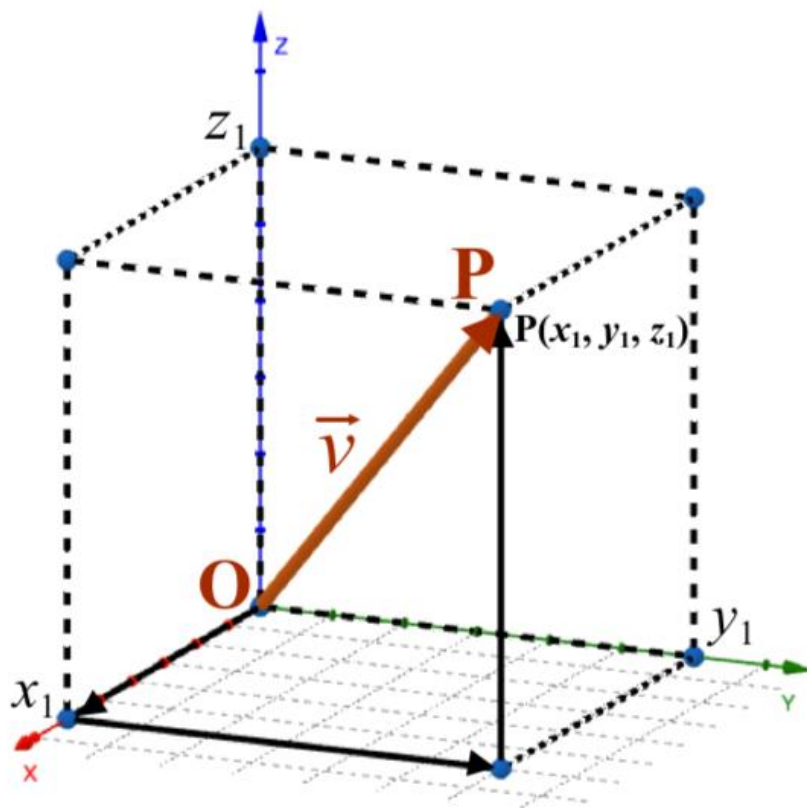


Unit 5- Cartesian Vectors

Lesson Package

MCV4U



Unit 5 Outline

Unit Goal: By the end of this unit, you will be able to demonstrate an understanding of vectors in two-space by representing them geometrically and by recognizing their applications.

Section	Subject	Learning Goals	Curriculum Expectations
L1	Cartesian Vectors	- Represent a vector in two-space in Cartesian form - Perform operations of addition, subtraction, and scalar multiplication on vectors represented in Cartesian form	C1.3, C2.1, C2.2, C2.3
L2	Dot Product	- find the dot product of two vectors in geometric and Cartesian form	C2.4, C2.5
L3	Applications of Dot Product	- solve problems involving the dot product of two vectors including work and projections.	C2.8
L4	Vectors in 3-Space	- Recognize that vectors in 3-space can be represented using Cartesian coordinates $[x, y, z]$ - Perform vector operations with vectors in 3-space	C1.4
L5	Cross Product	- find the cross product of two vectors in geometric and Cartesian form.	C2.6, C2.7
L6	Applications of Dot and Cross Product	- Solve problems involving dot and cross product including the triple scalar product	C2.8

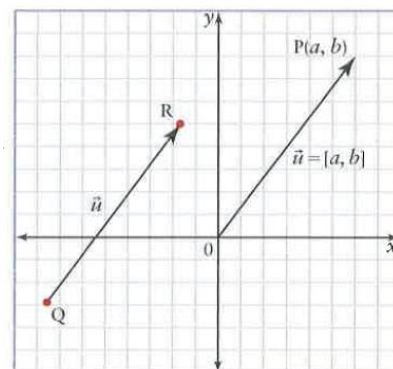
Assessments	F/A/O	Ministry Code	P/O/C	KTAC
Note Completion	A		P	
Practice Worksheet Completion	F/A		P	
Quiz - Dot and Cross Product	F		P	
PreTest Review	F/A		P	
Test - Cartesian Vectors	O	C1.3, C1.4, C2.1, C2.2, C2.3, C2.4, C2.5, C2.6, C2.7, C2.8	P	K(25%), T(25%), A(25%), C(25%)

Mathematicians started using coordinates to analyze physical situations in about the fourteenth century. However, a great deal of the credit for developing the methods used with coordinate systems should be given to the French mathematician Rene Descartes (1596-1650). Descartes was the first to realize that using a coordinate system would allow for the use of algebra in geometry. Since then, this idea has become important in the development of mathematical ideas in many areas. For our purposes, using algebra in this way leads us to the consideration of ideas involving vectors that otherwise would not be possible.

Part 1: What are Cartesian (algebraic) Vectors?

Suppose \vec{u} is any vector with endpoints Q and R. We identify \overrightarrow{QR} as a Cartesian vector because its endpoints can be defined using Cartesian coordinates.

If we translate \vec{u} so that its tail is at the origin, O, then its head will be at some point $P(a, b)$. Then we define this Cartesian vector as position vector $[a, b]$.



Note: Use square brackets to distinguish between a point (a, b) and a position vector $[a, b]$

Part 2: Resolving Cartesian Vectors in to Unit Vectors

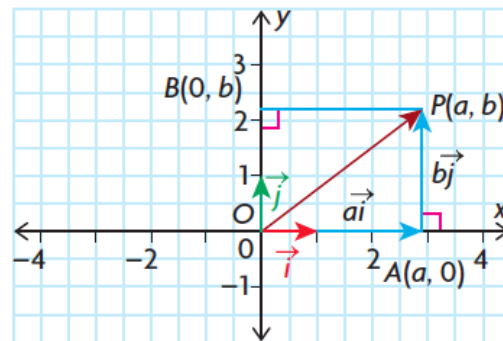
A second way of writing $\overrightarrow{OP} = [a, b]$ is with the use of unit vectors \vec{i} and \vec{j} .

$\vec{i} = [1, 0]$ and $\vec{j} = [0, 1]$

Both of these vectors have magnitude of 1 and lie along the positive x and y axes respectively.

In the diagram, $\overrightarrow{OA} = [a, 0] = a\vec{i}$

$\overrightarrow{OB} = [0, b] = b\vec{j}$



Therefore, using the triangle law of addition, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} = a\vec{i} + b\vec{j}$

It follows that $[a, b] = a\vec{i} + b\vec{j}$

Representations of Vectors in R^2

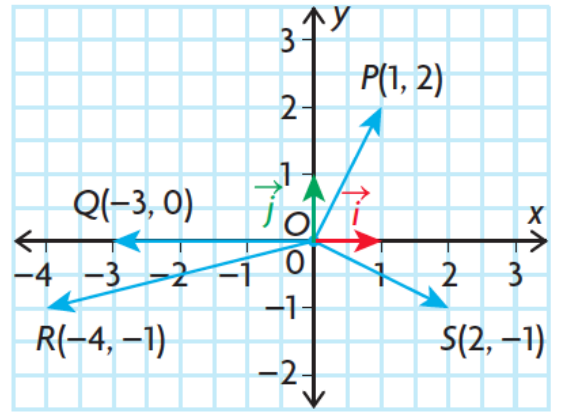
The position vector \overrightarrow{OP} can be represented as either $\overrightarrow{OP} = [a, b]$ or $\overrightarrow{OP} = a\vec{i} + b\vec{j}$ where $O(0,0)$ is the origin and $P(a, b)$ is any point on the plane.

Example 1: Practice representing vectors in two equivalent forms.

a) Four position vectors, $\overrightarrow{OP} = [1, 2]$, $\overrightarrow{OQ} = [-3, 0]$, $\overrightarrow{OR} = [-4, -1]$, and $\overrightarrow{OS} = [2, -1]$ are shown. Write each of these vectors using the unit vectors \vec{i} and \vec{j} .

$$\overrightarrow{OP} = \vec{i} + 2\vec{j} \quad \overrightarrow{OQ} = -3\vec{i}$$

$$\overrightarrow{OR} = -4\vec{i} - \vec{j} \quad \overrightarrow{OS} = 2\vec{i} - \vec{j}$$



b) The vectors $\overrightarrow{OA} = -\vec{i}$, $\overrightarrow{OB} = \vec{i} + 5\vec{j}$, and $\overrightarrow{OC} = -5\vec{i} + 2\vec{j}$ have been written using the unit vectors \vec{i} and \vec{j} . Write them in component form $[a, b]$.

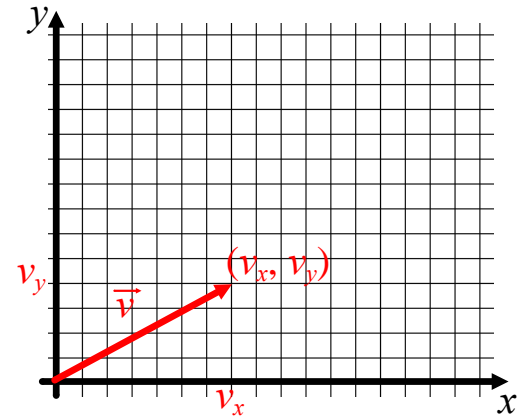
$$\overrightarrow{OA} = [-1, 0] \quad \overrightarrow{OB} = [1, 5] \quad \overrightarrow{OC} = [-5, 2]$$

Part 3: Magnitude of Vectors

Any Cartesian vector $\vec{v} = [v_x, v_y]$ can be translated so its tail is at the origin, $(0, 0)$, and its head is at the point (v_x, v_y) . To find the magnitude of the vector, use the formula for the distance between two points:

$$|\vec{v}| = \sqrt{(v_x - 0)^2 + (v_y - 0)^2}$$

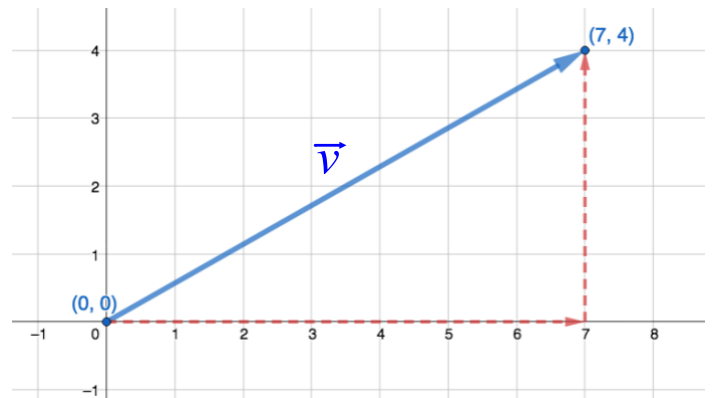
$$|\vec{v}| = \sqrt{(v_x)^2 + (v_y)^2}$$



Example 2: Find the magnitude of vector $\vec{v} = [7, 4]$

$$|\vec{v}| = \sqrt{(7)^2 + (4)^2}$$

$$|\vec{v}| = \sqrt{65}$$



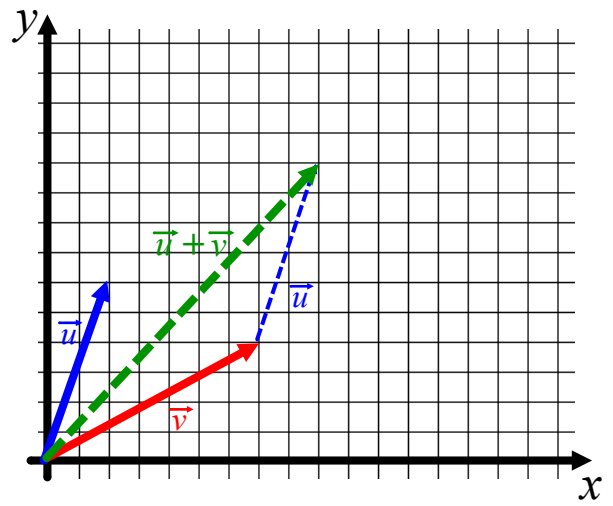
Part 4: Adding and Subtracting Vectors

Rule: $\vec{v} + \vec{u} = [v_x + u_x, v_y + u_y]$

Example 3: If $\vec{v} = [7, 4]$ and $\vec{u} = [2, 6]$. Find $\vec{v} + \vec{u}$.

$$\vec{v} + \vec{u} = [7 + 2, 4 + 6]$$

$$\vec{v} + \vec{u} = [9, 10]$$

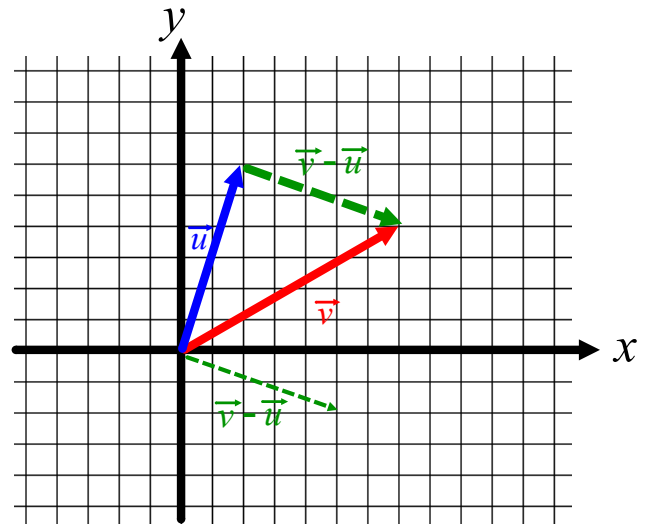


Rule: $\vec{v} - \vec{u} = [v_x - u_x, v_y - u_y]$

Example 4: If $\vec{v} = [7, 4]$ and $\vec{u} = [2, 6]$. Find $\vec{v} - \vec{u}$.

$$\vec{v} - \vec{u} = [7 - 2, 4 - 6]$$

$$\vec{v} - \vec{u} = [5, -2]$$



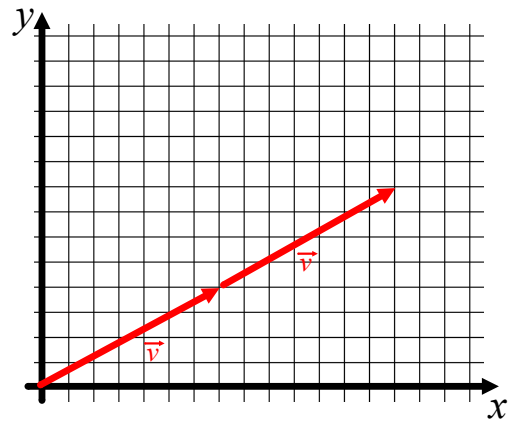
Part 5: Multiplying a Vector by a Scalar

For $\vec{v} = [v_x, v_y]$, $k\vec{v} = [kv_x, kv_y]$

Example 5: If $\vec{v} = [7, 4]$, find $2\vec{v}$.

$$2\vec{v} = [2(7), 2(4)]$$

$$2\vec{v} = [14, 8]$$



Recall: Two vectors that are **collinear** can be drawn so they lie on the same line. In other words, they are scalar multiples of one another.

$$\text{If } \vec{u} = k\vec{v} \text{ then } \frac{u_x}{v_x} = \frac{u_y}{v_y}$$

Example 6: Are $\vec{u} = \left[3, \frac{5}{6}\right]$ and $\vec{v} = \left[\frac{-3}{2}, \frac{-5}{12}\right]$

$$\frac{u_x}{v_x} = \frac{u_y}{v_y}$$

$$\frac{3}{\left(\frac{-3}{2}\right)} = \frac{\left(\frac{5}{6}\right)}{\left(\frac{-5}{12}\right)}$$

$$\frac{6}{-3} = \frac{60}{-30}$$

$$-2 = -2$$

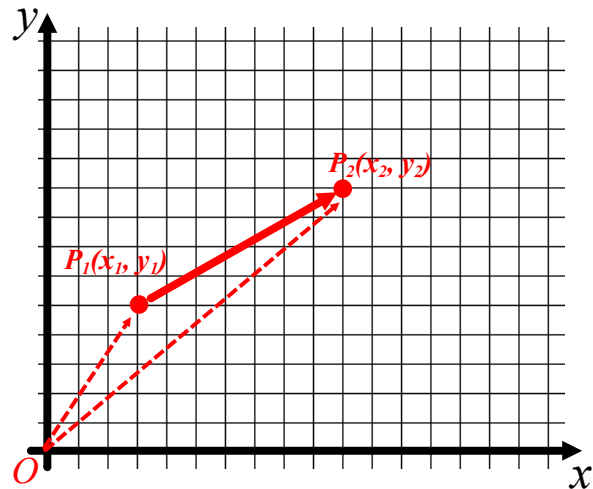
$\vec{u} = -2\vec{v}$, therefore \vec{u} and \vec{v} are collinear.

Part 6: Cartesian Vector Between 2 Points

Notice that $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$

Therefore, $\overrightarrow{P_1P_2} = [x_2 - x_1, y_2 - y_1]$

Or, $\overrightarrow{P_1P_2} = [x_{tip} - x_{tail}, y_{tip} - y_{tail}]$



Example 7: Find the coordinates and the magnitude of each vector.

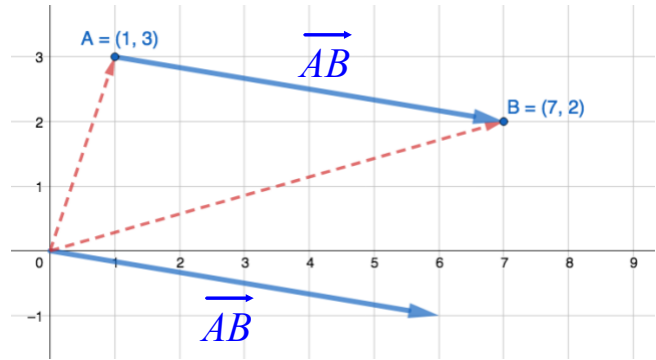
a) \overrightarrow{AB} , for $A(1,3)$ and $B(7,2)$

$$\overrightarrow{AB} = [7 - 1, 2 - 3]$$

$$\overrightarrow{AB} = [6, -1]$$

$$|\overrightarrow{AB}| = \sqrt{(6)^2 + (-1)^2}$$

$$|\overrightarrow{AB}| = \sqrt{37}$$



b) \overrightarrow{CD} , for $C(-10,0)$ and $D(0,10)$

$$\overrightarrow{CD} = [0 - (-10), 10 - 0]$$

$$\overrightarrow{CD} = [10, 10]$$

$$|\overrightarrow{CD}| = \sqrt{(10)^2 + (10)^2}$$

$$|\overrightarrow{CD}| = \sqrt{200}$$

$$|\overrightarrow{CD}| = 10\sqrt{2}$$

c) \overrightarrow{EF} , for $E(4, -3)$ and $F(1, -7)$

$$\overrightarrow{EF} = [1 - 4, -7 - (-3)]$$

$$\overrightarrow{EF} = [-3, -4]$$

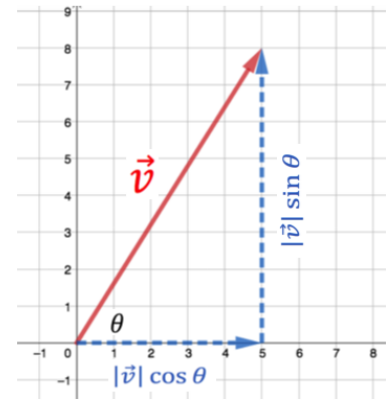
$$|\overrightarrow{EF}| = \sqrt{(-3)^2 + (-4)^2}$$

$$|\overrightarrow{EF}| = \sqrt{25}$$

$$|\overrightarrow{EF}| = 5$$

Part 7: Geometric Vectors in Cartesian Form

To write a geometric vector \vec{v} in Cartesian form, you need to use trigonometry. In the previous unit, we found that the magnitude of the horizontal component is $|\vec{v}| \cos \theta$, and the magnitude of the vertical component is $|\vec{v}| \sin \theta$, where θ is the angle \vec{v} makes with the horizontal, or the positive x -axis.



Example 8: Write a force of 200 N at 20° to the horizontal in Cartesian form.

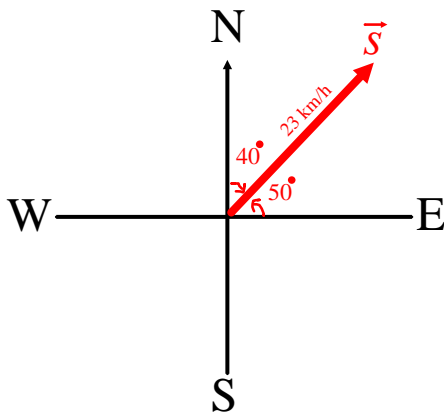
$$\vec{F} = [|\vec{F}| \cos \theta, |\vec{F}| \sin \theta]$$

$$\vec{F} = [200 \cos 20, 200 \sin 20]$$

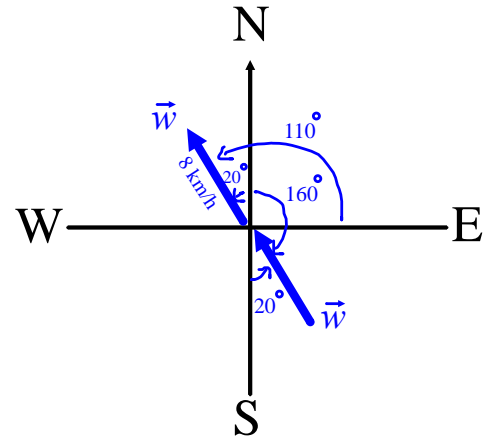
$$\vec{F} \cong [187.9, 68.4]$$

Example 9: A ship's course is set to travel at 23 km/h, relative to the water, on a heading of 040° . A current of 8 km/h is flowing from a bearing of 160° .

a) Write each vector as a cartesian (algebraic) vector.



$$\vec{s} = [23 \cos 50, 23 \sin 50]$$



$$\vec{w} = [8 \cos 110, 8 \sin 110]$$

b) Determine the resultant velocity of the ship.

$$\vec{r} = \vec{s} + \vec{w}$$

$$\vec{r} = [23 \cos 50 + 8 \cos 110, 23 \sin 50 + 8 \sin 110]$$

$$\vec{r} \cong [12.048, 25.137]$$

Magnitude:

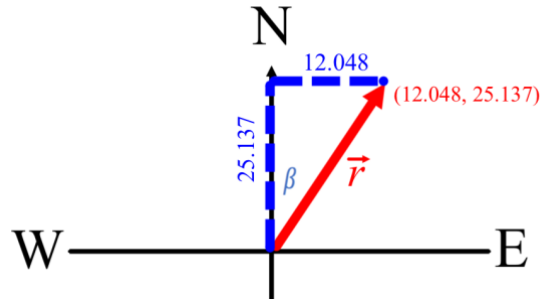
$$|\vec{r}| \cong \sqrt{(12.048)^2 + (25.137)^2}$$

$$|\vec{r}| \cong 27.875$$

Direction:

$$\beta = \tan^{-1}\left(\frac{12.048}{25.137}\right)$$

$$\beta \cong 25.6^\circ$$



The ship is travelling at a velocity of about 27.875 km/h at a bearing of 025.6° .

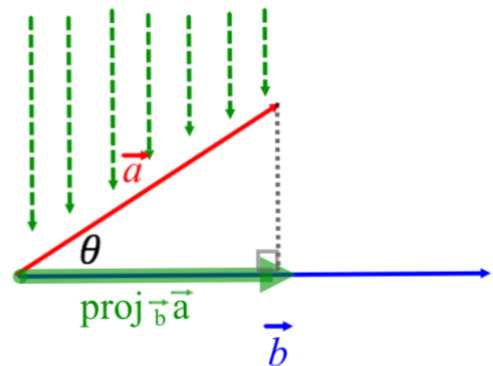
Part 1: Dot Product of Geometric Vectors

The dot product of two vectors is the product of the magnitude of one vector, vector \vec{b} , with the magnitude of the other vector, \vec{a} that is applied in the same direction as \vec{b} . To determine the magnitude of \vec{a} that is applied in the same direction as \vec{b} , we can consider the projection of vector \vec{a} on to vector \vec{b} ($\text{proj}_{\vec{b}} \vec{a}$).

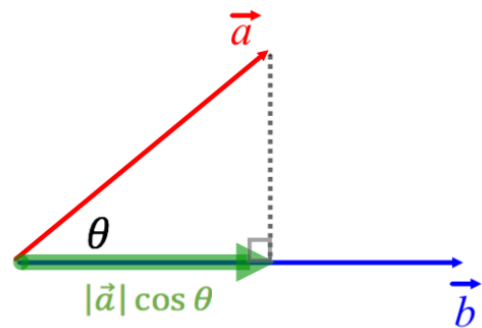
In other words, the dot product of $\vec{a} \cdot \vec{b} = |\vec{b}| |\text{proj}_{\vec{b}} \vec{a}|$

Notice: $\cos \theta = \frac{|\text{proj}_{\vec{b}} \vec{a}|}{|\vec{a}|}$, therefore $|\text{proj}_{\vec{b}} \vec{a}| = |\vec{a}| \cos \theta$

From this, the dot product of $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$

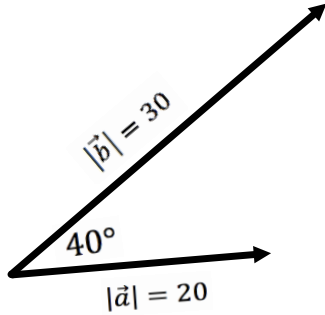
**The Dot Product:**

For two vectors \vec{a} and \vec{b} , the dot product is defined as $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$, where θ is the angle between \vec{a} and \vec{b} when the vectors are arranged tail to tail, and $0 \leq \theta \leq 180^\circ$. The dot product is a scalar, not a vector, and the units depend on the application.



Example 1: Determine the dot product of each pair of vectors.

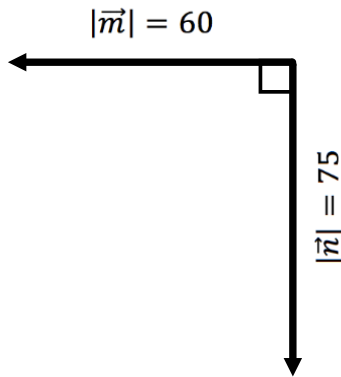
a)



$$\vec{a} \cdot \vec{b} = (30)(20) \cos 40$$

$$\vec{a} \cdot \vec{b} \cong 459.6$$

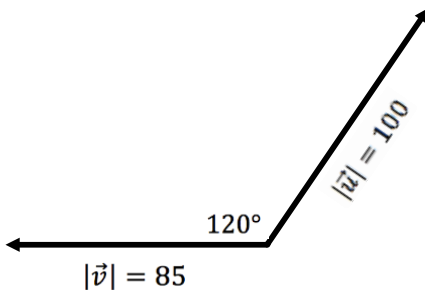
b)



$$\vec{m} \cdot \vec{n} = (60)(75) \cos 90$$

$$\vec{m} \cdot \vec{n} = 0$$

c)



$$\vec{u} \cdot \vec{v} = (85)(100) \cos 120$$

$$\vec{u} \cdot \vec{v} = -4250$$

Properties of the Dot Product

- For non-zero vectors \vec{u} and \vec{v} , \vec{u} and \vec{v} are perpendicular if and only if $\vec{u} \cdot \vec{v} = 0$
- For any vectors \vec{u} and \vec{v} , $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$. This is the commutative property.
- For any vector \vec{u} , $\vec{u} \cdot \vec{u} = |\vec{u}|^2$
- For any vectors \vec{u} and \vec{v} and scalar $k \in \mathbb{R}$, $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$. This is the associative property of the dot product.
- For any vectors \vec{u} , \vec{v} , and \vec{w} , $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$. This is the distributive property of the dot product.

Based on the angle θ , we can predict whether our answer will be +, -, or 0:

$$\text{If } \theta < 90^\circ \text{ then } \vec{u} \cdot \vec{v} > 0$$

$$\text{If } \theta > 90^\circ \text{ then } \vec{u} \cdot \vec{v} < 0$$

$$\text{If } \theta = 90^\circ \text{ then } \vec{u} \cdot \vec{v} = 0$$

Part 2: Dot Product of Cartesian Vectors

The dot product of two Cartesian vectors $\vec{a} = [a_x, a_y]$ and $\vec{b} = [b_x, b_y]$ is $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y$

Example 2: Calculate $\vec{u} \cdot \vec{v}$

a) $\vec{u} = [5, -3], \vec{v} = [4, 7]$

$$\vec{u} \cdot \vec{v} = 5(4) + (-3)(7)$$

$$\vec{u} \cdot \vec{v} = -1$$

b) $\vec{u} = [-2, 9], \vec{v} = [-1, 0]$

$$\vec{u} \cdot \vec{v} = (-2)(-1) + (9)(0)$$

$$\vec{u} \cdot \vec{v} = 2$$

Part 3: Mechanical Work

One of the applications of the dot product is to calculate the mechanical work (or simply the work) performed. Mechanical work is the product of the magnitude of the displacement travelled by an object and the magnitude of the force applied in the direction of the motion. The units are newton-meters ($N \cdot m$), also known as joules (J).

Example 3: Max is pulling his golf cart up a hill with a force of 120 N at an angle of 20° to the surface of the hill. This hill is 100 meters long. Find the work that Max performs.

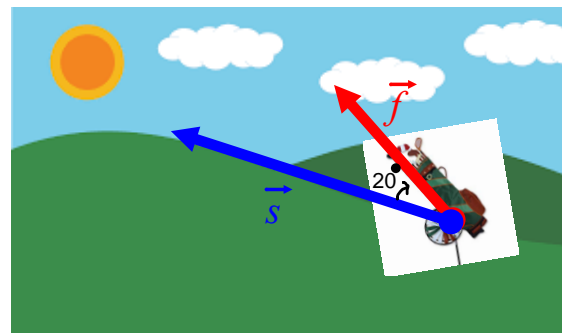
Let \vec{f} represent the force

Let \vec{s} represent the displacement along the hill

The work done equals $\vec{f} \cdot \vec{s}$

$$W = (120)(100) \cos 20$$

$$W \cong 11276.3 \text{ Joules}$$



Warm-Up

Example 1: A desk is pushed with a force of 50 N at an angle of 45 degrees below the horizontal. If the desk is pushed 5 meters, how much work is done?

$$W = \vec{f} \cdot \vec{s}$$

$$W = |\vec{f}| |\vec{s}| \cos \theta$$

$$W = 50(5) \cos 45$$

$$W = 250 \left(\frac{1}{\sqrt{2}} \right)$$

$$W = \frac{250}{\sqrt{2}} \text{ or } 125\sqrt{2} \text{ joules}$$

Remember: Mechanical work is the product of the magnitude of the displacement travelled by an object and the magnitude of the force applied in the direction of the motion.

Part 1: Angle Between 2 Vectors

To determine the angle between two vectors, you can rearrange the dot product formula, $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$, to isolate $\cos \theta$:

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

Example 2: Determine the angle between each pair of vectors.

a) $\vec{g} = [5, 1]$ and $\vec{h} = [-3, 8]$

b) $\vec{a} = [-3, 6]$ and $\vec{b} = [4, 2]$

$$\cos \theta = \frac{\vec{g} \cdot \vec{h}}{|\vec{g}| |\vec{h}|}$$

$$\cos \theta = \frac{5(-3) + 1(8)}{(\sqrt{5^2 + 1^2})(\sqrt{(-3)^2 + 8^2})}$$

$$\cos \theta = \frac{-7}{(\sqrt{26})(\sqrt{73})}$$

$$\theta \cong 99.2^\circ$$

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

$$\cos \theta = \frac{(-3)(4) + 6(2)}{(\sqrt{(-3)^2 + 6^2})(\sqrt{4^2 + 2^2})}$$

$$\cos \theta = \frac{0}{(\sqrt{45})(\sqrt{20})}$$

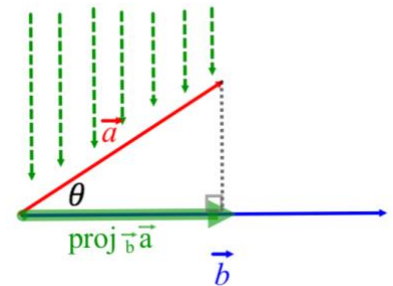
$$\theta = 90^\circ$$

Part 2: Vector Projections

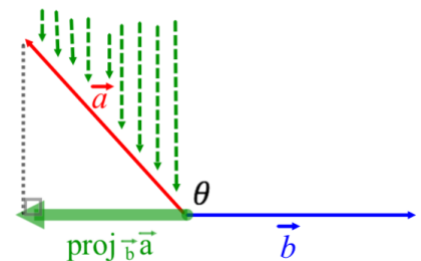
You can think of a vector projection like a shadow. The vertical arrows in the diagrams represent light from above.

Think of the projection of \vec{a} on \vec{b} as the shadow that \vec{a} casts on \vec{b} .

If the angle between \vec{a} and \vec{b} is less than 90° , then the projection of \vec{a} on \vec{b} , or $proj_{\vec{b}} \vec{a}$, is the vector component of \vec{a} in the direction of \vec{b} .

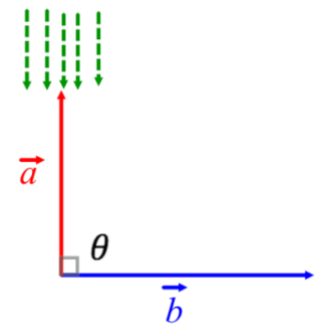


If the angle between \vec{a} and \vec{b} is between 90° and 180° , the direction of $proj_{\vec{b}} \vec{a}$ is in the opposite direction of \vec{b} .



If \vec{a} is perpendicular to \vec{b} , then \vec{a} casts 'no shadow' on to \vec{b} . So if $\theta = 90^\circ$, $proj_{\vec{b}} \vec{a} = 0$.

Note: This is why the dot product $\vec{a} \cdot \vec{b}$ would be zero for perpendicular vectors.



Formulas for Vector Projection:

Geometric Formulas:

$$proj_{\vec{b}} \vec{a} = |\vec{a}| \cos \theta (\hat{b})$$

$$proj_{\vec{b}} \vec{a} = |\vec{a}| \cos \theta \left(\frac{\vec{b}}{|\vec{b}|} \right)$$

OR

Cartesian Formulas:

$$proj_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} (\hat{b})$$

$$proj_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} \left(\frac{\vec{b}}{|\vec{b}|} \right)$$

$$proj_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{b \cdot b} (\vec{b})$$

Formulas for Magnitude of Vector Projection:

If $0^\circ < \theta < 90^\circ$ $|proj_{\vec{b}} \vec{a}| = |\vec{a}| \cos \theta$

OR

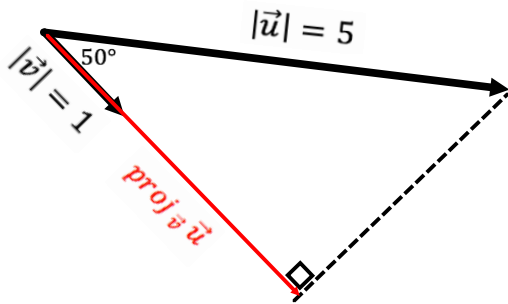
$$|proj_{\vec{b}} \vec{a}| = \left| \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} \right|$$

If $90^\circ < \theta < 180^\circ$ $|proj_{\vec{b}} \vec{a}| = -|\vec{a}| \cos \theta$

Note: $\frac{\vec{b}}{|\vec{b}|}$ is a unit vector in the direction of \vec{b} . Sometimes the symbol \hat{b} is used to denote a unit vector in the direction of \vec{b} .

Example 3: Determine the following projections of one vector on another.

a) Determine the projection of \vec{u} on \vec{v}



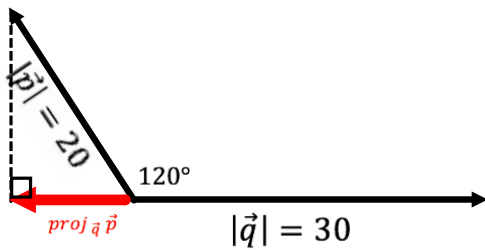
$$proj_{\vec{v}} \vec{u} = |\vec{u}| \cos \theta(\hat{v})$$

$$proj_{\vec{v}} \vec{u} = 5 \cos 50(\hat{v})$$

$$proj_{\vec{v}} \vec{u} \cong 3.2\hat{v}$$

3.2 units in the same direction as \vec{v}

b) Determine $proj_{\vec{q}} \vec{p}$



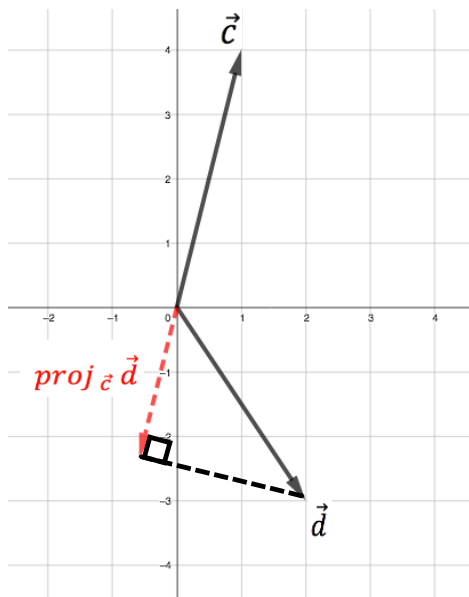
$$proj_{\vec{q}} \vec{p} = |\vec{p}| \cos \theta(\hat{q})$$

$$proj_{\vec{q}} \vec{p} = 20 \cos 120(\hat{q})$$

$$proj_{\vec{q}} \vec{p} \cong -10\hat{q}$$

10 units in the opposite direction as \vec{q}

c) Determine the projection of $\vec{d} = [2, -3]$ on $\vec{c} = [1, 4]$



$$proj_{\vec{c}} \vec{d} = \frac{\vec{d} \cdot \vec{c}}{\vec{c} \cdot \vec{c}} (\vec{c}) = \frac{\vec{d} \cdot \vec{c}}{|\vec{c}|^2} (\vec{c})$$

$$proj_{\vec{c}} \vec{d} = \left(\frac{2(1) + (-3)(4)}{1^2 + 4^2} \right) [1, 4]$$

$$proj_{\vec{c}} \vec{d} = \left(\frac{-10}{17} \right) [1, 4]$$

$$proj_{\vec{c}} \vec{d} = \left[\frac{-10}{17}, \frac{-40}{17} \right]$$

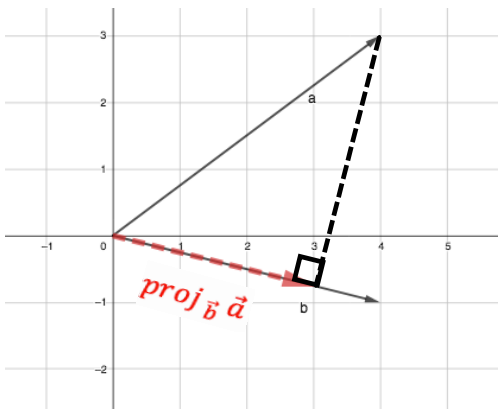
d) Find the magnitude of the projection of $\vec{a} = [4,3]$ on $\vec{b} = [4, -1]$

$$|\text{proj}_{\vec{b}} \vec{a}| = \frac{|\vec{a} \cdot \vec{b}|}{|\vec{b}|}$$

$$|\text{proj}_{\vec{b}} \vec{a}| = \frac{|4(4)+(3)(-1)|}{\sqrt{(4)^2+(-1)^2}}$$

$$|\text{proj}_{\vec{b}} \vec{a}| = \frac{13}{\sqrt{17}}$$

e) Find the projection of $\vec{a} = [4,3]$ on $\vec{b} = [4, -1]$



$$\text{proj}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} (\vec{b})$$

$$\text{proj}_{\vec{b}} \vec{a} = \frac{4(4)+(3)(-1)}{(4)^2+(-1)^2} [4, -1]$$

$$\text{proj}_{\vec{b}} \vec{a} = \frac{13}{17} [4, -1]$$

$$\text{proj}_{\vec{b}} \vec{a} = \left[\frac{52}{17}, \frac{-13}{17} \right]$$

Part 3: Dot Product with Sales

Example 4: A shoe store sold 350 pairs of Nike shoes and 275 pairs of Adidas shoes in a year. Nike shoes sell for \$175 and Adidas shoes sell for \$250.

a) Write a Cartesian vector, \vec{s} , to represent the numbers of pairs of shoes sold.

$$\vec{s} = [350, 275]$$

b) Write a Cartesian vector, \vec{p} , to represent the prices of the shoes.

$$\vec{p} = [175, 250]$$

c) Find the dot product $\vec{s} \cdot \vec{p}$. What does this dot product represent?

$$\vec{s} \cdot \vec{p} = 350(175) + 275(250)$$

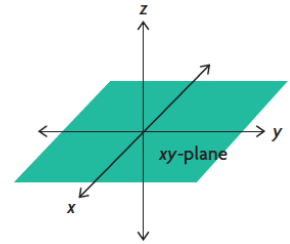
$$\vec{s} \cdot \vec{p} = 130\,000$$

The dot product represents the revenue, \$130 000, from sales of the shoes.

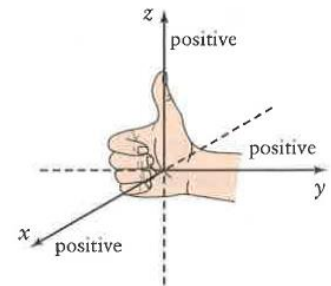
Part 1: Plotting Points in 3-Dimensions

<https://www.geogebra.org/3d?lang=en>

In placing points in 3-Dimensions (R^3), we choose three axes called x -, y -, and z -axis. Each axis is perpendicular. Each point is written using ordered triples (x, y, z) .

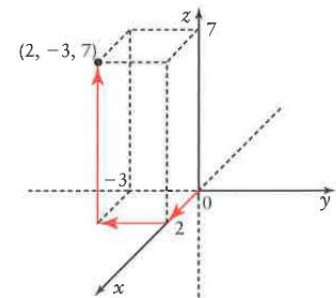


There are several ways to choose the orientation of the positive axes, but we will use what is called the right-handed system. If we imagine ourselves looking down the positive z -axis onto the xy plane so that, when the positive x -axis is rotated 90° counterclockwise it becomes coincident with the positive y -axis, then this is called the right-handed system.



If you curl the fingers of your right hand from the positive x -axis to the positive y -axis your thumb will point along the positive z -axis.

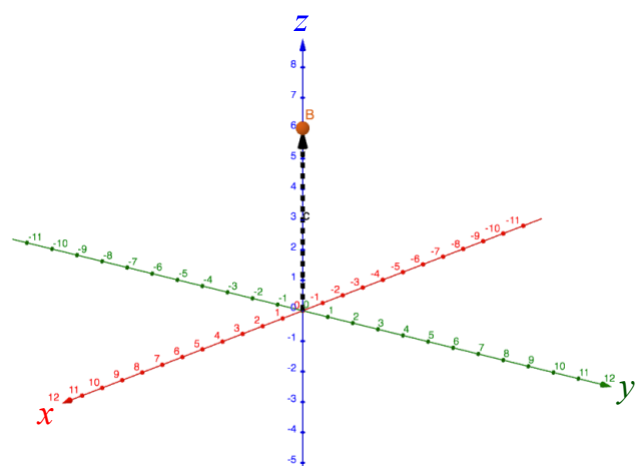
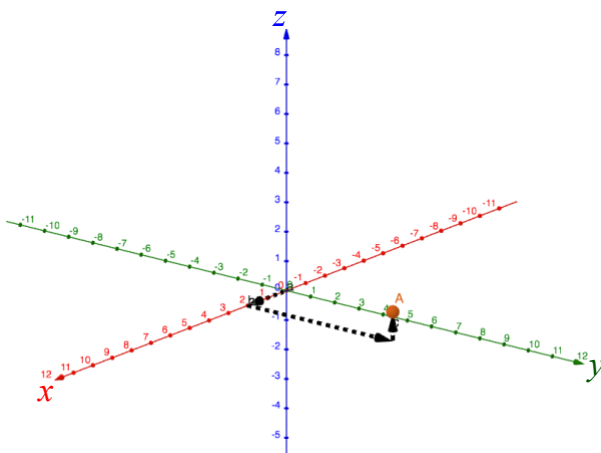
To plot the point of $(2, -3, 7)$, start at the origin. Move two units along the positive x -axis, then 3 units parallel to the negative y -axis, and then 7 units parallel to the positive z -axis.



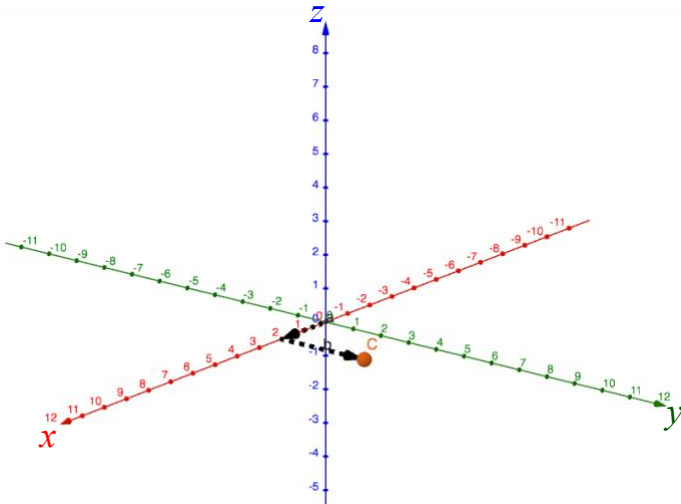
Example 1: Plot the following points in R^3

a) $A(2, 6, 1)$

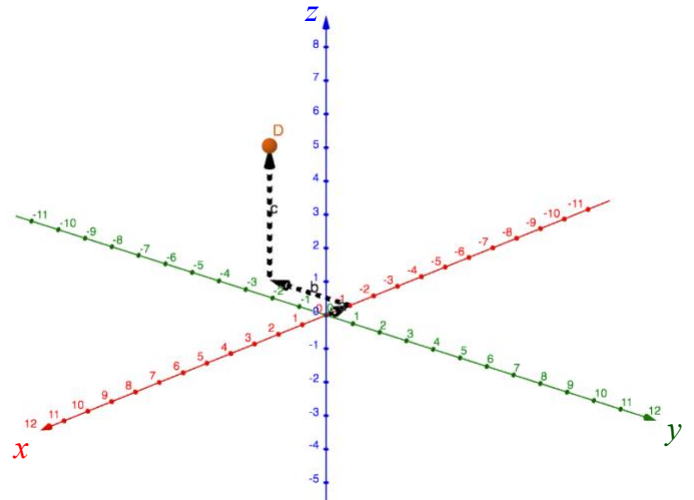
b) $B(0, 0, 6)$



c) C(2, 3, 0)

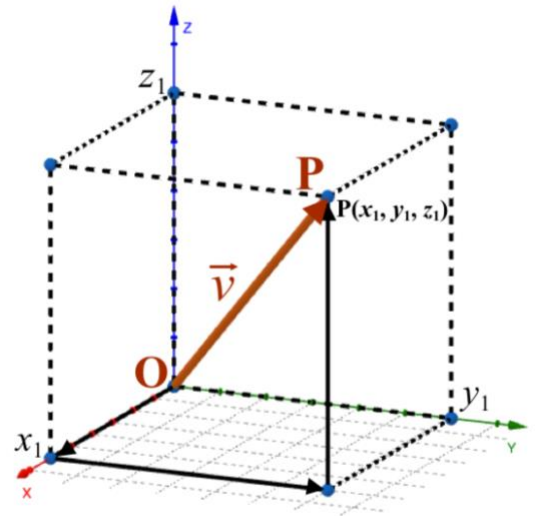


d) D(-1, -3, 4)



Part 2: 3-D Cartesian Vectors

Let \vec{v} represent a vector in space. If \vec{v} is translated so that its tail is at the origin, O , then its tip will be at some point $P(x_1, y_1, z_1)$. Then \vec{v} is the position vector of the point P , and $\vec{v} = \overrightarrow{OP} = [x_1, y_1, z_1]$.



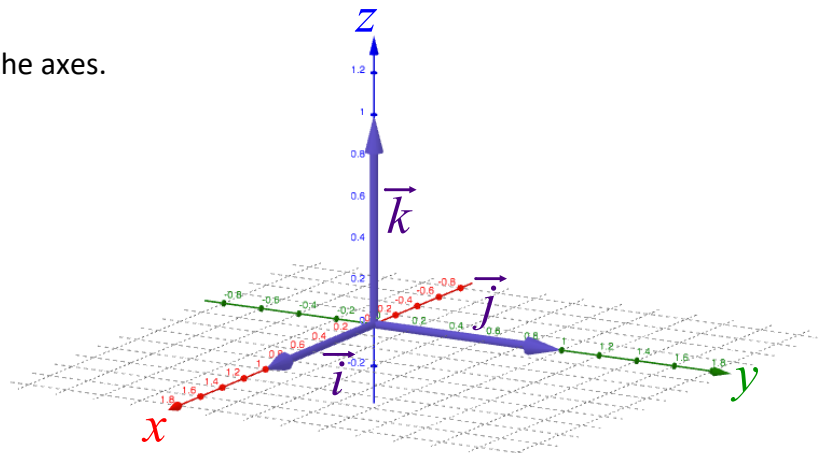
Unit Vectors in R^3

Unit vectors all have a magnitude of 1 and along the axes. In 3-Dimensions, there are 3 unit vectors:

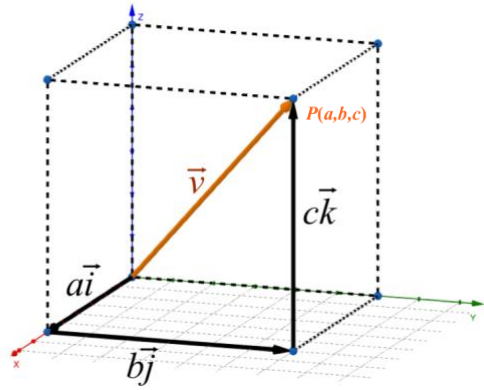
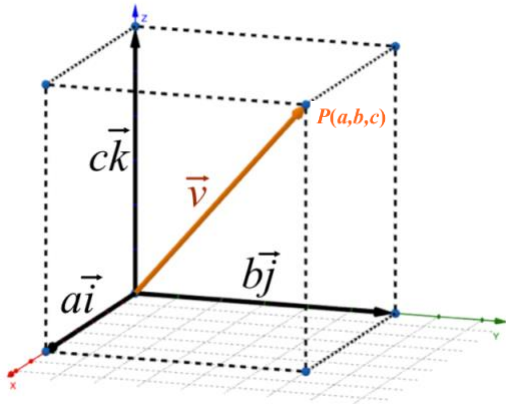
x-axis is $\vec{i} = [1,0,0]$

y-axis is $\vec{j} = [0,1,0]$

z-axis is $\vec{k} = [0,0,1]$

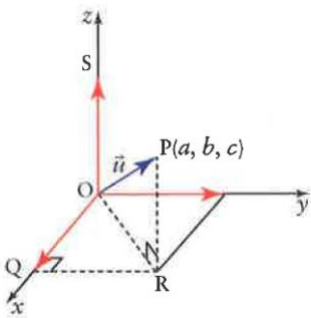


3-D vectors can be written as the sum of multiples of \vec{i} , \vec{j} , and \vec{k} .



$$\vec{v} = [a, b, c] = [a, 0, 0] + [0, b, 0] + [0, 0, c] = a\vec{i} + b\vec{j} + c\vec{k}$$

Part 3: Magnitude of Vectors in R^3



$$|\vec{u}|^2 = |\overline{OP}|^2$$

$$|\vec{u}|^2 = |\overline{OR}|^2 + |\overline{RP}|^2$$

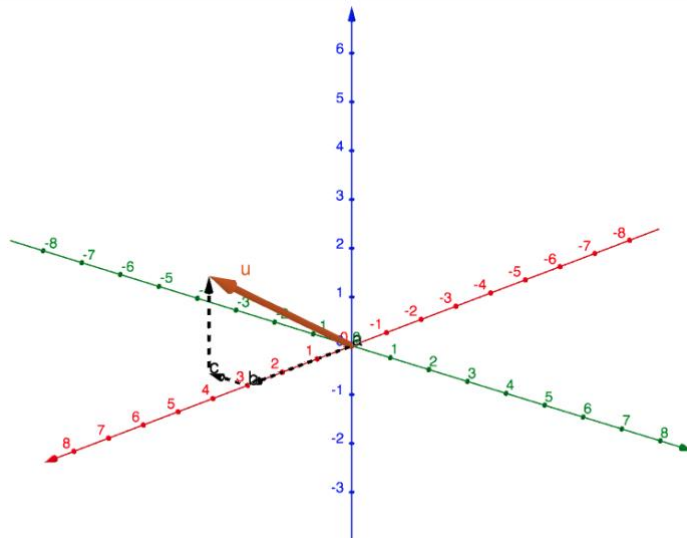
$$|\vec{u}|^2 = (a^2 + b^2) + c^2$$

$$|\vec{u}|^2 = a^2 + b^2 + c^2$$

$$|\vec{u}| = \sqrt{a^2 + b^2 + c^2}$$

Example 2: For $\vec{u} = [3, -1, 2]$...

a) sketch the position vector



b) write the vector in terms of \vec{i} , \vec{j} , and \vec{k}

$$\vec{u} = 3\vec{i} - 1\vec{j} + 2\vec{k}$$

c) find the magnitude

$$|\vec{u}| = \sqrt{(3)^2 + (-1)^2 + (2)^2}$$

$$|\vec{u}| = \sqrt{14}$$

Example 3: For the points $A(1,3,1)$ and $B(5, 4, -2)$

a) Find the magnitude of \overrightarrow{AB}

$$\overrightarrow{AB} = [5 - 1, 4 - 3, -2 - 1]$$

$$\overrightarrow{AB} = [4, 1, -3]$$

$$|\overrightarrow{AB}| = \sqrt{(4)^2 + (1)^2 + (-3)^2}$$

$$|\overrightarrow{AB}| = \sqrt{26}$$

b) Find a unit vector, \vec{u} , in the same direction as \overrightarrow{AB}

$$\vec{u} = \frac{1}{|\overrightarrow{AB}|} \overrightarrow{AB}$$

$$\vec{u} = \frac{1}{\sqrt{26}} [4, 1, -3]$$

$$\vec{u} = \left[\frac{4}{\sqrt{26}}, \frac{1}{\sqrt{26}}, \frac{-3}{\sqrt{26}} \right]$$

Vector between 2 points:

$$\overrightarrow{P_1P_2} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$$

Tools for 2-D vectors modified for 3-D vectors:

Vector Addition: $\vec{u} + \vec{v} = [u_x + v_x, u_y + v_y, u_z + v_z]$

Vector Subtraction: $\vec{u} - \vec{v} = [u_x - v_x, u_y - v_y, u_z - v_z]$

Vector between 2 points: $\overrightarrow{P_1P_2} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$

Magnitude of a vector between 2 points: $|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Dot Product: for $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$, $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

Example 4: Given the vectors $\vec{u} = [2, 3, -5]$, $\vec{v} = [8, -4, 3]$, and $\vec{w} = [-6, -2, 0]$, simplify each vector expression.

a) $-3\vec{v}$

$$= -3[8, -4, 3]$$

$$= [-24, 12, -9]$$

b) $\vec{u} + \vec{v} + \vec{w}$

$$= [2, 3, -5] + [8, -4, 3] + [-6, -2, 0]$$

$$= [2 + 8 + (-6), 3 + (-4) + (-2), -5 + 3 + 0]$$

$$= [4, -3, -2]$$

c) $|\vec{u} - \vec{v}|$

$$\vec{u} - \vec{v} = [2, 3, -5] - [8, -4, 3]$$

$$\vec{u} - \vec{v} = [2 - 8, 3 - (-4), -5 - 3]$$

$$\vec{u} - \vec{v} = [-6, 7, -8]$$

$$|\vec{u} - \vec{v}| = \sqrt{(-6)^2 + (7)^2 + (-8)^2}$$

$$|\vec{u} - \vec{v}| = \sqrt{149}$$

d) $\vec{u} \cdot \vec{v}$

$$= [2, 3, -5] \cdot [8, -4, 3]$$

$$= 2(8) + 3(-4) + (-5)(3)$$

$$= 16 - 12 - 15$$

$$= -11$$

Example 5: Determine if the vectors $\vec{a} = [6,2,4]$ and $\vec{b} = [9,3,6]$ are collinear.

Check if \vec{b} is a scalar multiple of \vec{a}

$$[6,2,4] = k[9,3,6]$$

$x:$

$$6 = 9k$$

$$k = \frac{2}{3}$$

$y:$

$$2 = 3k$$

$$k = \frac{2}{3}$$

$z:$

$$4 = 6k$$

$$k = \frac{2}{3}$$

Therefore, \vec{a} and \vec{b} are collinear.

Example 6: Find a such that $[1,2,3]$ and $[2,a,6]$ are collinear.

$$[1,2,3] = k[2,a,6]$$

$x:$

$$1 = 2k$$

$$k = \frac{1}{2}$$

$y:$

$$2 = ak$$

$$2 = a\left(\frac{1}{2}\right)$$

$$a = 4$$

$z:$

$$3 = 6k$$

$$k = \frac{1}{2}$$

Example 7: Calculate the angle between $\vec{u} = [0, -1, -4]$ and $\vec{v} = [6, 1, -2]$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

$$\cos \theta = \frac{0(6) + (-1)(1) + (-4)(-2)}{\left[\sqrt{(0)^2 + (-1)^2 + (-4)^2}\right] \left[\sqrt{(6)^2 + (1)^2 + (-2)^2}\right]}$$

$$\cos \theta = \frac{7}{(\sqrt{17})(\sqrt{41})}$$

$$\theta \cong 74.6^\circ$$

Angle between 2 vectors can be found using:

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Example 8: Find a vector that is orthogonal to $[3,4,5]$

If $[x, y, z]$ is orthogonal to $[3,4,5]$,

$$[x, y, z] \cdot [3,4,5] = 0$$

$$3x + 4y + 5z = 0$$

An infinite number of vectors would satisfy this equation. Select values for any two variables and then solve for the third.

$$3(2) + 4(1) + 5z = 0$$

$$5z = -10$$

$$z = -2$$

$[2,1,-2]$ is orthogonal to $[3,4,5]$

Shortcut:

Make one variable equal 0, then swap the other two and change one sign.

Examples: $[0,5,-4]$, $[0,-5,4]$, $[5,0,-3]$, $[-5,0,3]$, $[-4,3,0]$, and $[4,-3,0]$ would all be orthogonal to $[3,4,5]$

Example 9: Find the magnitude of the projection of $\vec{u} = [3, -3, 2]$ onto $\vec{v} = [5, 2, 0]$

$$|\text{proj}_{\vec{v}} \vec{u}| = \frac{|\vec{u} \cdot \vec{v}|}{|\vec{v}|}$$

$$|\text{proj}_{\vec{v}} \vec{u}| = \frac{|3(5) + (-3)(2) + 2(0)|}{\sqrt{5^2 + 2^2 + 0^2}}$$

$$|\text{proj}_{\vec{v}} \vec{u}| = \frac{9}{\sqrt{29}}$$

$$|\text{proj}_{\vec{v}} \vec{u}| = \frac{9}{\sqrt{29}} \cong 1.67$$

Formula reminder:

$$|\text{proj}_{\vec{b}} \vec{a}| = \frac{|\vec{a} \cdot \vec{b}|}{|\vec{b}|}$$

L5 – Cross Product of Vectors

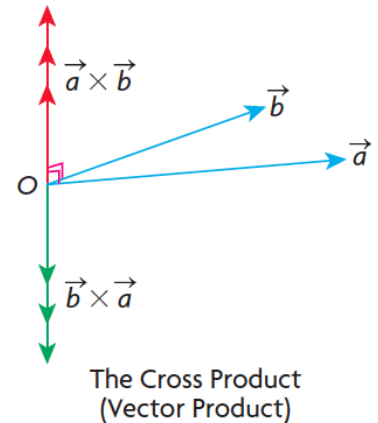
MCV4U

Jensen

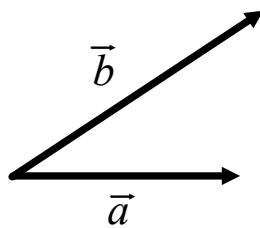
Unit 5

The cross product of two vectors \vec{a} and \vec{b} in R^3 is the vector that is **perpendicular** to these vectors such that the vectors \vec{a} , \vec{b} , and $\vec{a} \times \vec{b}$ form a right-handed system.

Right handed system tells you to point your hand along vector \vec{a} and curl your fingers towards vector \vec{b} . Your thumb will be pointing in the direction of $\vec{a} \times \vec{b}$. Notice that $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ point in opposite directions.



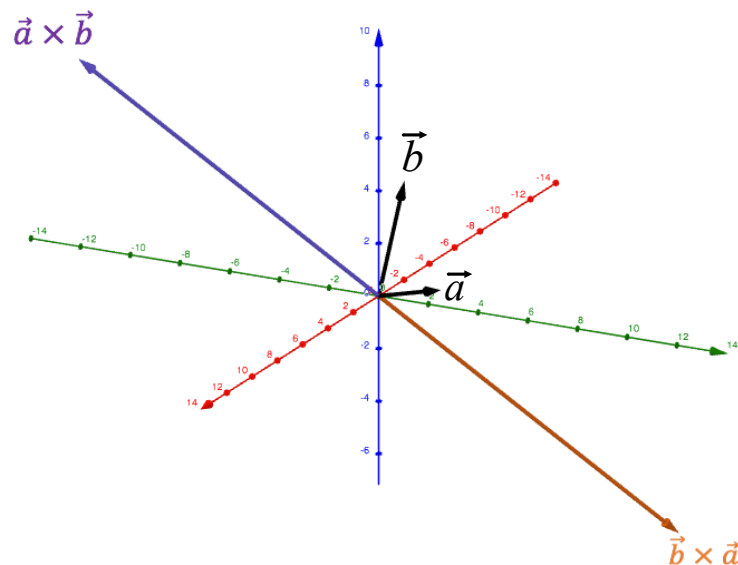
Sometimes the direction of the cross product is defined by either ‘in to the page’ or ‘out of the page’:



$\vec{a} \times \vec{b}$	$\vec{b} \times \vec{a}$
“Out of the page”	“In to the page”

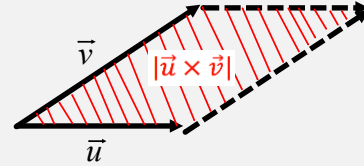
3D visualization:

<https://www.geogebra.org/3d/jyqcr3bf>



Properties of Cross Product:

- $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
- If $\vec{u} \times \vec{v} = 0$, and \vec{u} and \vec{v} are non-zero, then \vec{u} and \vec{v} are collinear.
- $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$
- $|\vec{u} \times \vec{v}| =$ the area of the parallelogram defined by \vec{u} and \vec{v}



Part 2: Cross Product of Geometric Vectors

Formula: $\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}| \sin \theta) \hat{n}$

θ is the angle between the vectors

\hat{n} is a unit vector perpendicular to both \vec{a} and \vec{b}

Example 1: If $|\vec{u}| = 30$, $|\vec{v}| = 20$, the angle between \vec{u} and \vec{v} is 40° , and \vec{u} and \vec{v} are in the plane of the page, find...

a) $\vec{u} \times \vec{v}$

$$\vec{u} \times \vec{v} = (|\vec{u}||\vec{v}| \sin \theta) \hat{n}$$

$$\vec{u} \times \vec{v} = [(30)(20) \sin 40] \hat{n}$$

$$\vec{u} \times \vec{v} \cong 385.7 \hat{n}$$

$$\vec{u} \times \vec{v} \cong 385.7 \text{ out of the page}$$

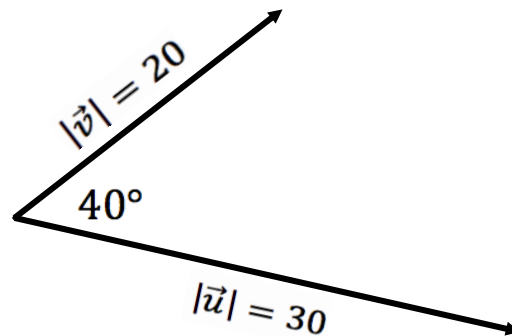
b) $\vec{v} \times \vec{u}$

$$\vec{v} \times \vec{u} = (|\vec{v}||\vec{u}| \sin \theta) \hat{n}$$

$$\vec{v} \times \vec{u} = [(20)(30) \sin 40] (-\hat{n})$$

$$\vec{v} \times \vec{u} \cong -385.7 \hat{n}$$

$$\vec{v} \times \vec{u} \cong 385.7 \text{ in to the page}$$



Part 3: Cross Product of Algebraic Vectors

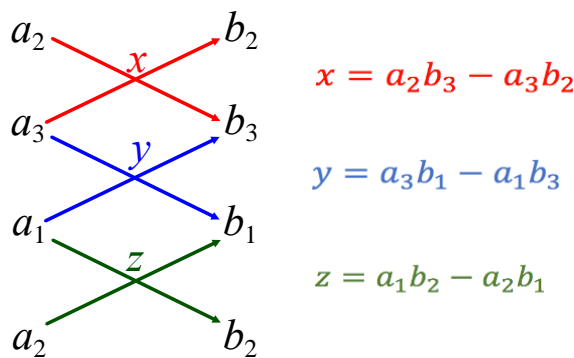
Formula:

If $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$

$$\vec{a} \times \vec{b} = [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1]$$

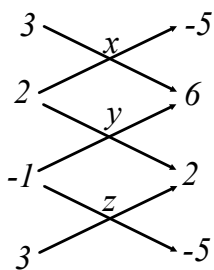
$$= (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$

How to set it up:



Example 2: If $\vec{p} = [-1, 3, 2]$ and $\vec{q} = [2, -5, 6]$, calculate each of the following:

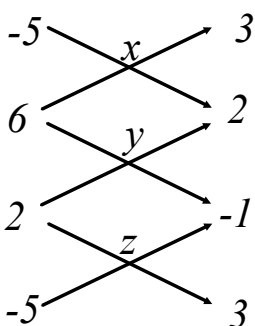
a) $\vec{p} \times \vec{q}$



$$\vec{p} \times \vec{q} = [3(6) - 2(-5), 2(2) - (-1)(6), (-1)(-5) - (3)(2)]$$

$$\vec{p} \times \vec{q} = [28, 10, -1]$$

b) $\vec{q} \times \vec{p}$

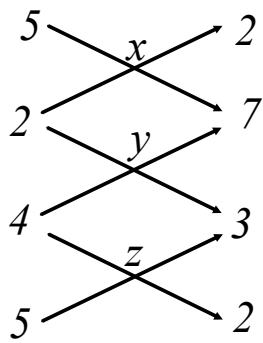


$$\vec{q} \times \vec{p} = [(-5)(2) - 6(3), 6(-1) - 2(2), 2(3) - (-5)(-1)]$$

$$\vec{q} \times \vec{p} = [-28, -10, 1]$$

Notice $\vec{q} \times \vec{p} = -1(\vec{p} \times \vec{q})$

Example 3a: Determine the area of the parallelogram defined by the vectors $\vec{u} = [4,5,2]$ and $\vec{v} = [3,2,7]$.



$$\vec{u} \times \vec{v} = [5(7) - 2(2), 2(3) - 4(7), 4(2) - 5(3)]$$

$$\vec{u} \times \vec{v} = [31, -22, -7]$$

$$|\vec{u} \times \vec{v}| = \sqrt{(31)^2 + (-22)^2 + (-7)^2}$$

$$|\vec{u} \times \vec{v}| = \sqrt{1494}$$

$$|\vec{u} \times \vec{v}| \cong 38.7 \text{ units}^2$$

Example 3b: Determine the angle between the vectors \vec{u} and \vec{v} .

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin \theta$$

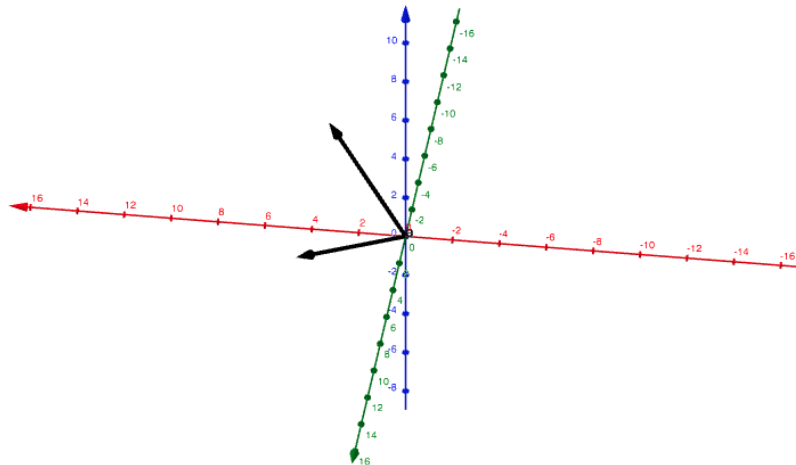
$$\sin \theta = \frac{|\vec{u} \times \vec{v}|}{|\vec{u}||\vec{v}|}$$

$$\sin \theta = \frac{\sqrt{1494}}{(\sqrt{4^2 + 5^2 + 2^2})(\sqrt{3^2 + 2^2 + 7^2})}$$

$$\sin \theta = \frac{\sqrt{1494}}{(\sqrt{45})(\sqrt{62})}$$

$$\theta \cong 47^\circ \quad \text{OR} \quad \theta \cong 180 - 47 = 133^\circ$$

From an inspection of the vectors we can tell that $\theta \cong 47^\circ$.



Part 1: Torque

Torque, τ , is a measure of the force acting on an object that causes it to rotate. Torque is the cross product of the force and the torque arm. Torque is measured in $N \cdot m$ (Newton meters).

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$|\vec{\tau}| = |\vec{r}| |\vec{F}| \sin \theta$$

\vec{F} is the force acting on the object

\vec{r} represent the arm and points from the axis of rotation to the point where the force is applied

θ is the angle between \vec{r} and \vec{F}

Use the right-hand rule to determine the direction of the torque vector.

Example 1: A wrench is used to tighten a bolt. A force of 60 N is applied in a clockwise direction at 80° to the handle, 20 cm from the centre of the bolt.

a) Calculate the magnitude of the torque

Note: make sure to convert displacement vector to meters.

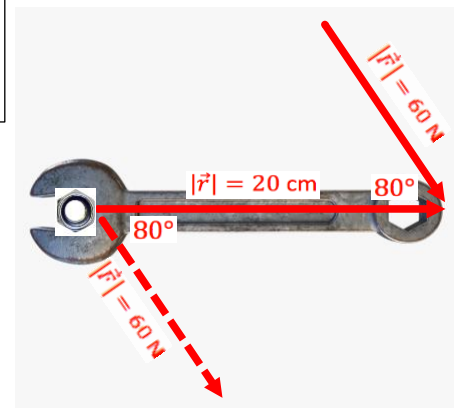
$$|\vec{\tau}| = |\vec{r}| |\vec{F}| \sin \theta$$

$$|\vec{\tau}| = (0.2)(60) \sin 80$$

$$|\vec{\tau}| \cong 11.8 \text{ N} \cdot \text{m}$$

b) In what direction does the torque vector point?

Downward into the material. Therefore, the bolt is being tightened.



Part 2: Projection and Work

Projections in R^3 are similar to projections in R^2 , so you can use the same formula: $proj_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} (\vec{b})$

Example 2: For the vectors $\vec{v} = [4,2,7]$ and $\vec{u} = [6,3,8]$

a) Find $proj_{\vec{u}} \vec{v}$

$$proj_{\vec{u}} \vec{v} = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} (\vec{u})$$

$$proj_{\vec{u}} \vec{v} = \frac{4(6)+2(3)+7(8)}{6^2+3^2+8^2} [6,3,8]$$

$$proj_{\vec{u}} \vec{v} = \frac{86}{109} [6,3,8]$$

b) Find the magnitude of the projection

$$|proj_{\vec{u}} \vec{v}| = \frac{|\vec{v} \cdot \vec{u}|}{|\vec{u}|}$$

$$|proj_{\vec{u}} \vec{v}| = \frac{86}{\sqrt{109}}$$

Or just find magnitude of vector found in part a)

$$= \left| \frac{86}{109} [6,3,8] \right|$$

$$= \frac{86}{109} |[6,3,8]|$$

$$= \frac{86}{109} \sqrt{109}$$

Reminder: Work performed is the product of the magnitude of the displacement travelled by an object and the magnitude of the force applied in the direction of the motion (use dot product to find this). The units are joules (J).

Example 3: A force with units in newtons and defined by $\vec{F} = [300,700,500]$ acts on an object with displacement, in meters, defined by $\vec{d} = [3,1,12]$.

a) Determine the work done in the direction of travel.

$$W = \vec{F} \cdot \vec{d}$$

$$W = 300(3) + 700(1) + 500(12)$$

$$W = 7600 \text{ J}$$

b) Determine the work done against gravity, which is a force in the direction of the negative z-axis.

$$W = [0,0,500] \cdot [0,0,12]$$

$$W = 0(0) + 0(0) + 500(12)$$

$$W = 6000 \text{ J}$$

Use only the vertical components to calculate the work against gravity.

Part 3: Triple Scalar Product and Volume of a Parallelepiped

The Triple Scalar Product

Certain situations require a combination of the dot and cross products. The triple scalar product, $\vec{a} \cdot \vec{b} \times \vec{c}$, is one such combination. Because the dot product is a scalar, this combination is only meaningful if the cross product is performed first.

Example 4: Consider the vectors $\vec{u} = [4,3,1]$, $\vec{v} = [2,5,6]$, and $\vec{w} = [10, -3, -14]$. Evaluate the expression $\vec{u} \times \vec{v} \cdot \vec{w}$

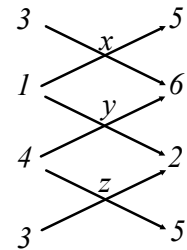
$$\vec{u} \times \vec{v} \cdot \vec{w} = [4,3,1] \times [2,5,6] \cdot [10, -3, -14]$$

$$\vec{u} \times \vec{v} \cdot \vec{w} = [3(6) - 1(5), 1(2) - 4(6), 4(5) - 3(2)] \cdot [10, -3, -14]$$

$$\vec{u} \times \vec{v} \cdot \vec{w} = [13, -22, 14] \cdot [10, -3, -14]$$

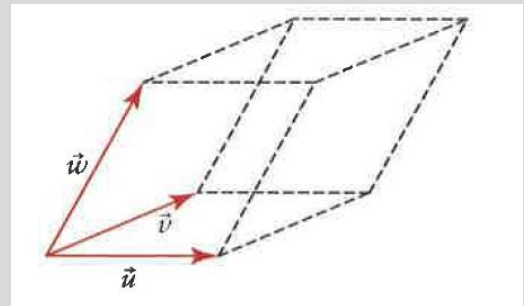
$$\vec{u} \times \vec{v} \cdot \vec{w} = 13(10) + (-22)(-3) + 14(-14)$$

$$\vec{u} \times \vec{v} \cdot \vec{w} = 0$$



Volume of a Parallelepiped

A parallelepiped is a three-dimensional figure formed by six parallelograms. In general, the volume of the parallelepiped defined by the vectors \vec{u} , \vec{v} , and \vec{w} is given by $V = |\vec{w} \cdot \vec{u} \times \vec{v}|$.



Example 5: Find the volume of the parallelepiped defined by $\vec{a} = [6,3, -2]$, $\vec{b} = [-4,6,9]$ and $\vec{c} = [3,3, -11]$

$$V = |\vec{a} \cdot \vec{b} \times \vec{c}|$$

$$V = |[6,3, -2] \cdot [-4,6,9] \times [3,3, -11]|$$

$$V = |[6,3, -2] \cdot [6(-11) - 9(3), 9(3) - (-4)(-11), (-4)(3) - 6(3)]|$$

$$V = |[6,3, -2] \cdot [-93, -17, -30]|$$

$$V = |6(-93) + 3(-17) + (-2)(-30)|$$

$$V = |-549|$$

$$V = 549 \text{ units}^3$$

