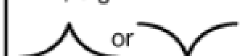
Name:

Unit 2- Curve Sketching

Lesson Package

MCV4U

graph feature	f(x)	f'(x)	f''(x)	Notes
rising (L to R)	slope > 0	+		
falling (L to R)	slope < 0	-		
extrema: maximum	slope = 0	= 0 + on L, - on R	- at x_{max}	derivative may not exist at a max or min, e.g. 
extrema: minimum	slope = 0	= 0 - on L, + on R	+ at x_{min}	
inflection pt.	Curvature changes: 		= 0 potential inflection point	Check f''(x) on either side of a potential inflection point.
concave up	 	- | +	+	
concave down	 	+ | -	-	

Unit 2 Outline

Unit Goal: Make connections, graphically and algebraically, between the key features of a function and its first and second derivatives, and use the connections in curve sketching.

Section	Subject	Learning Goals	Curriculum Expectations
L1	Increasing and Decreasing	- use the first derivative to determine when a function is increasing or decreasing	A2.1
L2	Max and Min	- use the first derivative test to see if a critical point is a max, min, or neither	A2.1
L3	Second Derivative and Concavity	- Use the second derivative to find intervals of concavity and points of inflection	B1.1,1.2,B1.4
L4	Rational Functions	- Sketch the graph of rational functions using critical points and knowledge of asymptotes	B1.3
L5	Curve Sketching	- Sketch the graph of various polynomial and rational functions using the algorithm for curve sketching	B1.5
L6	Optimization	- solve optimization problems using mathematical models and derivatives	B2.4, B2.5

Assessments	F/A/O	Ministry Code	P/O/C	KTAC
Note Completion	A		P	
Practice Worksheet Completion	F/A		P	
Quiz – Curve Sketching	F		P	
PreTest Review	F/A		P	
Test – Curve Sketching	O	A2.1, B1.1-B1.5, B2.4, B2.5	P	K(25%), T(25%), A(25%), C(25%)

L1 – Increasing / Decreasing Unit 2
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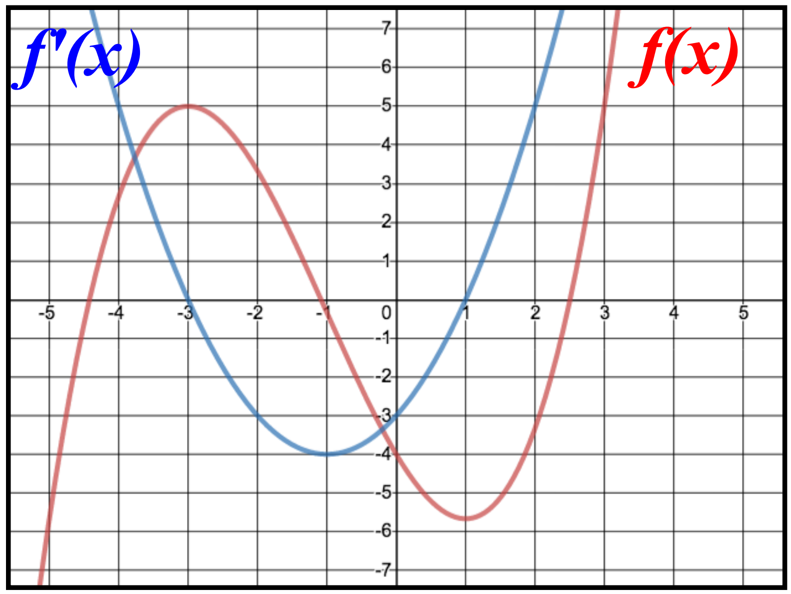
Increasing: As x-values increase, y-values are increasing

Decreasing: As x-values increase, y-values are decreasing

Part 1: Discovery

$$f(x) = \frac{1}{3}x^3 + x^2 - 3x - 4$$

$$f'(x) = x^2 + 2x - 3$$



a) Over which values of x is $f(x)$ increasing?

b) Over which values of x is $f(x)$ decreasing?

c) What is true about the graph of $f'(x)$ when $f(x)$ is increasing?

d) What is true about the graph of $f'(x)$ when $f(x)$ is decreasing?

Effects of $f'(x)$ on $f(x)$: When the graph of $f'(x)$ is positive, or above the x-axis, on an interval, then the function $f(x)$ _________________ over that interval. Similarly, when the graph of $f'(x)$ is negative, or below the x-axis, on an interval, then the function $f(x)$ ___________________ over that interval.

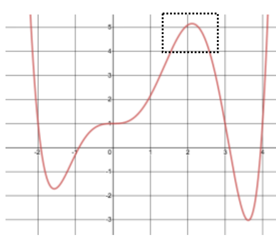
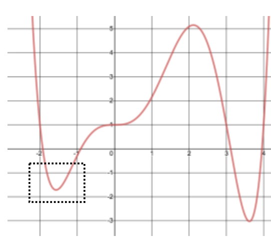
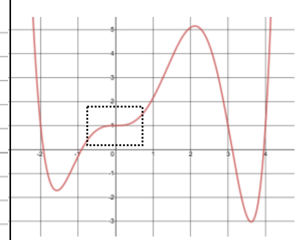
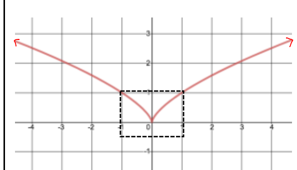
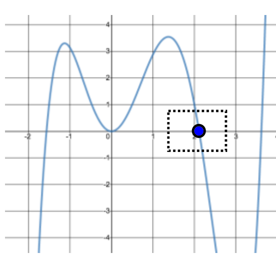
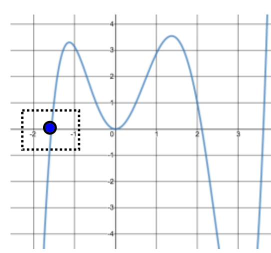
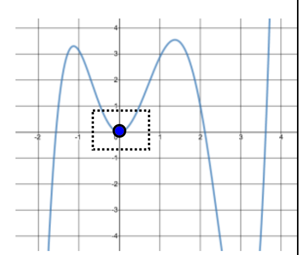
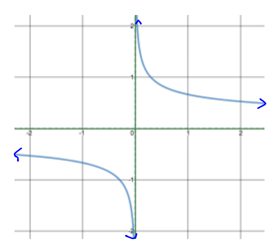
If $f'(x) > 0$ on an interval, $f(x)$ is increasing on that interval

If $f'(x) < 0$ on an interval, $f(x)$ is decreasing on that interval

Part 2: Properties of graphs of $f(x)$ and $f'(x)$

A critical number is a value $'a'$ in the domain where $f'(a) = 0$ or $f'(a)$ does not exist.

A critical number could yield…

	A local max	A local min	Neither	Max/Min at Cusp
$f(x)$				
$f'(x)$				

Conclusion:

Local extrema occur when the sign of the derivative CHANGES. If the sign of the derivative does not change, you do not have a local extrema.

A critical number of a function is a value of a in the domain of the function where either $f'(a) = 0$ or $f'(a)$ does not exist. If a is a critical number, $(a, f(a))$ is a critical point. Critical points could be local extrema but not necessarily. You must test around the critical points to see if the derivative changes sign. This is called the ____.

Example 1: Determine all local extrema for the function below using critical numbers and the first derivative test. State when the function is increasing or decreasing.

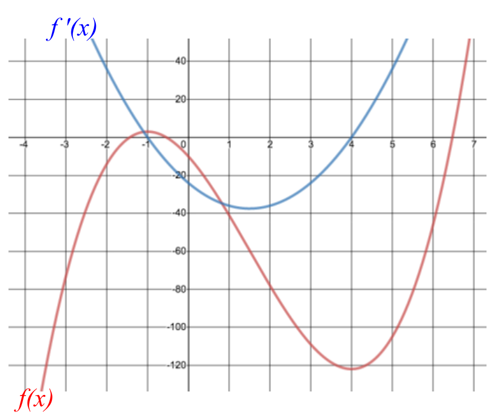
$f(x) = 2x^3 - 9x^2 - 24x - 10$

Critical Numbers: Critical Points:

Sign Chart:

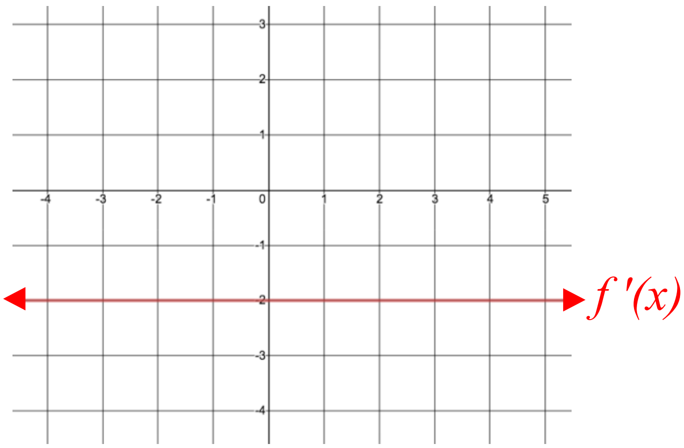
Test value for x			
$f'(x)$			
$f(x)$			

Notice how we could use the graph of the derivative to verify our solution:

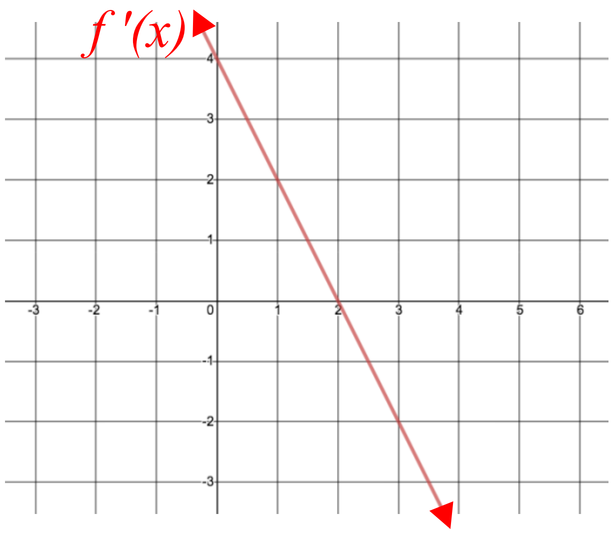


Example 2: For each function, use the graph of $f'(x)$ to sketch a possible function $f(x)$.

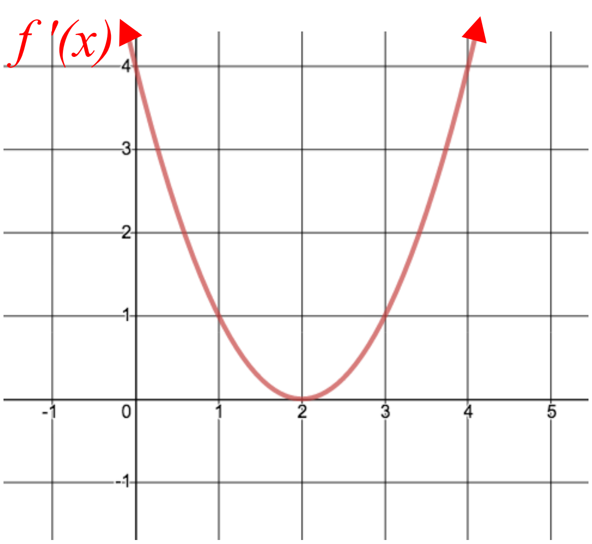
a)



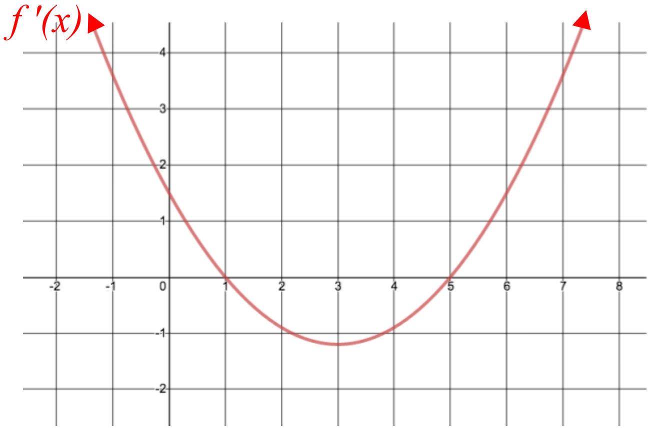
b)

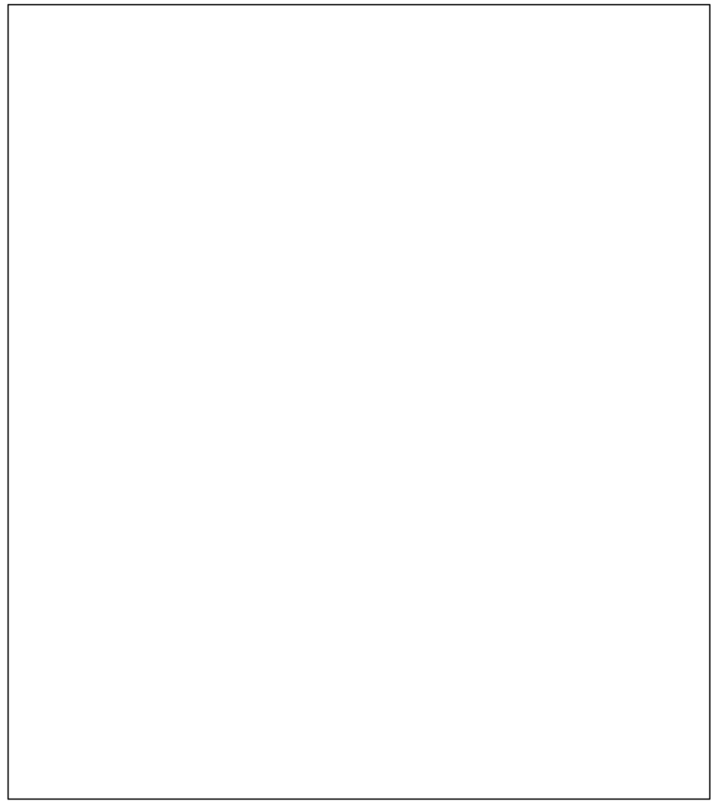


c)



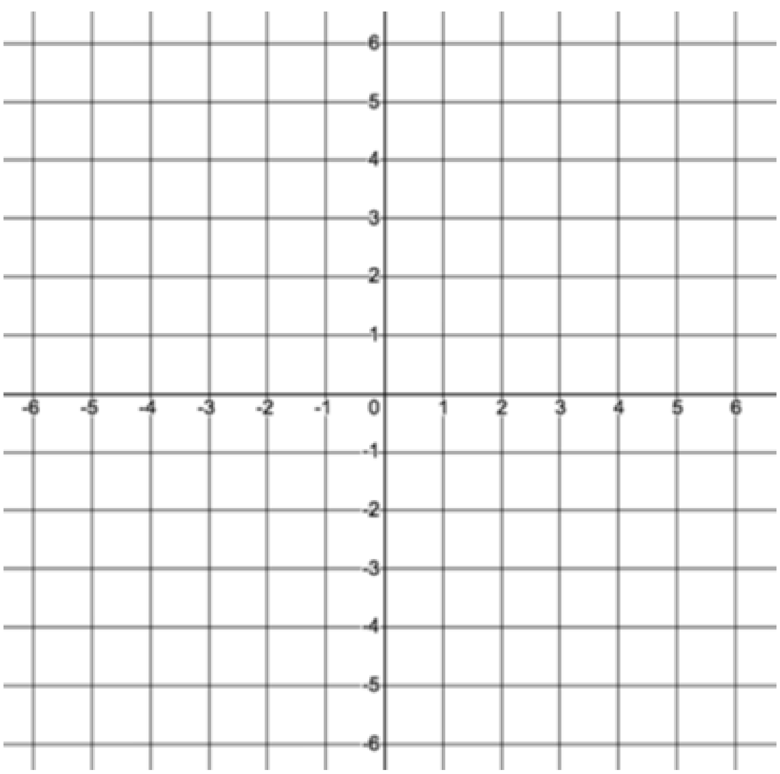
d)

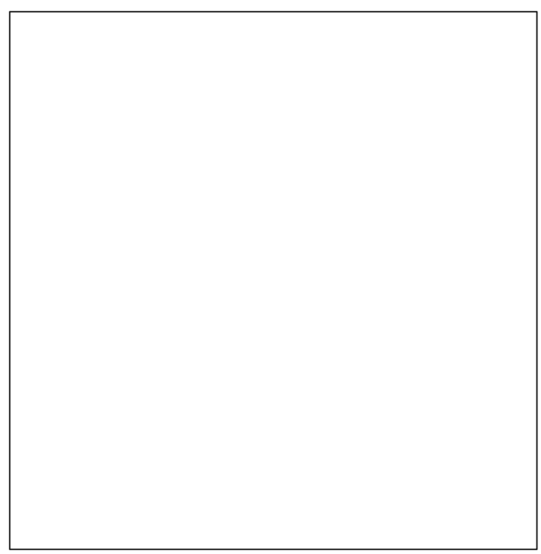




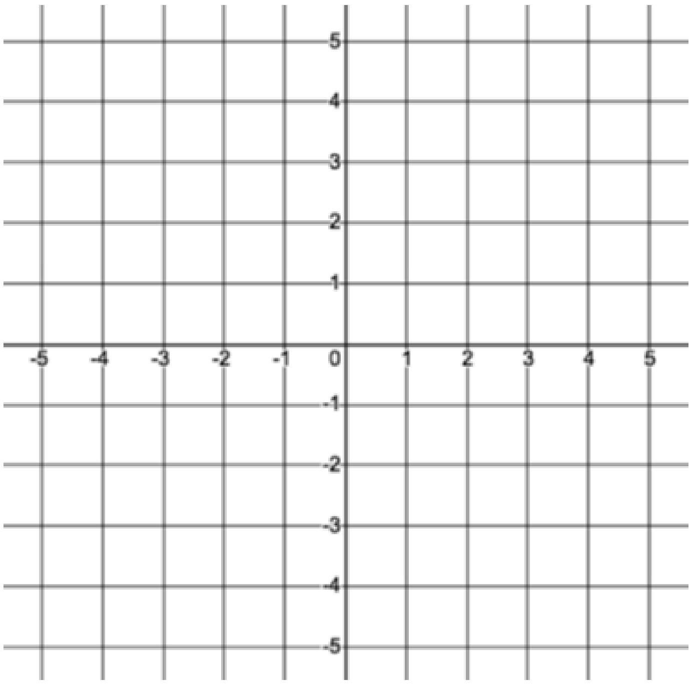
Example 3: Sketch a continuous function for each set of conditions

a) $f'(x) > 0$ when $x < 0$, $f'(x) < 0$ when $x > 0$, $f(0) = 4$





b) $f'(x) > 0$ when $x < -1$ and when $x > 2$, $f'(x) < 0$ when $-1 < x < 2$, $f(0) = 0$



Example 4: The temperature of a person with a certain strain of flu can be approximated by the function $T(d) = -\frac{5}{18}d^2 + \frac{15}{9}d + 37$, where $0 < d < 6$; T represents the person's temperature, in degrees Celsius and d is the number of days after the person first shows symptoms. During what interval will the person's temperature be increasing?

L2 – Maxima and Minima Unit 2
MCV4U
Jensen

Part 1: Review

Determine all local extrema for the function below using critical numbers and the first derivative test. State when the function is increasing and decreasing.

$f(x) = 2x^3 + 3x^2 - 36x + 5$

Remember: *Local extrema occur when the sign of the derivative CHANGES. If the sign of the derivative does not change, you have neither local extrema.*

Part 2: Local vs Absolute Extrema

Local max or min values of a function are also called local extrema or turning points.

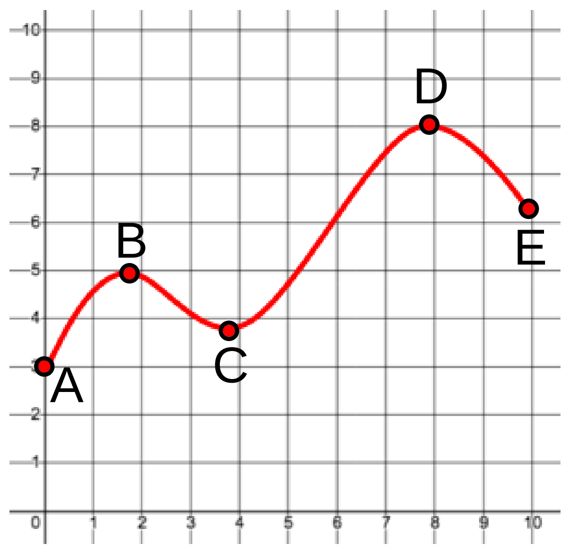
Local max: If the y-coordinate of all points in the vicinity are less than the y-coordinate of the point. The sign of the derivative would change from positive before the point, to zero at the point, to negative after.

Local min: If the y-coordinate of all points in the vicinity are greater than the y-coordinate of the point. The sign of the derivative would change from negative before the point, to zero at the point, to positive after.

Absolute max/min: A function $f(x)$ has an ABSOLUTE max or min at point a if $f(a)$ is the biggest or smallest value of $f(x)$ for ALL x in the domain.

Example 1: Consider the graph of a function on the interval $[0, 10]$.

a) Identify the local maximum points.



b) Identify the local minimum points.

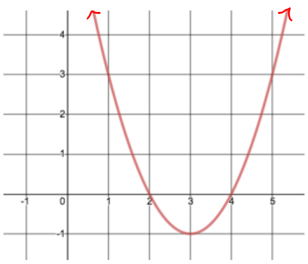
c) What do all the points identified in parts a) and b) have in common?

d) Identify the absolute max and min points in the interval $[0,10]$

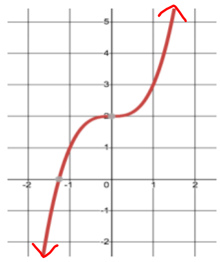
Reminder: A critical number of a function is a value of a **in the domain of the function** where either $f'(a) = 0$ or $f'(a)$ does not exist. If a is a critical number, $(a, f(a))$ is a critical point.

Scenarios for critical numbers:

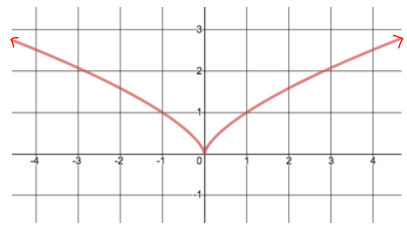
1) $f'(a) = 0$
Local extrema at $(a, f(a))$
$f(x) = x^2 - 6x + 8$



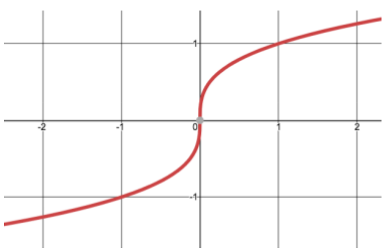
2) $f'(a) = 0$
No local extrema at $(a, f(a))$
$f(x) = x^3 + 2$



3) $f'(a)$ does not exist
local extrema at $(a, f(a))$ (cusp)
$f(x) = x^{\frac{2}{3}}$

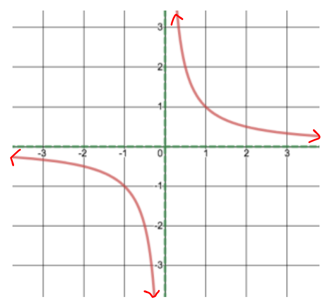


4) $f'(a)$ does not exist
No local extrema at $(a, f(a))$
$f(x) = x^{\frac{1}{3}}$



5) $\boldsymbol{f'(a)}$ **does not exist**
No local extrema
$f(a)$ does not exist either, therefore a is NOT a critical number

$f(x) = \frac{1}{x}$



To determine the absolute extrema values of a function on an interval, find the critical numbers, then substitute the critical numbers and also the x-coordinates of the endpoints of the interval into the function.

Example 2: Find the absolute max and min of the function $f(x) = x^3 - 12x - 3$ on the interval $-3 \leq x \leq 4$.

Test critical numbers AND endpoints of interval.

Example 3: The surface area of a cylindrical container is to be 100 cm^2. Its volume is given by the function $V(r) = 50r - \pi r^3$, where r is the radius of the cylinder in cm. Find the max volume of the cylinder if the radius cannot exceed 3 cm.

L3 – Concavity and the Second Derivative

Unit 2

MCV4U

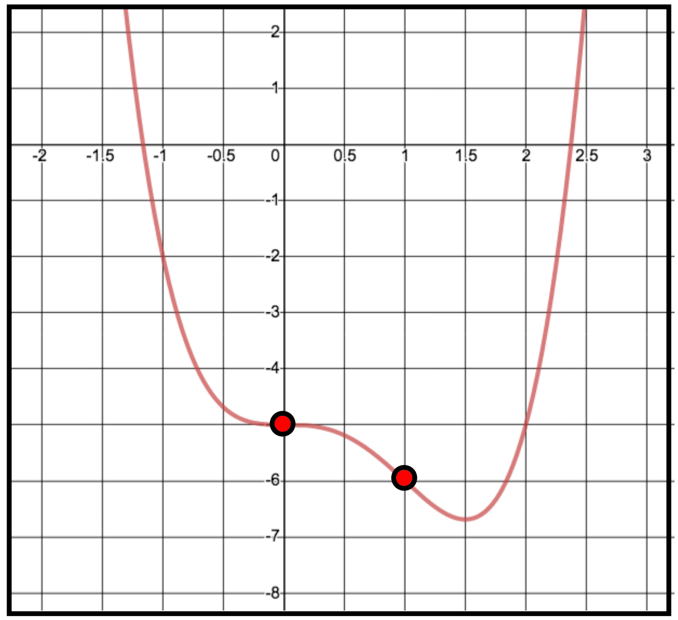
Jensen

The ___________________________ is the derivative of the first derivative. It is the rate of change of the slope of the tangent.

Part 1: Discovery

Example 1:

a) Given the graph of $f(x) = x^4 - 2x^3 - 5$



$f'(x) =$

$f''(x) =$

When is $f''(x) = 0$?

b) Use your pencil to simulate a tangent line to the function when $x = -1$. Drag the pencil slowly to the right, keeping it tangent to the curve, approaching $x = 0$. What is happening to the slope of the tangent? Is it above or below the curve? What is the value of $f''(-0.5)$?

c) Drag the pencil slowly to the right, keeping it tangent to the curve, moving through $x = 0$. What is happening to the slope of the tangent as it moves through $x = 0$? What is the value of $f''(0.5)$?

d) What happens to the slope of the tangent as it moves through $x = 1$?

Summary of findings:

How $f''(x)$ effects $f(x)$:

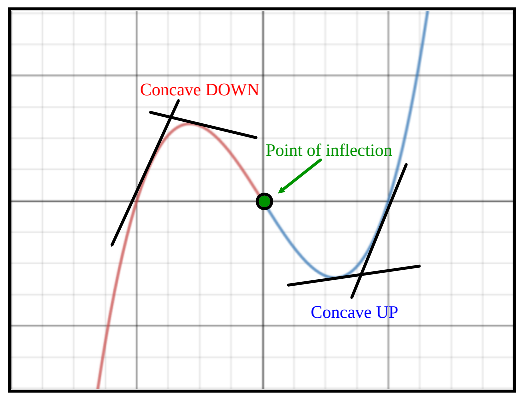
The graph of a function is concave up over an interval if the curve is above all of the tangents on the interval. The slopes of the tangent lines are increasing, therefore $f''(x) > 0$ over this interval.

The graph of a function is concave down over an interval if the curve is below all of the tangents on the interval. The slopes of the tangent lines are decreasing, therefore $f''(x) < 0$ over this interval.

$f(x)$ is concave _______ on an interval if $f''(x) > 0$ over that interval (tangent line slopes are increasing)

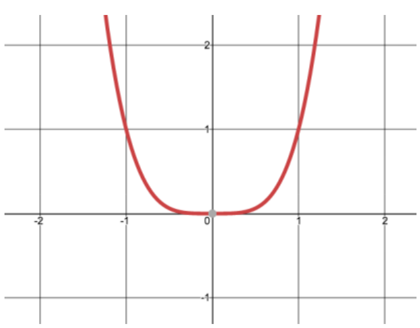
$f(x)$ is concave _______ on an interval if $f''(x) < 0$ over that interval (tangent line slopes are decreasing)

A ________________________________ is a point in the domain of the function at which the graph changes from being concave up to concave down or vice versa. The second derivative, $f''(x)$, is equal to zero at this point (or is undefined) and changes sign on either side. The tangent lines change from increasing to decreasing OR from decreasing to increasing.



However, just like that not every critical point is a local max / min, not every zero or restriction of the second derivative is an inflection point either. They are just the pool of points you need to check in order to find the inflection point(s) of a curve.

$f(x) = x^4$

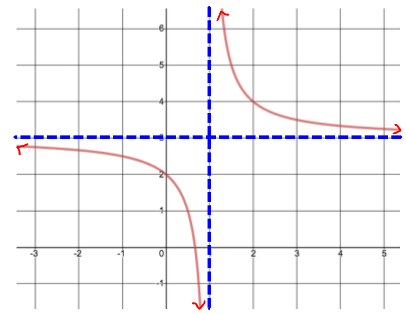


$f''(x) = 12x^2$

$f''(0) = 0$

But $x = 0$ is not a point of inflection; the function has no change in concavity. Tangent slopes are always increasing.

Note: It often happens that a graph has different concavity on the two sides of a vertical asymptote. However, because a curve is not continuous at a vertical asymptote, it can never have an inflection point there. We will look at these types of functions next lesson (rational functions).



The ___________________________________ can also be used to help check for local min/max points.

In the second derivative test we check the critical points themselves (those where $f'(x) = 0$), by evaluating $f''(x)$ AT each critical point.

If $f'(c) = 0$ and $f''(c) > 0$, then f has a local ___________________ at *c*.

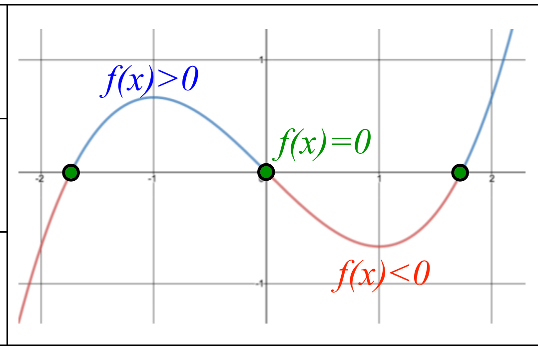
If $f'(c) = 0$ and $f''(c) < 0$, then f has a local ___________________ at *c*.

Note: *Even though it is often easier to use than the first derivative test, the second derivative test can fail at some points (eg.* $y = x^4$*). If the second derivative test fails, then the first derivative test must be used to classify the point in question.*

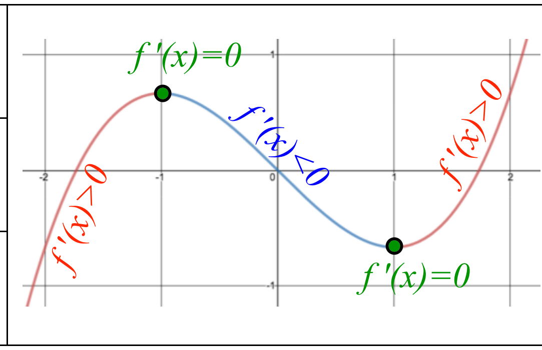
Summary Page of what we know so far

Relationship between $f(x)$, $f'(x)$, and $f''(x)$

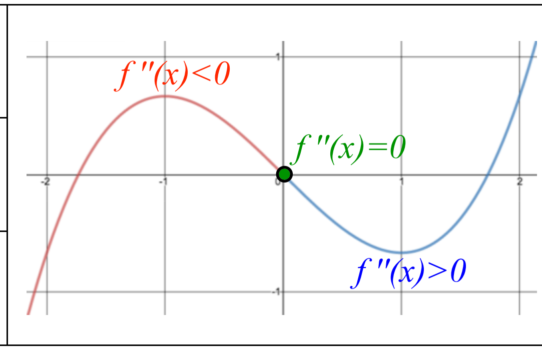
$f(x) = 0$	Zeros (x-intercepts) of the function
$f(x) > 0$	Function is positive (above x-axis)
$f(x) < 0$	Function is negative (below x-axis)



$f'(x) = 0$	Horizontal tangent; possible local extrema (turning point)
$f'(x) > 0$	$f(x)$ is increasing
$f'(x) < 0$	$f(x)$ is decreasing



$f''(x) = 0$	Possible point of inflection (change in concavity)
$f''(x) > 0$	$f(x)$ is concave up
$f''(x) < 0$	$f(x)$ is concave down



Tests of Critical Numbers:

Absolute Extrema on an Interval $[a, b]$	1. Find CN $x = c$, at $f'(x) = 0$ or undefined 2. Check endpoints and critical numbers; $f(a), f(c), f(b)$ 3. Choose the minimum and maximum values
Local Extrema – First Derivative Test of Critical Numbers	1. Find CN $x = c$, at $f'(x) = 0$ or undefined 2. Make a sign chart for $f'(x)$. Use test values. 3. Draw conclusions about $f(x)$ - If $f(x)$ changes from increasing to decreasing, $(c, f(c))$ is a local max - If $f(x)$ changes from decreasing to increasing, $(c, f(c))$ is a local min
Local Extrema – Second Derivative Test of Critical Numbers	1. Find CN $x = c$, at $f'(x) = 0$ or undefined 2. Calculate the second derivative $f''(x)$ 3. Test the critical numbers in $f''(x)$ - if $f''(c) > 0$, $f(x)$ is concave up and $(c, f(c))$ is a local min - if $f''(c) < 0$, $f(x)$ is concave down and $(c, f(c))$ is a local max - if $f''(c) = 0$, the test fails and you must use the First Derivative Test

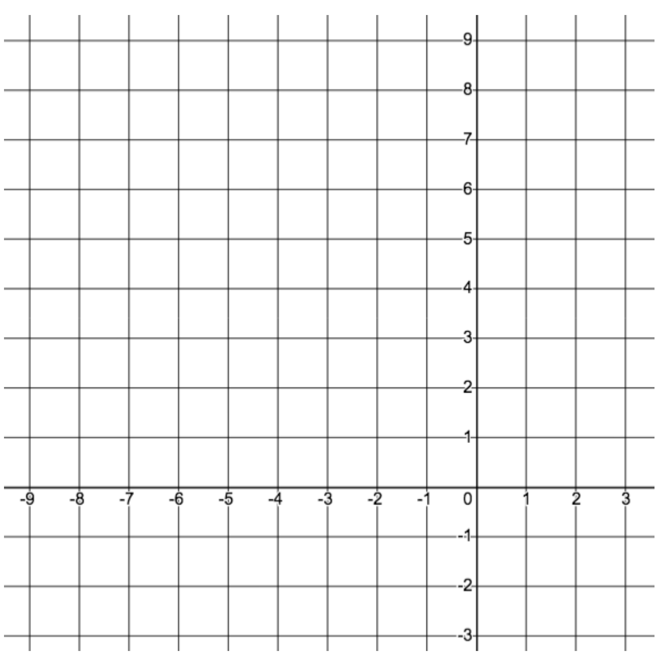
Example 2: For the function $f(x) = x^4 - 6x^2 - 5$, find all points of inflection (POI) and the intervals of concavity.

Example 3: For the function below, find the critical points. Then, classify them using the second derivative test.

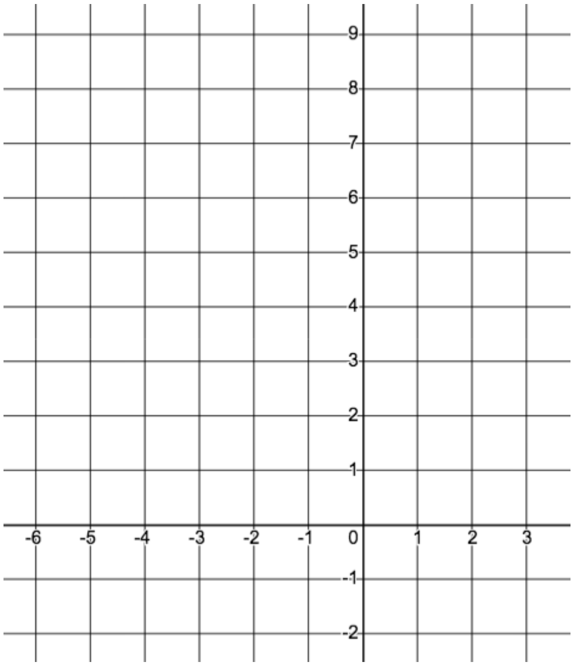
$g(x) = x^3 - 3x^2 + 2$

Example 4: Sketch a graph of a function that satisfies each set of conditions.

a) $f''(x) = -2$ for all x, $f'(-3) = 0$, $f(-3) = 9$



b) $f''(x) < 0$ when $x < -1$, $f''(x) > 0$ when $x > -1$, $f'(-3) = 0$, $f'(1) = 0$



L4 – Rational Functions Unit 2
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Part 1: Warm-Up

Find the intervals of concavity and the coordinates of any points of inflection for $y = \frac{1}{3}x^3 - 12x^2 + 5$

Remember:

$f''(x) = 0$ or undefined is a possible POI

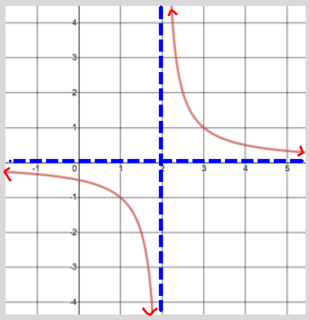
If $f''(x) < 0$, $f(x)$ is concave DOWN

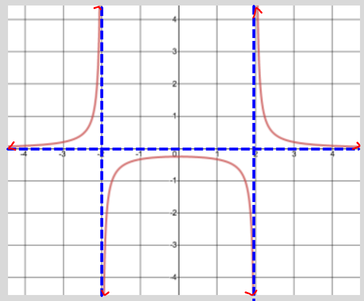
If $f''(x) > 0$, $f(x)$ is concave UP

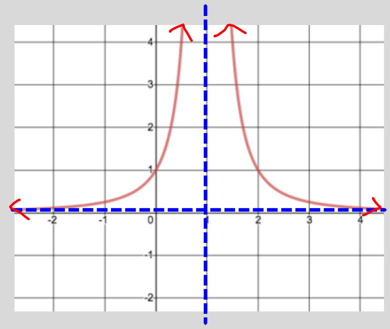
<u>Part 2: Reminder of some simple rational functions</u>

Degree of denominator > degree of numerator:

$y = \frac{1}{x-2}$ $\qquad$ $y = \frac{1}{x^2-4}$ $\qquad$ $y = \frac{1}{(x-1)^2}$





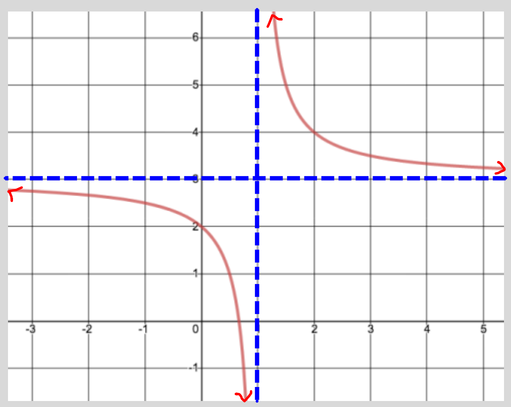


Notice: Horizontal asymptotes all are at $y = 0$

Vertical asymptotes are at zeros of the denominator

Degree of denominator = degree of numerator:

$y = \frac{3x-2}{x-1}$

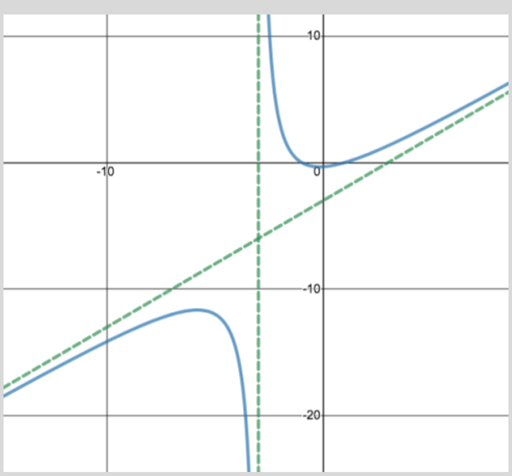


Notice: HA at quotient of leading coefficients

VA at zero of the denominator

Degree of denominator < degree of numerator:

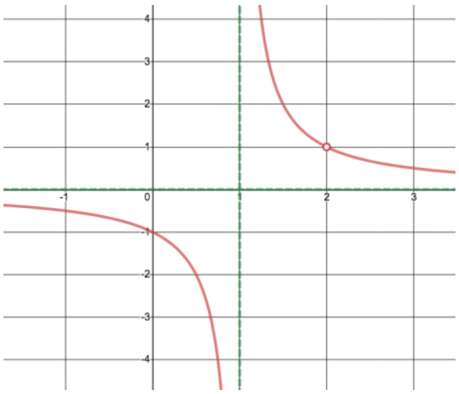
$y = \frac{x^2-1}{x+3}$



Notice: Oblique asymptote at quotient of numerator and denominator; VA at zero of the denominator

Vertical Asymptote vs. Hole in Graph

$f(x) = \frac{(x-2)}{(x-1)(x-2)}$



Notice: VA at $x = 1$; $f(1) = \frac{-1}{0}$

Hole at $(2, 1)$; $f(2) = \frac{0}{0}$

(remove discontinuity to find y-value of hole)

Conclusion: If $f(a) = \frac{\#}{0}$, $x = a$ is a VA

If $f(a) = \frac{0}{0}$, there is a hole in the graph when $x = a$

Limit Definition of Asymptotes:

For the rational function $y = \frac{f(x)}{g(x)}$

There is a Vertical Asymptote at $x = a$ when $g(a) = 0$ and $\lim_{x \to a} \frac{f(x)}{g(x)} = \pm\infty$

There is a Horizontal Asymptote at $y = L$ when $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = L$

Note: Horizontal asymptote only exists if the degree of the numerator is ________________________________ the degree of the denominator.

Part 3: Apply What You Know to Graph Rational Functions

Example 1: State the Horizontal Asymptotes of the following functions:

a) $y = \frac{3x^2+2}{6x^2-4x-1}$

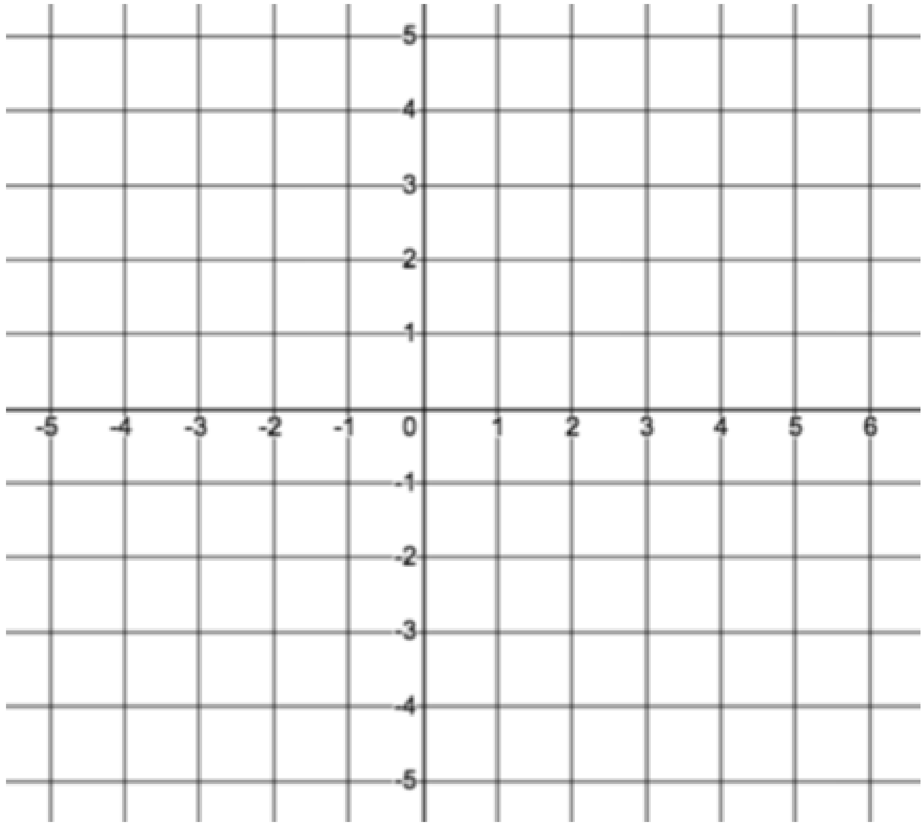
b) $y = \frac{3x^2+2}{6x^3-4x-1}$

Example 2: Consider the function $f(x) = \frac{1}{(x+2)(x-3)}$

a) Find the asymptotes

b) Find the one-sided limits as the x-values approach the vertical asymptotes (sub values very close to the limit for x, and find what the value of the function is approaching)

c) Sketch the graph



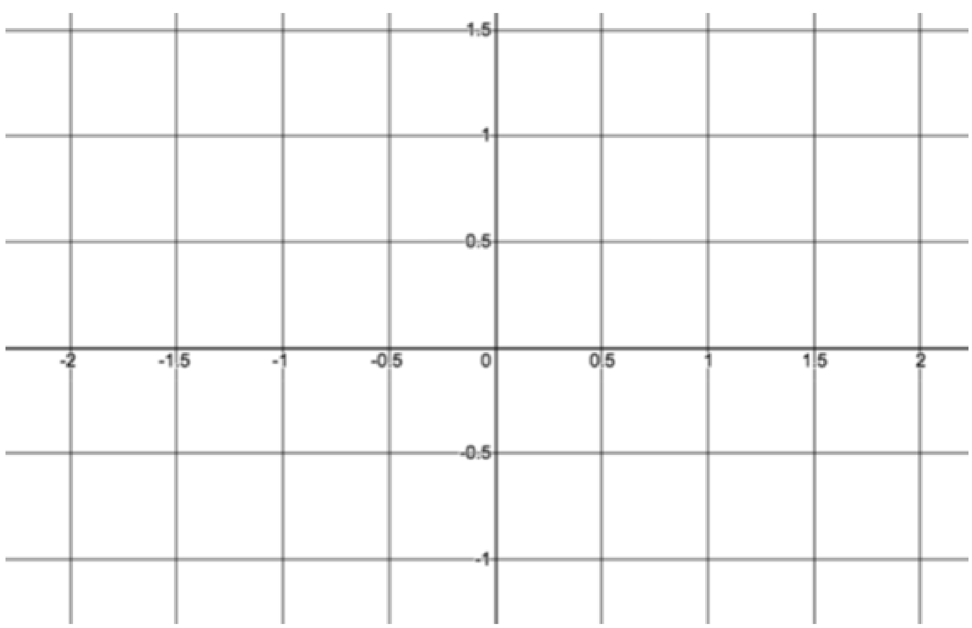
Example 3: Consider the function $f(x) = \frac{1}{x^2+1}$

a) Where are the vertical and horizontal asymptotes?

b) Find any local max/min points and the intervals of increase/decrease

c) Find the points of inflection

d) Sketch a graph of the function



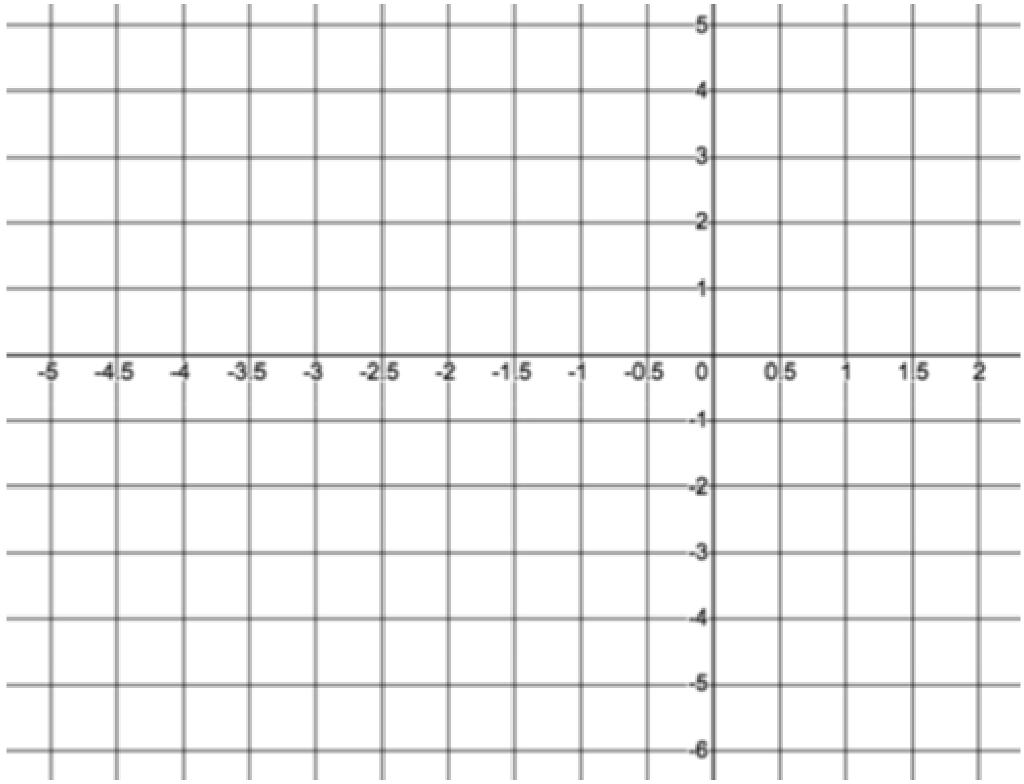
L5 – Curve Sketching Unit 2
MCV4U
Jensen

Algorithm for Curve Sketching

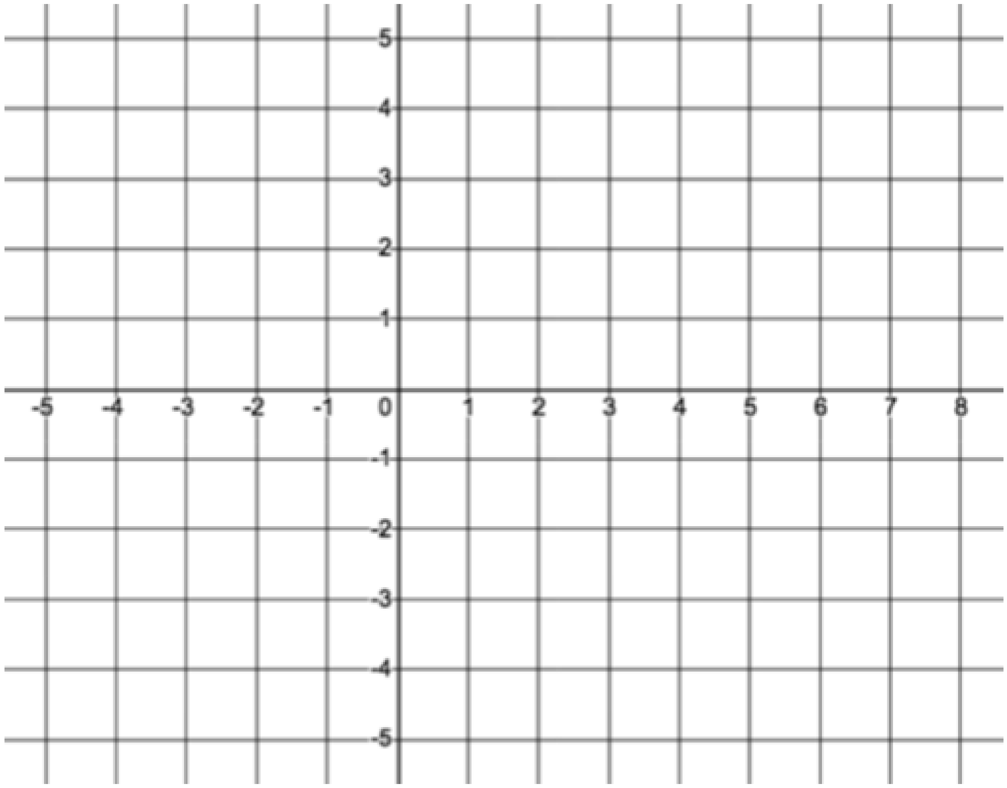
1. Determine any restrictions in the domain. State any horizontal and vertical asymptotes or holes in the graph.
2. Determine the intercepts of the graph
3. Determine the critical numbers of the function (where is $f'(x) = 0$ or undefined)
4. Determine the possible points of inflection (where is $f''(x)$=0 or undefined)
5. Create a sign chart that uses the critical numbers and possible points of inflection as dividing points.
6. Use the sign chart to find intervals of increase/decrease and the intervals of concavity. Use all critical numbers, possible points of inflection, and vertical asymptotes as dividing points.
7. Identify local extrema and points of inflection
8. Sketch the function

Example 1: Use the algorithm for curve sketching to analyze the key features of each of the following functions and to sketch a graph of them.

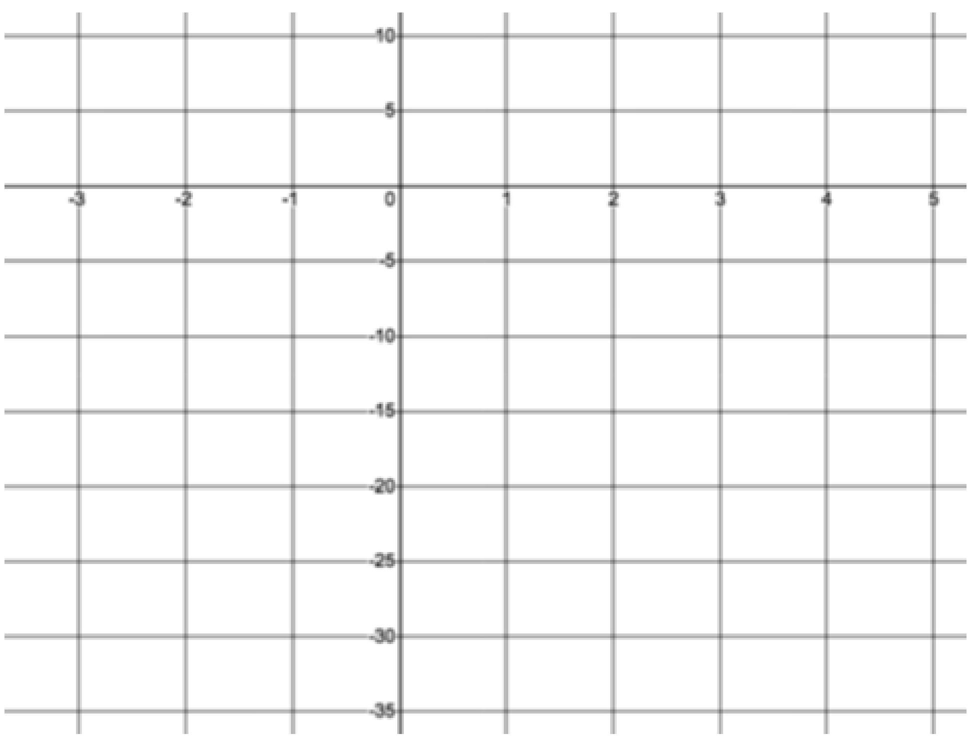
a) $g(x) = x^3 + 6x^2 + 9x$



b) $f(x) = \frac{1}{(x+1)(x-4)}$



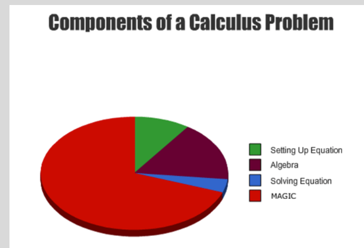
c) $h(x) = x^4 - 5x^3 + x^2 + 21x - 18$


10
5
-3
-2
-1
0
1
2
3
4
5
-5
-10
-15
-20
-25
-30
-35

L6 – Optimization Problems Unit 2
MCV4U
Jensen

Tips for Optimization Problems:

- Diagrams can be helpful
- Identify the independent variable and express all other variables in terms of it
- Define a function in terms of the independent variable
- Identify any restriction on the variable
- Solve for $f'(x) = 0$ to identify critical points
- Check critical points and endpoints



Optimization Warm Up:

A lifeguard has 200 meters of rope and some buoys with which she intends to enclose a rectangular area at a lake for swimming. The beach will form one side of the rectangle, with the rope forming the other 3 sides. Find the dimensions that will produce the maximum enclosed area.

Example 1: A cardboard box with a square base is to have a volume of 8 Liters (1 L = 1000 cm^3)
Find the dimensions that will minimize the amount of cardboard to be used. What is the minimum surface area?

Example 2: A soup can of volume 500 cm^3 is to be constructed. The material for the top costs 0.4¢/cm^2 while the material for the bottom and sides costs 0.2¢/cm^2. Find the dimensions that will minimize the cost of producing the can. What is the min cost?

Example 3: Ian and Ada are both training for a marathon. Ian's house is located 20 km north of Ada's house. At 9:00 am one Saturday, Ian leaves his house and jogs south at 8 km/h. At the same time, Ada leaves her house and jogs east at 6 km/h. When are Ian and Ada closest together, given that they both run for 2.5 hours?