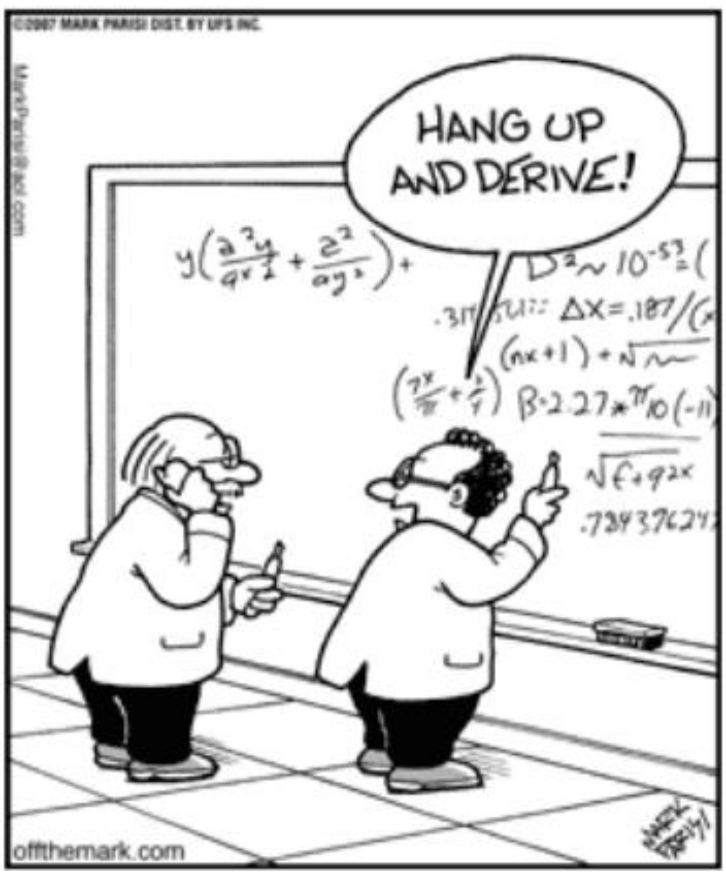
UNIT 1- Derivative Rules

Lesson Package

MCV4U



UNIT 1 Outline

Unit Goal: By the end of this unit, you will be able to verify graphically and algebraically the rules for determining derivatives; apply these rules to determine the derivatives of polynomial and radical functions, and solve related problems.

Section	Subject	Learning Goals	Curriculum Expectations
L1	Derivatives of Polynomial Functions	- use Newton's Quotient to calculate instantaneous rates of change - determine derivatives of polynomial functions	A2.3, A3.1-3.4
L2	Product Rule	- find the derivative of a product of functions using the product rule	A3.5
L3	Displacement, Velocity, Acceleration	- make connections between the concept of motion and derivatives	B2.1
L4	Quotient Rule	- find the derivative of a quotient of functions using the quotient rule	A3.5
L5	Chain Rule	- Find the derivative of a function using the chain rule	A3.4, A3.5
L6	Applications of Rates of Change	- solve problems using mathematical models and derivatives	B2.2-2.5

Assessments	F/A/O	Ministry Code	P/O/C	KTAC
Note Completion	A		P	
Practice Worksheet Completion	F/A		P	
Quiz – Derivative Rules	F		P	
PreTest Review	F/A		P	
Test – Derivative Rules	O	A2.3, A3.1-A3.5, B2.1-B2.5	P	K(21%), T(34%), A(10%), C(34%)

L1 – Derivative of a Polynomial Functions Unit 1
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In advanced functions, you should have been introduced to the idea that the instantaneous rate of change is represented by the slope of the **tangent** at a point on the curve. You also learned that you can determine this value by taking the derivative of the function using the Newton Quotient.

Newton's Quotient:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Newton Quotient Example:

a) Find the equation of the derivative of $f(x) = 3x^2 + 2x + 4$

$$f'(x) = \lim_{h \to 0} \frac{3(x+h)^2 + 2(x+h) + 4 - (3x^2 + 2x + 4)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) + 2x + 2h + 4 - 3x^2 - 2x - 4}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 + 2x + 2h + 4 - 3x^2 - 2x - 4}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{6xh + 3h^2 + 2h}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{h(6x + 3h + 2)}{h}$$

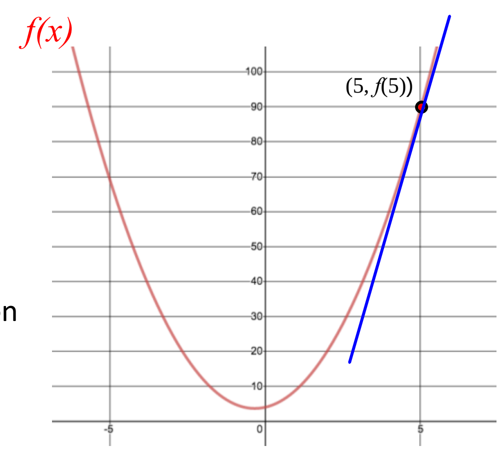
$$f'(x) = 6x + 3(0) + 2$$

$$f'(x) = 6x + 2$$

b) Calculate $f'(5)$. What does it represent?

$$f'(5) = 6(5) + 2$$

$$f'(5) = 32$$



This tells us that the instantaneous rate of change of the original function when $x = 5$ is 32. Graphically speaking, this means the slope of the tangent line drawn on the original function at $(5, f(5))$ is 32.

Mathematicians have derived a set of rules for calculating derivatives that make this process more efficient.

Rule	Derivative	Example
Constant Rule If $f(x) = c$ where c is a constant	$f'(x) = 0$	$f(x) = 87$ $f'(x) = 0$
Power Rule If $f(x) = x^n$	$f'(x) = nx^{n-1}$	$f(x) = x^5$ $f'(x) = 5x^4$
Constant Multiple Rule If $f(x) = c \cdot g(x)$ where c is a constant	$f'(x) = c \cdot g'(x)$	$f(x) = 3x^5$ $f'(x) = 3 \cdot 5x^4$ $f'(x) = 15x^4$
Sum Rule If $h(x) = f(x) + g(x)$	$h'(x) = f'(x) + g'(x)$	$h(x) = x^5 + x^4$ $h'(x) = 5x^4 + 4x^3$
Difference Rule If $h(x) = f(x) - g(x)$	$h'(x) = f'(x) - g'(x)$	$h(x) = x^5 - x^4$ $h'(x) = 5x^4 - 4x^3$

Proof of Power Rule:

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{\binom{n}{0}x^n h^0 + \binom{n}{1}x^{n-1}h^1 + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n}x^0 h^n - x^n}{h}$$

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{\binom{n}{1}x^{n-1}h^1 + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n}x^0 h^n}{h}$$

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{h\left[\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h^1 + \cdots + \binom{n}{n}x^0 h^{n-1}\right]}{h}$$

$$\frac{d}{dx}x^n = \lim_{h\to 0}\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h^1 + \cdots + \binom{n}{n}x^0 h^{n-1}$$

$$\frac{d}{dx}x^n = \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}0^1 + \cdots + \binom{n}{n}x^0 0^{n-1}$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

Use Binomial Theorem:

$$t_{r+1} = \binom{n}{r}a^{n-r}b^r$$

Example 1: Determine the equation of the derivative of each of the following functions:

a) $f(x) = 3x^5$

$f'(x) = 15x^4$

b) $f(x) = 71$

$f'(x) = 0$

c) $f(x) = \sqrt{x}$

$f(x) = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}x^{\frac{1}{2}-1}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$f'(x) = \frac{1}{2\sqrt{x}}$

d) $y = \sqrt[3]{x}$

$y = x^{\frac{1}{3}}$

$y' = \frac{1}{3}x^{\frac{1}{3}-1}$

$y' = \frac{1}{3}x^{-\frac{2}{3}}$

$y' = \frac{1}{3x^{\frac{2}{3}}}$

e) $y = \frac{1}{x}$

$y = x^{-1}$

$y' = -1x^{-2}$

$y' = \frac{-1}{x^2}$

f) $y = -\frac{1}{x^5}$

$y = -1x^{-5}$

$\frac{dy}{dx} = 5x^{-6}$

$\frac{dy}{dx} = \frac{5}{x^6}$

Example 2: Differentiate each function

a) $y = 5x^6 - 4x^3 + 6$

$y' = 30x^5 - 12x^2 + 0$

$y' = 30x^5 - 12x^2$

b) $f(x) = -3x^5 + 8\sqrt{x} - 9.3$

$f(x) = -3x^5 + 8x^{\frac{1}{2}} - 9.3$

$f'(x) = -15x^4 + 4x^{-\frac{1}{2}} - 0$

$f'(x) = -15x^4 + \frac{4}{\sqrt{x}}$

c) $g(x) = (2x-3)(x+1)$

$g(x) = 2x^2 - x - 3$

$g'(x) = 4x - 1$

d) $h(x) = \frac{-8x^6+8x^2}{4x^5}$

$h(x) = \frac{-8x^6}{4x^5} + \frac{8x^2}{4x^5}$

$h(x) = -2x + 2x^{-3}$

$h'(x) = -2 - 6x^{-4}$

$h'(x) = -2 - \frac{6}{x^4}$

Example 3: Determine an equation for the tangent to the curve $f(x) = 4x^3 + 3x^2 - 5$ at $x = -1$.

Point on the tangent line:

$f(-1) = 4(-1)^3 + 3(-1)^2 - 5$

$f(-1) = -6$

$(-1, -6)$

Slope of tangent line:

$f'(x) = 12x^2 + 6x$

$f'(-1) = 12(-1)^2 + 6(-1)$

$f'(-1) = 6$

Slope of the tangent is 6

Remember: *The equation of the derivative tells you the slope of the tangent to the original function.*

Equation of tangent line:

$y = mx + b$

$-6 = 6(-1) + b$

$b = 0$

$y = 6x$

Example 4: Determine the point(s) on the graph of $y = x^2(x + 3)$ where the slope of the tangent is 24.

$y = x^3 + 3x^2$

$\frac{dy}{dx} = 3x^2 + 6x$

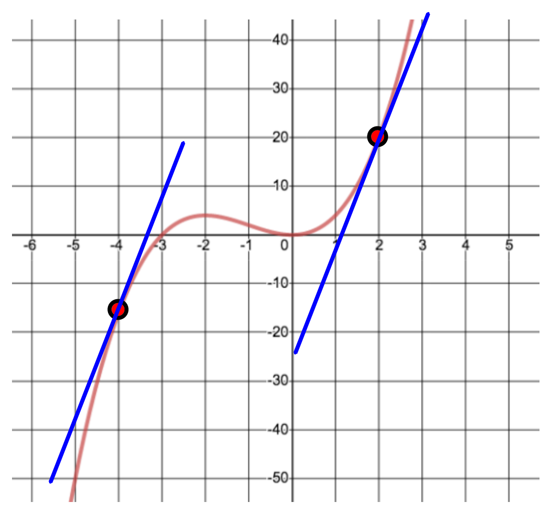
$24 = 3x^2 + 6x$

$0 = 3x^2 + 6x - 24$

$0 = 3(x^2 + 2x - 8)$

$0 = (x + 4)(x - 2)$

$x_1 = -4 \qquad x_2 = 2$



Now find y-coordinates of points:

Point 1:

$y = (-4)^3 + 3(-4)^2$

$y = -16$

$(-4, 16)$

Point 2:

$y = (2)^3 + 3(2)^2$

$y = 20$

$(2, 20)$

L2 – The Product Rule Unit 1
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Part 1: Proof of the Product Rule

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0}\frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h}$$

$$\frac{d}{dx}[f(x)g(x)] = \left[\lim_{h\to 0} f(x+h)\right]\left[\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right] + \left[\lim_{h\to 0} g(x)\right]\left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right]$$

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

The Product Rule:

If $P(x) = f(x)g(x)$, then $P'(x) = f'(x)g(x) + g'(x)f(x)$

"Derivative of the first times the second plus derivative of the second times the first"

Part 2: Apply the Product Rule

Example 1: Use the product rule to differentiate each function.

a) $P(x) = (3x-5)(x^2+1)$

$$P'(x) = 3(x^2+1) + 2x(3x-5)$$

$$P'(x) = 3x^2 + 3 + 6x^2 - 10x$$

$$P'(x) = 9x^2 - 10x + 3$$

b) $y = (2x+3)(1-x)$

$$\frac{dy}{dx} = 2(1-x) + (-1)(2x+3)$$

$$\frac{dy}{dx} = 2 - 2x - 2x - 3$$

$$\frac{dy}{dx} = -4x - 1$$

Example 2: Find $h'(-1)$ where $h(x) = (5x^3 + 7x^2 + 3)(2x^2 + x + 6)$

$h'(x) = (15x^2 + 14x)(2x^2 + x + 6) + (4x + 1)(5x^3 + 7x^2 + 3)$

$h'(-1) = [15(-1)^2 + 14(-1)][2(-1)^2 + (-1) + 6] + [4(-1) + 1][5(-1)^3 + 7(-1)^2 + 3]$

$h'(-1) = (1)(7) + (-3)(5)$

$h'(-1) = -8$

Example 3: Find the derivative of $g(x) = (x-1)(2x)(x^2+3)$

$g'(x) = (4x - 2)(x^2 + 3) + (2x)(x - 1)(2x)$

$g'(x) = 4x^3 + 12x - 2x^2 - 6 + 4x^2(x - 1)$

$g'(x) = 4x^3 + 12x - 2x^2 - 6 + 4x^3 - 4x^2$

$g'(x) = 8x^3 - 6x^2 + 12x - 6$

Consider $(x-1)(2x)$ as the 1st function

Consider $x^2 + 3$ as the 2nd function

$\frac{d}{dx}$ 1st $= 1(2x) + 2(x - 1) = 4x - 2$

Note: In example 3, expanding first would probably be easier, but that is not always the case such as with $h(x) = (2x) \cdot \sqrt{x+1}$

Example 4: Determine an equation for the tangent to the curve $y = (x^2 - 1)(x^2 - 2x + 1)$ at $x = 2$.

Point on the tangent line:

$y = [2^2 - 1][2^2 - 2(2) + 1]$

$y = (3)(1)$

$y = 3$

$(2, 3)$

Slope of tangent line:

$\frac{dy}{dx} = 2x(x^2 - 2x + 1) + (2x - 2)(x^2 - 1)$

$\left.\frac{dy}{dx}\right|_{x=2} = 2(2)[2^2 - 2(2) + 1] + [2(2) - 2][2^2 - 1]$

$\left.\frac{dy}{dx}\right|_{x=2} = 4(1) + 2(3)$

$\left.\frac{dy}{dx}\right|_{x=2} = 10$

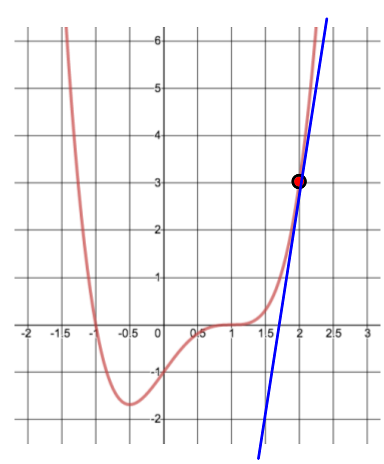
Equation of Tangent Line:

$y = mx + b$

$3 = 10(2) + b$

$b = -17$

$y = 10x - 17$



Example 5: Student council is organizing its annual trip to an out-of-town concert. For the past 3 years, the cost of the trip has been $140 per person. At this price, all 200 seats on the train were filled. This year, student council plans to increase the price of the trip. Based on a student survey, council estimates that for every $10 increase in price, five fewer students will attend the concert.

a) Write an equation to represent revenue, R, in dollars, as a function of the number of $10 increases, n.

$R(n) = (price)(\# \; of \; students)$
$R(n) = (140 + 10n)(200 - 5n)$

b) Determine an expression, in simplified form, for $\frac{dR}{dn}$ and interpret it for this situation.

$R'(n) = 10(200 - 5n) + (-5)(140 + 10n)$

$R'(n) = 2000 - 50n - 700 - 50n$

$R'(n) = -100n + 1300$

c) Determine when $R'(n) = 0$. What information does this give the manager?

$0 = -100n + 1300$

$100n = 1300$

$n = 13$

The tangent slope is 0 when $n = 13$. Therefore, there is a maximum revenue when there are 13 price increases.

$R(13) = [140 + 10(13)][200 - 5(13)]$

$R(13) = (270)(135)$

$R(13) = 36450$

With 13 price increases, the manager will sell 135 tickets for $270 each and make a max revenue of $36450.

L3 – Velocity, Acceleration, and Second Derivatives Unit 1
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Part 1: Second Derivatives

The second derivative of a function is determined by differentiating the first derivative of the function.

Example 1: For the function $f(x)=\frac{1}{3}x^3-x^2-3x+4$

a) Calculate $f'(x)$

$f'(x)=x^2-2x-3$

b) When is $f'(x)=0$?

$0=x^2-2x-3$

$0=(x-3)(x+1)$

$x_1=3 \qquad x_2=-1$

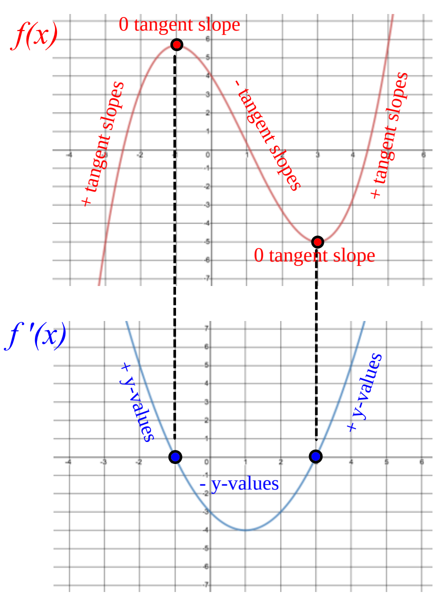
c) Calculate $f''(x)$

$f''(x)=2x-2$

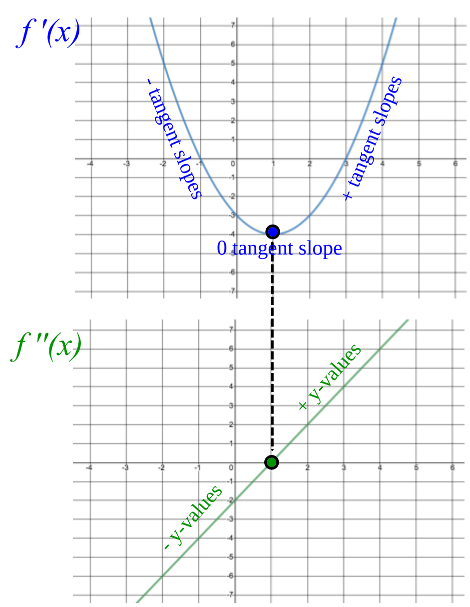
d) When is $f''(x)=0$?

$0=2x-2$

$x=1$



Negative y-values from $-1<x<3$ on the graph of $f'(x)$ correspond to the negative tangent slopes on the graph of $f(x)$. The x-intercepts of -1 and 3 on the graph of $f'(x)$ correspond to the points on $f(x)$ that has a tangent slope of zero.



Negative y-values for $x<1$ on the graph of $f''(x)$ correspond to the negative tangent slopes on the graph of $f'(x)$. The x-intercept of 1 on the graph of $f''(x)$ correspond to the point on $f'(x)$ that has a tangent slope of zero.

Part 2: Displacement, Velocity, and Acceleration

	Displacement (s)	Velocity (v)	Acceleration (a)
Definition	Distance an object has moved from the origin over a period of time (t)	Rate of change of displacement (s) with respect to time (t). Speed with direction.	Rate of change of velocity(v) with respect to time (t)
Relationship	$s(t)$	$v(t) = s'(t)$	$a(t) = v'(t) = s''(t)$
Possible Units	m	m/s	m/s^2

Important: *speed and velocity are often confused for one another. Speed is a scalar quantity. It describes the magnitude of motion but does not describe the direction. Velocity has both magnitude and direction. The sign indicates the direction the object is travelling relative to the origin.*

Example 2: A construction worker accidentally drops a hammer from a height of 90 meters. The height, s, in meters, of the hammer t seconds after it is dropped can be modelled by the function $s(t) = 90 - 4.9t^2$.

a) What is the velocity of the hammer at 1s vs. 4s?

$v(t) = s'(t) = -9.8t$

$v(1) = -9.8(1) = -9.8\ m/s$

$v(4) = -9.8(4) = 39.2\ m/s$

b) When does the hammer hit the ground? When is $s(t) = 0$?

$0 = 90 - 4.9t^2$

$4.9t^2 = 90$

$$t = \pm\sqrt{\frac{90}{4.9}}$$

$t \cong 4.3\ seconds$

c) What is the velocity of the hammer when it hits the ground?

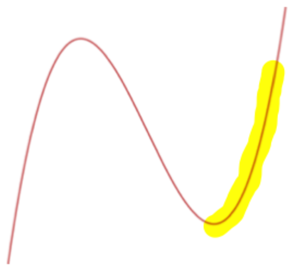
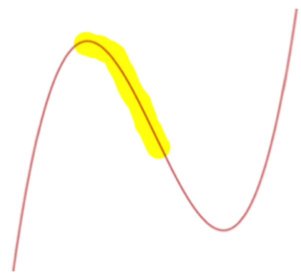
$$v(4.3) = -9.8\left(\sqrt{\frac{90}{4.9}}\right) = -42\ m/s$$

d) Determine the acceleration function.

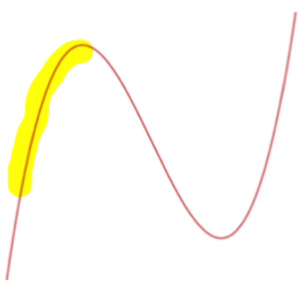
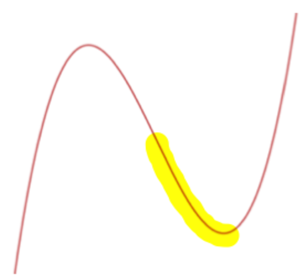
$a(t) = v'(t) = s''(t) = -9.8$ (acceleration due to gravity)

Speeding Up vs. Slowing Down

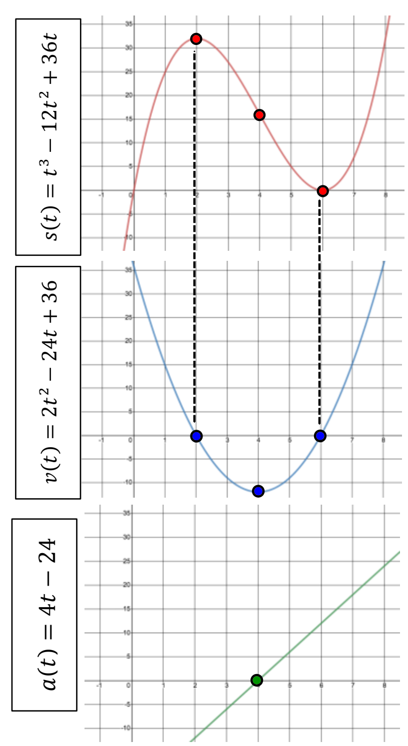
An object is **speeding up** if the graph of $s(t)$ has a positive slope that is increasing OR has a negative slope that is decreasing. In these scenarios, $v(t) \times a(t) > 0$.

Positive slope increasing $v(t) \times a(t) = + \times + = +$ 	Negative slope decreasing $v(t) \times a(t) = - \times - = +$ 

An object is **slowing down** if the graph of $s(t)$ has a positive slope that is decreasing OR has a negative slope that is increasing. In these scenarios, $v(t) \times a(t) < 0$.

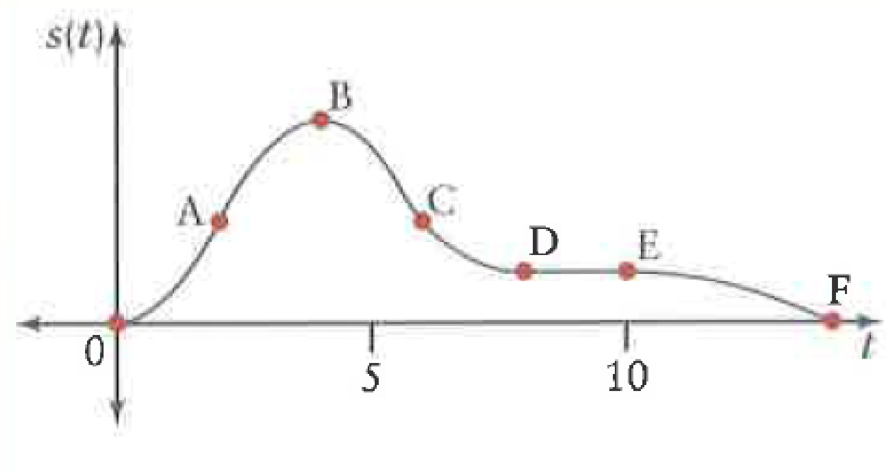
Positive slope decreasing $v(t) \times a(t) = + \times - = -$ 	Negative slope increasing $v(t) \times a(t) = - \times + = -$ 

Example 3: The position of a particle moving along a straight line can be modelled by the function below where t is the time in seconds and s is the displacement in meters. Use the graphs of $s(t)$, $v(t)$, and $a(t)$ to determine when the particle is speeding up and slowing down.



Interval	$v(t)$	$a(t)$	$v(t) \times a(t)$	Slope of $s(t)$	Motion of particle
$(0,2)$	+	-	-	positive slope that is decreasing	Slowing down and moving forward
$(2,4)$	-	-	+	Negative slope that is decreasing	Speeding up and moving in reverse
$(4,6)$	-	+	-	Negative slope that is increasing	Slowing down and moving in reverse
$(6,8)$	+	+	+	Positive slope that is increasing	Speeding up and moving forward

Example 4: Given the graph of $s(t)$, figure out where $v(t)$ and $a(t)$ are + or – and use this information to state when the particle is speeding up and slowing down.



Interval	$v(t)$	$a(t)$	$v(t) \times a(t)$	Slope of $s(t)$	Motion of particle
$(0, A)$	+	+	+	Positive and increasing	Speeding up and moving forward
(A, B)	+	-	-	Positive and decreasing	Slowing down and moving forward
(B, C)	-	-	+	Negative and decreasing	Speeding up and moving in reverse
(C, D)	-	+	-	Negative and increasing	Slowing down and moving in reverse
(D, E)	0	0	0	0	Not moving
(E, F)	-	-	+	Negative and decreasing	Speeding up and moving in reverse

L4 – The Quotient Rule Unit 1
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The Quotient Rule:

If $h(x) = \frac{f(x)}{g(x)}$, then $h'(x) = \frac{f'(x)g(x)-g'(x)f(x)}{[g(x)]^2}$

Proof:

$$h(x) = \frac{f(x)}{g(x)}$$

$$h(x)g(x) = f(x)$$

$$h'(x)g(x) + g'(x)h(x) = f'(x)$$

$$h'(x) = \frac{f'(x) - g'(x)h(x)}{g(x)}$$

$$h'(x) = \frac{f'(x) - g'(x)\frac{f(x)}{g(x)}}{g(x)}$$

$$h'(x) = \left[\frac{f'(x) - g'(x)\frac{f(x)}{g(x)}}{g(x)}\right]\left[\frac{g(x)}{g(x)}\right]$$

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

Example 1: Find the derivative of each of the following:

a) $f(x) = \frac{3x-4}{x^2+5}$

$$f'(x) = \frac{3(x^2+5) - 2x(3x-4)}{(x^2+5)^2}$$

$$f'(x) = \frac{3x^2 + 15 - 6x^2 + 8x}{(x^2+5)^2}$$

$$f'(x) = \frac{-3x^2 + 8x + 15}{(x^2+5)^2}$$

b) $g(x) = \frac{6x-5}{x^3+4}$

$$g'(x) = \frac{6(x^3+4) - 3x^2(6x-5)}{(x^3+4)^2}$$

$$g'(x) = \frac{6x^3 + 24 - 18x^3 + 15x^2}{(x^3+4)^2}$$

$$g'(x) = \frac{-12x^3 + 15x^2 + 24}{(x^3+4)^2}$$

$$g'(x) = \frac{-3(4x^3 - 5x^2 - 8)}{(x^3+4)^2}$$

c) $h(x) = \frac{2x+8}{\sqrt{x}}$

$$h'(x) = \frac{2(\sqrt{x}) - \frac{1}{2}x^{-\frac{1}{2}}(2x+8)}{(\sqrt{x})^2}$$

$$h'(x) = \frac{2\sqrt{x} - \frac{1}{2\sqrt{x}}(2)(x+4)}{x}$$

$$h'(x) = \frac{2\sqrt{x} - \frac{1}{\sqrt{x}}(x+4)}{x} \times \frac{\sqrt{x}}{\sqrt{x}}$$

$$h'(x) = \frac{2x - 1(x+4)}{x\sqrt{x}}$$

$$h'(x) = \frac{x-4}{x^{\frac{3}{2}}}$$

d) $r(x) = \frac{x+3}{\sqrt{x^2-1}}$

$$r'(x) = \frac{1(x^2-1)^{\frac{1}{2}} - \frac{1}{2}(x^2-1)^{-\frac{1}{2}}(2x)(x+3)}{\left(\sqrt{x^2-1}\right)^2}$$

$$r'(x) = \frac{(x^2-1)^{-\frac{1}{2}}\left[1(x^2-1)^1 - \frac{1}{2}(2x)(x+3)\right]}{x^2-1}$$

$$r'(x) = \frac{[1(x^2-1) - 1(x)(x+3)]}{(x^2-1)^{\frac{1}{2}}(x^2-1)^1}$$

$$r'(x) = \frac{x^2 - 1 - x^2 - 3x}{(x^2-1)^{\frac{3}{2}}}$$

$$r'(x) = \frac{-3x-1}{(x^2-1)^{\frac{3}{2}}}$$

Note: *$\frac{d}{dx}\sqrt{x^2-1}$ from part d) uses the 'chain rule'. We will learn this in depth in the next lesson.*

Example 2: Determine an equation for the tangent to the curve $y = \frac{x^2-3}{5-x}$ at $x = 2$.

Point on Tangent Line:

$$y = \frac{2^2 - 3}{5 - 2}$$

$$y = \frac{1}{3}$$

$$\left(2, \frac{1}{3}\right)$$

Slope of tangent line:

$$\frac{dy}{dx} = \frac{(2x)(5-x) - (-1)(x^2-3)}{(5-x)^2}$$

$$\frac{dy}{dx} = \frac{-x^2 + 10x - 3}{(5-x)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{-(2)^2 + 10(2) - 3}{(5-2)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{13}{9}$$

Equation of tangent line:

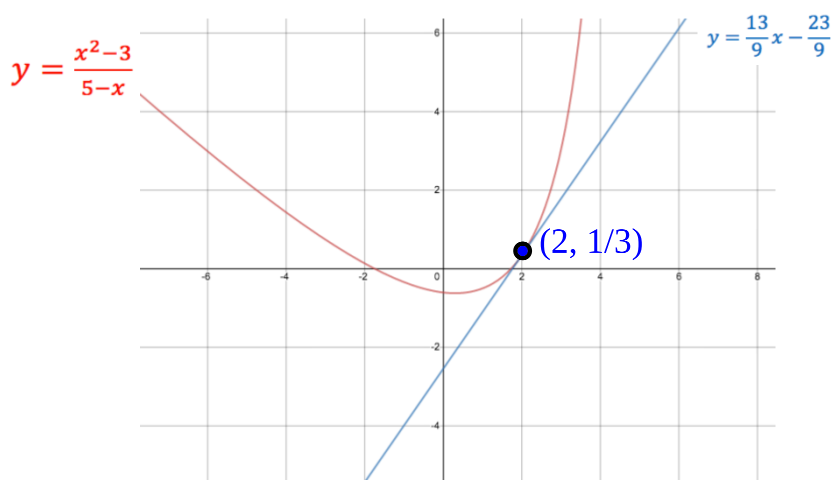
$$y = mx + b$$

$$\frac{1}{3} = \frac{13}{9}(2) + b$$

$$\frac{3}{9} - \frac{26}{9} = b$$

$$b = -\frac{23}{9}$$

$$y = \frac{13}{9}x - \frac{23}{9}$$



Example 3: Determine the coordinates of each point on the graph of $h(x) = \frac{2x+8}{\sqrt{x}}$ where the tangent is horizontal.

The tangent will be horizontal when $h'(x) = 0$

$h'(x) = \frac{x-4}{x^{\frac{3}{2}}}$

$0 = \frac{x-4}{x^{\frac{3}{2}}}$

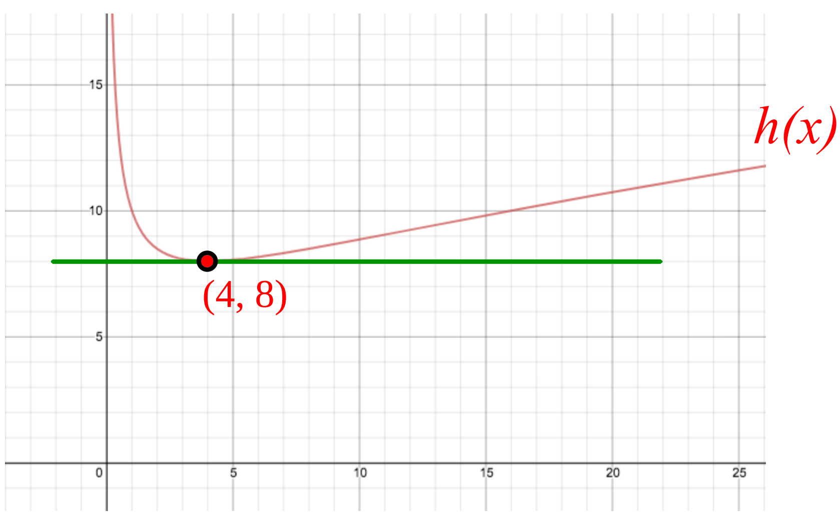
$0 = x - 4$

$x = 4$

$$h(4) = \frac{2(4) + 8}{\sqrt{4}}$$

$h(4) = 8$

There will be a horizontal tangent to the point (4, 8) on the graph of $h(x)$.



L5 – The Chain Rule **Unit 1**
MCV4U
Jensen

A composite function consists of an outer function, $g(x)$, and an inner function, $h(x)$. The chain rule says to differentiate outer function with respect to the inner function, and then multiply by the derivative of the inner function.

Given two differentiable functions $g(x)$ and $h(x)$, the derivative of the composite function $f(x) = g[h(x)]$ is $f'(x) = g'[h(x)] \times h'(x)$

A special case of the chain rule is the 'Power of a Function Rule'. This occurs when the outer function is a power function:

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$$

Summary of Derivative Rules:

Rule	Derivative
Power Rule If $f(x) = x^n$	$f'(x) = nx^{n-1}$
Constant Multiple Rule If $f(x) = c \cdot g(x)$ where c is a constant	$f'(x) = c \cdot g'(x)$
Sum Rule If $h(x) = f(x) + g(x)$	$h'(x) = f'(x) + g'(x)$
Difference Rule If $h(x) = f(x) - g(x)$	$h'(x) = f'(x) - g'(x)$
Product Rule If $h(x) = f(x)g(x)$	$h'(x) = f'(x)g(x) + f(x)g'(x)$
Quotient Rule If $h(x) = f(x) \div g(x)$	$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$
Power of a Function Rule If $h(x) = [f(x)]^n$	$h'(x) = n[f(x)]^{n-1} \times f'(x)$
Chain Rule If $h(x) = f[g(x)]$	$h'(x) = f'[g(x)] \times g'(x)$

Example 1: Differentiate each function using the chain rule. Express in simplified factored form.

a) $f(x) = (3x - 5)^4$

$f'(x) = 4(3x - 5)^3(3)$

$f'(x) = 12(3x - 5)^3$

b) $g(x) = \sqrt{4 - x^2}$

$g'(x)$
$= \frac{1}{2}(4 - x^2)^{-\frac{1}{2}}(-2x)$

$g'(x) = \frac{-x}{\sqrt{4 - x^2}}$

c) y=$\left(2\sqrt{x}\right)^3 - 4\sqrt{x} + 1$

$$\frac{dy}{dx} = 3\left(2\sqrt{x}\right)^2 (x)^{-\frac{1}{2}} - 2x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = 3(4x)(x)^{-\frac{1}{2}} - 2x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = (12x)(x)^{-\frac{1}{2}} - 2x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{2(6x-1)}{\sqrt{x}}$$

d) $h(x) = (x^2+3)^4(4x-5)^3$

$$h'(x) = 4(x^2+3)^3(2x)(4x-5)^3 + 3(4x-5)^2(4)(x^2+3)^4$$

$$h'(x) = 4(x^2+3)^3(4x-5)^2[2x(4x-5) + 3(x^2+3)]$$

$$h'(x) = 4(x^2+3)^3(4x-5)^2[11x^2 - 10x + 9]$$

Example 2: Determine an equation for the tangent to $f(x) = 3x(1-x)^2$ at $x = 0.5$

Point on Tangent Line:

$$f(0.5) = 3(0.5)(1-0.5)^2$$

$$f(0.5) = \frac{3}{8}$$

Equation of line:

$$y = mx + b$$

$$\frac{3}{8} = \left(-\frac{3}{4}\right)\left(\frac{1}{2}\right) + b$$

$$b = \frac{3}{4}$$

$$y = -\frac{3}{4}x + \frac{3}{4}$$

Slope of tangent line:

$$f'(x) = 3(1-x)^2 + 2(1-x)(-1)(3x)$$

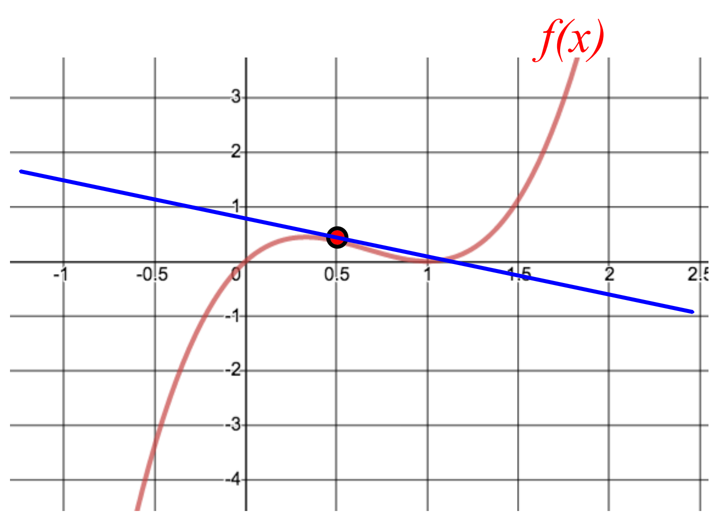
$$f'(x) = 3(1-x)^2 - 6x(1-x)$$

$$f'(x) = 3(1-x)[(1-x) - 2x]$$

$$f'(x) = 3(1-x)(1-3x)$$

$$f'(0.5) = 3(1-0.5)[1-3(0.5)]$$

$$f'(0.5) = -\frac{3}{4}$$



L6 – Applications of Rates of Change Unit 1
MCV4U
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Part 1: Rates of Change Applications

Example 1: Suppose the function $V(t) = \frac{50\,000 + 6t}{1+0.4t}$ represents the value, V, in dollars, of a new car t years after it is purchased.

a) What is the rate of change of the value of the car at 2 years? 5 years? And 7 years?

$$V'(t) = \frac{6(1+0.4t) - 0.4(50\,000 + 6t)}{(1+0.4t)^2}$$

$$V'(t) = \frac{6 + 2.4t - 20\,000 - 2.4t}{(1+0.4t)^2}$$

$$V'(t) = \frac{-19994}{(1+0.4t)^2}$$

$$V'(2) = \frac{-19994}{[1+0.4(2)]^2} \cong -6170.99 \text{ \$/year}$$

$$V'(5) = \frac{-19994}{[1+0.4(5)]^2} \cong -2221.56 \text{ \$/year}$$

$$V'(7) = \frac{-19994}{[1+0.4(7)]^2} \cong -1384.63 \text{ \$/year}$$

b) What was the initial value of the car?

$$V(0) = \frac{50\,000 + 6(0)}{1+0.4(0)} = \$50\,000$$

Example 2: Kinetic energy, K, is the energy due to motion. When an object is moving, its kinetic energy is determined by the formula $K(v) = 0.5mv^2$, where K is in joules, m is the mass of the object, in kilograms; and v is the velocity of the object, in meters per second.

Suppose a ball with a mass of 0.35 kg is thrown vertically upward with an initial velocity of 40 m/s. Its velocity function is $v(t) = 40 - 9.8t$, where t is time, in seconds.

a) Express the kinetic energy of the ball as a function of time.

$$K[v(t)] = K(t) = 0.5(0.35)(40 - 9.8t)^2$$

$$K(t) = 0.175(40 - 9.8t)^2$$

b) Determine the rate of change of the kinetic energy of the ball at 3 seconds.

$$K'(t) = 2(0.175)(40 - 9.8t)(-9.8)$$

$$K'(t) = -3.43(40 - 9.8t)$$

$$K'(3) = -3.43[40 - 9.8(3)]$$

$$K'(3) = -36.358$$

At 3 seconds, the rate of change of kinetic energy of the ball is decreasing by 36.358 J/s.

Linear Density:

The linear density of an object refers to the mass of an object per unit length. Suppose the function $f(x)$ gives the mass, in kg, of the first x meters of an object. For the part of the object that lies between x_1 and x_2, the average linear density $= \frac{f(x_2)-f(x_1)}{x_2-x_1}$. The corresponding derivative function $f'(x)$ is the linear density, the rate of change of mass at a particular length x.

Example 3: The mass, in kg, of the first x meters of wire can be modelled by the function $f(x) = \sqrt{3x+1}$.

a) Determine the average linear density of the part of the wire from $x = 5$ to $x = 8$.

$average\ linear\ density = \frac{f(8)-f(5)}{8-5} = \frac{\sqrt{3(8)+1}-\sqrt{3(5)+1}}{3} = \frac{1}{3}$ or about 0.333 kg/m.

b) Determine the linear density at $x = 5$ and $x = 8$. What do these results tell you about the wire.

$$f'(x) = \frac{1}{2}(3x+1)^{-\frac{1}{2}}(3)$$

$$f'(x) = \frac{3}{2\sqrt{3x+1}}$$

$$f'(5) = \frac{3}{2\sqrt{3(5)+1}}$$

$f'(5) = \frac{3}{8}$ or 0.375 kg/m

$$f'(8) = \frac{3}{2\sqrt{3(8)+1}}$$

$f'(5) = \frac{3}{10}$ or 0.3 kg/m

The linear densities are different. Therefore, the material of which the wire is composed is non-homogenous.

Part 2: Business Applications

Terminology:

- The demand functions, or price function, is $p(x)$, where x is the number of units of a product or service that can be sold at a particular price, p.
- The revenue function is $R(x) = x \cdot p(x)$, where x is the number of units of a product or service sold at a price per unit of $p(x)$.
- The cost function, $C(x)$, is the total cost of producing x units of a product or service.
- The profit function, $P(x)$, is the profit from the sale of x units of a product or service. The profit function is the difference between the revenue function and the cost function: $P(x) = R(x) - C(x)$

Economists use the word marginal to indicate the derivative of a business function.

- $C'(x)$ is the marginal cost function and refers to the instantaneous rate of change of total cost with respect to the number of items produced.
- $R'(x)$ is the marginal revenue function and refers to the instantaneous rate of change of total revenue with respect to the number of items sold.
- $P'(x)$ is the marginal profit function and refers to the instantaneous rate of change of total profit with respect to the number of items sold.

Example 4: A company sells 1500 movie DVDs per month at $10 each. Market research has shown that sales will decrease by 125 DVDs per month for each $0.25 increase in price.

a) Determine a demand (or price) function.

Let x represent number of DVDs sold per month
Let p be the price of one DVD
Let n be the number of $0.25 price increases

Equation 1: $x = 1500 - 125n$
Equation 2: $p = 10 + 0.25n$

Re-write price (p) in terms of number of DVDS sold per month (x):

From Equation 1: $n = \frac{1500-x}{125}$

Sub in to Equation 2: $p = 10 + 0.25\left(\frac{1500-x}{125}\right) = 10 + 0.002(1500 - x) = 10 + 3 - 0.002x = 13 - 0.002x$

The demand (price) function is $p(x) = 13 - 0.002x$. This gives the price for one DVD when x DVDs are sold.

b) Determine the marginal revenue when sales are 1000 DVDs per month.

$R(x) = x \cdot p(x)$

$R(x) = x(13 - 0.002x)$

$R(x) = -0.002x^2 + 13x$

$R'(x) = -0.004x + 13$

$R'(1000) = -0.004(1000) + 13$

$R'(1000) = 9$

When sales are 1000 DVDs, the revenue is increasing at a rate of $9 per additional DVD sold.

c) The cost of producing x DVDs is $C(x) = -0.004x^2 + 9.2x + 5000$. Determine the marginal cost when production is 1000 DVDs per month.

$C'(x) = -0.008x + 9.2$

$C'(1000) = -0.008(1000) + 9.2$

$C'(1000) = 1.2$

When producing 1000 DVDs per month, the cost is increasing by $1.20 for each additional DVD produced.

d) Determine the actual cost of producing the 1001st DVD.

$C(1001) - C(1000) = [-0.004(1001)^2 + 9.2(1001) + 5000] - [-0.004(1000)^2 + 9.2(1000) + 5000]$

$C(1001) - C(1000) = 10201.196 - 10200.00$

$C(1001) - C(1000) = 1.196$

The actual cost of producing the 1001st DVD is $1.196. Notice the similarity between the marginal cost of the 1000th DVD and the actual cost of producing the 1001st DVD. For large values of x, the marginal cost when producing x items is approximately equal to the cost of producing one more item, the $(x + 1)$th item.

e) Determine the Profit and Marginal Profit for the monthly sales of 1000 DVDs.

$P(x) = R(x) - C(x)$

$P(x) = -0.002x^2 + 13x - (-0.004x^2 + 9.2x + 5000)$

$P(x) = 0.002x^2 + 3.8x - 5000$

$P(1000) = [0.002(1000)^2 + 3.8(1000) - 5000]$

$P(1000) = 800$

The profit if 1000 DVDs are sold is $800.

$P'(x) = 0.004x + 3.8$

$P'(1000) = 0.004(1000) + 3.8$

$P'(1000) = 7.80$

If 1000 DVDs are sold, the profit is increasing at a rate of $7.80 per additional DVD sold.