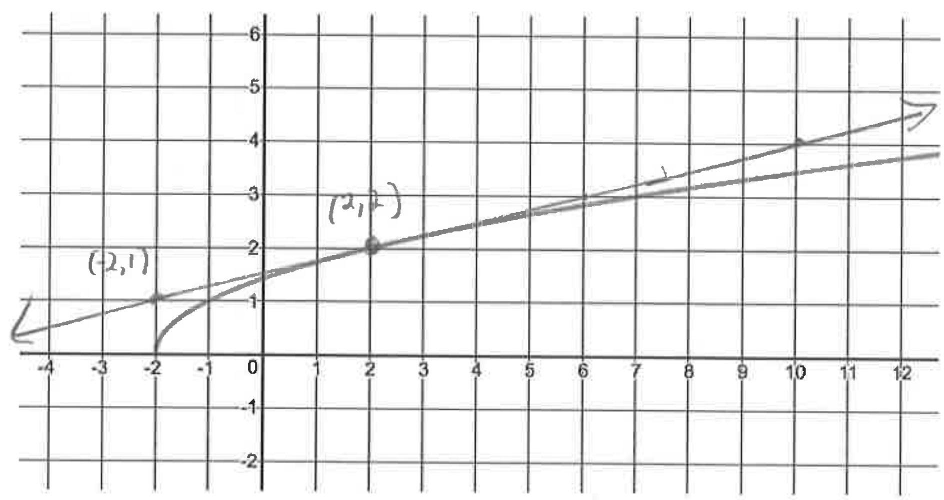
$\therefore \left.\frac{dy}{dx}\right|_{x=2} \simeq 7$



b) $\sqrt{x+2}$

$m = \frac{2-1}{2-(-2)} = \frac{1}{4}$

$\therefore \left.\frac{dy}{dx}\right|_{x=2} \simeq \frac{1}{4}$



4) Verify your answers from question #3 by calculating the LIMIT of the secant slopes as you approach $x = 2$.

a) $3x^2 - 5x + 1$

Interval	Δy	Δx	Slope of secant $= \frac{\Delta y}{\Delta x}$
$2 \le x \le 2.5$	$= f(2.5) - f(2)$ $= 7.25 - 3$ $= 4.25$	$= 2.5 - 2$ $= 0.5$	$= \frac{4.25}{0.5}$ $= 8.5$
$2 \le x \le 2.1$	$= f(2.1) - f(2)$ $= 3.73 - 3$ $= 0.73$	$= 2.1 - 2$ $= 0.1$	$= \frac{0.73}{0.1}$ $= 7.3$
$2 \le x \le 2.01$	$= f(2.01) - f(2)$ $= 3.0703 - 3$ $= 0.0703$	$= 2.01 - 2$ $= 0.01$	$= \frac{0.0703}{0.01}$ $= 7.03$
$2 \le x \le 2.001$	$= f(2.001) - f(2)$ $= 3.007003 - 3$ $= 0.007003$	$= 2.001 - 2$ $= 0.001$	$= \frac{0.007003}{0.001}$ $= 7.003$

$$\left.\frac{dy}{dx}\right|_{x=2} \simeq 7$$

b) $\sqrt{x+2}$

Interval	Δy	Δx	Slope of secant $= \frac{\Delta y}{\Delta x}$
$2 \le x \le 2.5$	$= f(2.5) - f(2)$ $= 2.121320344 - 2$ $= 0.121320344$	$= 2.5 - 2$ $= 0.5$	$= \frac{0.121320344}{0.5}$ $= 0.242640687$
$2 \le x \le 2.1$	$= f(2.1) - f(2)$ $= 2.024845673 - 2$ $= 0.024845673$	$= 2.1 - 2$ $= 0.1$	$= \frac{0.024845673}{0.1}$ $= 0.24845673$
$2 \le x \le 2.01$	$= f(2.01) - f(2)$ $= 2.002498439 - 2$ $= 0.002498439$	$= 2.01 - 2$ $= 0.01$	$= \frac{0.002498439}{0.01}$ $= 0.2498439$
$2 \le x \le 2.001$	$= f(2.001) - f(2)$ $= 2.000249984 - 2$ $= 0.000249984$	$= 2.001 - 2$ $= 0.001$	$= \frac{0.000249984}{0.001}$ $= 0.249984$

$$\left.\frac{dy}{dx}\right|_{x=2} \simeq 0.25$$

5) Use the chart below to estimate the slope of the tangent to the curve $y = \sqrt{2 - x}$ at $x = 1$.Have 4 (four) decimal place accuracy in the "slope of secant" column. (4 mks)

Interval	Change in y = Δy	Δx	$\frac{\Delta y}{\Delta x}$ = slope of secant
$0 \le x \le 1$	$= f(1) - f(0)$ $= 1 - 1.414213562$ $= -0.414213562$	$= 1 - 0$ $= 1$	$= \frac{-0.414213562}{1} \approx -0.4142$
$0.5 \le x \le 1$	$= f(1) - f(0.5)$ $= 1 - 1.224744871$ $= -0.224744871$	$= 1 - 0.5$ $= 0.5$	$= \frac{-0.224744871}{0.5} \approx -0.4495$
$0.9 \le x \le 1$	$= f(1) - f(0.9)$ $= 1 - 1.048808848$ $= -0.048808848$	$= 1 - 0.9$ $= 0.1$	$= \frac{-0.048808848}{0.1} \approx -0.4881$
$0.99 \le x \le 1$	$= f(1) - f(0.99)$ $= 1 - 1.004987562$ $= -0.004987562$	$= 1 - 0.99$ $= 0.01$	$= \frac{-0.004987562}{0.01} \approx 0.4988$
$0.999 \le x \le 1$	$= f(1) - f(0.999)$ $= 1 - 1.000499877$ $= -0.000499877$	$= 1 - 0.999$ $= 0.001$	$= \frac{-0.000499877}{0.001} \approx -0.4999$

Predicted Slope of the Tangent when $x = 1$ ... $\left.\frac{dy}{dx}\right|_{x=1} = -0.5$ (follow the trend in the 4th column)

6) The data shows the percent of households that play games over the internet.

'ear	1999	2000	2001	2002	2003
% of Households	12.3	18.2	24.4	25.7	27.9

a) Determine the average rate of change, in percent, of households that played games over the internet from 1999 to 2003.

$$m = \frac{\Delta \text{ \% households}}{\Delta \text{ year}} = \frac{27.9 - 12.3}{2003 - 1999} = \frac{15.6}{4} = 3.9\ \%/\text{year}$$

b) Estimate the instantaneous rate of change in percent of households that played games over the internet in the year 2000. Use the method of averaging a preceding and following interval AND the method of choosing a surrounding interval.

Method 1: averaging

for interval [1999, 2000]

$$m = \frac{\Delta y}{\Delta x} = \frac{18.2 - 12.3}{2000 - 1999} = 5.9\ \%/\text{year}$$

for interval [2000, 2001]

$$m = \frac{\Delta y}{\Delta x} = \frac{24.4 - 18.2}{2001 - 2000} = 6.2\ \%/\text{year}$$

$$\left.\frac{dy}{dx}\right|_{x=2000} \approx \frac{5.9 + 6.2}{2} = 6.05\%/\text{year}$$

Method 2: Surrounding

for interval [1999, 2001]

$$m = \frac{\Delta y}{\Delta x} = \frac{24.4 - 12.3}{2001 - 1999} = 6.05\ \%/\text{year}$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=2000} \approx 6.05\%/\text{year}.$$

7) Consider the data below describing the height of the world's tallest modern human, Robert Wadlow (1918-1940). At his death at 22 years of age, his height was 8 feet, 11.1 inches.

Age in years	4	8	10	13	16	18	19	21	22
Height in cm	160	190	200	220	240	250	260	268	272

a) Find average rate of change in Wadlow's height between the ages of 4 and 22. Show proper units and notation.

$$m = \frac{\Delta y}{\Delta x} = \frac{272-160}{22-4} = \frac{112}{18} \doteq 6.2 \text{ cm/year}$$

b) Estimate the instantaneous rate of change for Robert Wadlow's height when he was 16 years of age using 2 methods.

Method 1: averaging

for interval [16, 18]

$$m = \frac{\Delta y}{\Delta x} = \frac{250-240}{18-16} = \frac{10}{2} = 5 \text{ cm/year}$$

for interval [13, 16]

$$m = \frac{\Delta y}{\Delta x} = \frac{240-220}{16-13} = \frac{20}{3} = 6.67 \text{ cm/year}$$

$$\left.\frac{dy}{dx}\right|_{x=16} \approx \frac{5+6.67}{2} = 5.835 \text{ cm/year}$$

Method 2: surrounding

for interval [13, 18]

$$m = \frac{\Delta y}{\Delta x} = \frac{250-220}{18-13} = \frac{30}{5} = 6 \text{ cm/year}$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=16} \approx 6 \text{ cm/year}$$

Answer Key

1)a) (5, 3) **b)** (3, 7) **c)** $m = -2$ **d)** instantaneous rate of change at $x = 5$
2)a) at A the instantaneous rate of change is positive, at B the instantaneous rate of change is 0, and at C it is negative **b)** A: $m = 4$ m/s C: $m = -6$ m/s c) velocity at 2 seconds and 7 seconds
3&4)a) $\frac{dy}{dx} = 7$ **b)** $\frac{dy}{dx} = 0.25$
5) $\frac{dy}{dx} = -0.5$
6)a) $\frac{\Delta y}{\Delta x} = 3.9$ %/year **b)** averaging: $\frac{dy}{dx} = 6.05$ %/year surrounding: $\frac{dy}{dx} = 6.05$ %/year
7)a) $\frac{\Delta y}{\Delta x} = 6.2$ cm/year **b)** averaging: $\frac{dy}{dx} = 5.83$ cm/year surrounding: $\frac{dy}{dx} = 6$ cm/year

W3 – Newton Quotient
MHF4U
Jensen

SOLUTIONS

1. Find the equation of the derivative for each of the following functions. Also, find the instantaneous rate of change for the function when $x = 4$ and $x = -1$.

a) $f(x) = 3x - 8$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{3(x+h) - 8 - (3x-8)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{3x + 3h - 8 - 3x + 8}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{3h}{h}$$

$$\boxed{f'(x) = 3}$$

$$\boxed{\begin{array}{l} f'(4) = 3 \\ f'(-1) = 3 \end{array}}$$

b) $y = 20x + x^2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{20(x+h) + (x+h)^2 - (20x + x^2)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{20x} + 20h + \cancel{x^2} + 2xh + h^2 - \cancel{20x} - \cancel{x^2}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{20h + 2xh + h^2}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{h}(20 + 2x + h)}{\cancel{h}}$$

$$f'(x) = 20 + 2x + 0$$

$$\boxed{f'(x) = 2x + 20}$$

$$f'(4) = 2(4) + 20$$

$$\boxed{f'(4) = 28}$$

$$f'(-1) = 2(-1) + 20$$

$$\boxed{f'(-1) = 18}$$

c) $y = 2x^3 + 4$

$$f'(x) = \lim_{h\to 0} \frac{2(x+h)^3 + 4 - (2x^3 + 4)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2(x+h)(x^2 + 2xh + h^2) + 4 - 2x^3 - 4}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2(x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3) - 2x^3}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) - 2x^3}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{2x^3} + 6x^2h + 6xh^2 + 2h^3 - \cancel{2x^3}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{h}(6x^2 + 6xh + 2h^2)}{\cancel{h}}$$

$$f'(x) = 6x^2 + 6x(0) + 2(0)^2$$

$$\boxed{f'(x) = 6x^2}$$

$$\boxed{\begin{array}{l} f'(4) = 6(4)^2 \\ \quad = 96 \end{array}}$$

$$\boxed{\begin{array}{l} f'(-1) = 6(-1)^2 \\ \quad = 6 \end{array}}$$

d) $f(x) = x^2 - 9x + 17$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - 9(x+h) + 17 - (x^2 - 9x + 17)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{9x} - 9h + \cancel{17} - \cancel{x^2} + \cancel{9x} - \cancel{17}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2xh + h^2 - 9h}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{h}(2x + h - 9)}{\cancel{h}}$$

$$f'(x) = 2x + 0 - 9$$

$$\boxed{f'(x) = 2x - 9}$$

$$\boxed{\begin{array}{l} f'(4) = 2(4) - 9 \\ \quad = -1 \end{array}}$$

$$\boxed{\begin{array}{l} f'(-1) = 2(-1) - 9 \\ \quad = -11 \end{array}}$$

e) $f(x) = \frac{x(x+1)}{2}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{(x+h)(x+h+1)}{2} - \frac{x(x+1)}{2}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{x^2}+xh+\cancel{x}+xh+h^2+h-\cancel{x^2}-\cancel{x}}{2h}$$

$$f'(x) = \lim_{h\to 0} \frac{2xh+h^2+h}{2h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{h}(2x+h+1)}{2\cancel{h}}$$

$$f'(x) = \frac{2x+0+1}{2}$$

$$f'(x) = \frac{2x+1}{2}$$

$$\boxed{f'(x) = x+\frac{1}{2}}$$

$$\boxed{f'(4) = 4+\frac{1}{2} = \frac{9}{2}}$$

$$\boxed{f'(-1) = -1+\frac{1}{2} = \frac{-1}{2}}$$

f) $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{x-1(x+h)}{x(x+h)}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{x-x-h}{h(x)(x+h)}$$

$$f'(x) = \lim_{h\to 0} \frac{-1\cancel{h}}{\cancel{h}(x)(x+h)}$$

$$f'(x) = \frac{-1}{x(x+0)}$$

$$f'(x) = \frac{-1}{x^2}$$

2) State whether the functions are increasing, decreasing, or neither when $x = 4$ for each function in #1. How do you know?

a, b, c, e are increasing since f'(4) is positive

d, f are decreasing since f'(4) is negative

3)a) State the derivative of $f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)(x^2+2xh+h^2) - x^3}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{x^3}+2x^2h+xh^2+x^2h+2xh^2+h^3-\cancel{x^3}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{h}(3x^2+3xh+h^2)}{\cancel{h}}$$

$$f'(x) = 3x^2+3x(0)+(0)^2$$

$$\boxed{f'(x) = 3x^2}$$

b) Evaluate $f'(-6)$ $= 3(-6)^2$

$= 108$

c) Determine the equation of the tangent line at $x = 6$

$f(6) = (6)^3$
$= 216$

$\therefore$ point $(6, 216)$ is on the tangent line

$f'(6) = 3(6)^2$
$= 108$

$\therefore$ slope of the tangent line is 108

$y = mx + b$

$216 = 108(6) + b$

$b = -432$

$\boxed{y = 108x - 432}$

<u>**Answer Key**</u>

1)a) $f'(x) = 3, f'(4) = 3, f'(-1) = 3$ **b)** $f'(x) = 20 + 2x, f'(4) = 28, f'(-1) = 18$

c) $f'(x) = 6x^2, f'(4) = 96, f'(-1) = 6$ **d)** $f'(x) = 2x - 9, f'(4) = -1, f'(-1) = -11$

e) $f'(x) = x + \frac{1}{2}, f'(4) = \frac{9}{2}, f'(-1) = -\frac{1}{2}$ **f)** $f'(x) = -\frac{1}{x^2}, f'(4) = -\frac{1}{16}, f'(-1) = -1$

2) a,b,c and e are increasing functions when $x = 4$ since the instantaneous rate of change is positive

nd f are decreasing when $x = 4$

3)a) $f'(x) = 3x^2$ **b)** 108 **c)** $y = 108x - 432$

W4 – Limits
MHF4U
Jensen

SOLUTIONS

1) Evaluate each limit

a) $\lim_{x\to 2}\frac{3x}{x^2+2}$

$= \frac{3(2)}{(2)^2+2}$

$= \frac{6}{6}$

$= 1$

b) $\lim_{x\to -1}(x^4+x^3+x^2)$

$= (-1)^4+(-1)^3+(-1)^2$

$= 1-1+1$

$= 1$

c) $\lim_{x\to 9}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2$

$= \left(\sqrt{9}+\frac{1}{\sqrt{9}}\right)^2$

$= \left(3+\frac{1}{3}\right)^2$

$= \left(\frac{10}{3}\right)^2$

$= \frac{100}{9}$

2) Evaluate the limit of each

a) $\lim_{x\to 2}\frac{4-x^2}{2-x}$

$= \lim_{x\to 2}\frac{\cancel{(2-x)}(2+x)}{\cancel{2-x}}$

$= 2+2$

$= 4$

b) $\lim_{x\to -1}\frac{2x^2+5x+3}{x+1}$

$= \lim_{x\to -1}\frac{\cancel{(x+1)}(2x+3)}{\cancel{x+1}}$

$= 2(-1)+3$

$= 1$

c) $\lim_{x\to 3}\frac{x^3-27}{x-3}$

$= \lim_{x\to 3}\frac{\cancel{(x-3)}(x^2+3x+9)}{\cancel{x-3}}$

$= (3)^2+3(3)+9$

$= 27$

d) $\lim_{x\to 4}\frac{16-x^2}{x^3+64}$

$= \lim_{x\to 4}\frac{(4-x)\cancel{(4+x)}}{\cancel{(x+4)}(x^2-4x+16)}$

$= \frac{4-4}{(4)^2-4(4)+16}$

$= \frac{0}{16}$

$= 0$

e) $\lim_{x\to 4}\frac{x^2-16}{x^2-5x+6}$

$= \lim_{x\to 4}\frac{(x-4)(x+4)}{(x-2)(x-3)}$

$= \frac{(4-4)(4+4)}{(4-2)(4-3)}$

$= \frac{0(8)}{2(1)}$

$= \frac{0}{2}$

$= 0$

f) $\lim_{x\to -1}\frac{x^2+x}{x+1}$

$= \lim_{x\to -1}\frac{x\cancel{(x+1)}}{\cancel{x+1}}$

$= -1$

3) Complete the following table and use results to estimate $\lim_{x\to 2}\frac{x-2}{x^2-x-2}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$\frac{x-2}{x^2-x-2}$	0.3448	0.3344	0.3334	0.3332	0.3322	0.3226

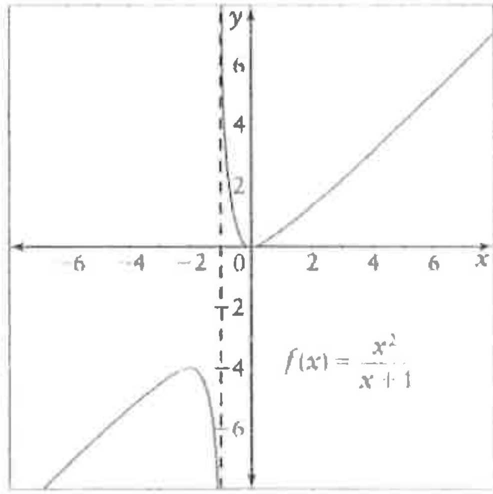
$\lim_{x\to 2}\frac{x-2}{x^2-x-2} \simeq \frac{1}{3}$

4) Use the graph to find the following limits:

a) $\lim_{x\to -1^+} \frac{x^2}{x+1}$ $= \infty$

b) $\lim_{x\to -1^-} \frac{x^2}{x+1}$ $= -\infty$

c) $\lim_{x\to -1} \frac{x^2}{x+1}$ Does not exist



5) Use the graph to determine the following limits

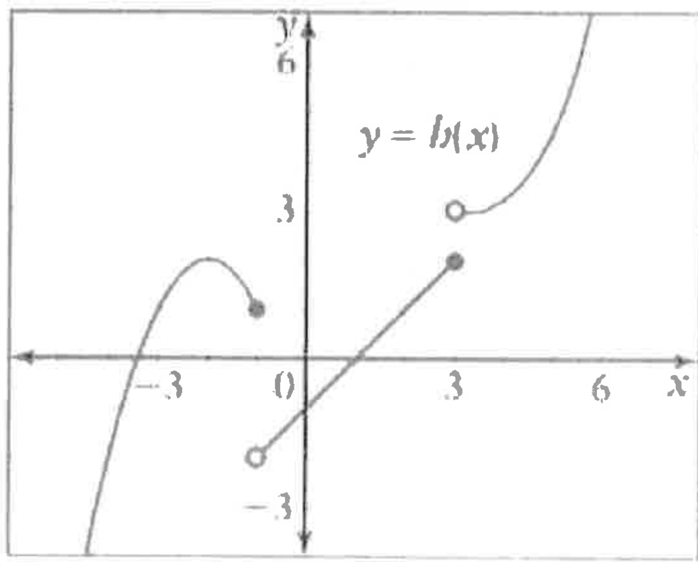
a) $\lim_{x\to -1^+} h(x)$ $= -2$

b) $\lim_{x\to -1^-} h(x)$ $= 1$

c) $\lim_{x\to -1} h(x)$ Does not exist

d) $\lim_{x\to 3^+} h(x)$ $= 3$

e) $\lim_{x\to 3^-} h(x)$ $= 2$



f) $\lim_{x\to 3} h(x)$ Does not exist

<u>**Answer Key**</u>

1)a) 1 **b)** 1 **c)** $\frac{100}{9}$ **2)a)** 4 **b)** 1 **c)** 27 **d)** 0 **e)** 0 **f)** -1 **3)** $\frac{1}{3}$ **4)a)** ∞ **b)** $-\infty$ **c)** does not exist
5)a) -2 **b)** 1 **c)** does not exist **d)** 3 **e)** 2 **f)** does not exist