Unit 2- Curve Sketching

Lesson Package

MCV4U

graph feature	f(x)		f'(x)		f''(x)	Notes
rising (L to R)	slope > 0		+			
falling (L to R)	slope < 0		-			
extrema — maximum	slope = 0		= 0	+ on L - on R	− at x_{max}	derivative may not exist at a max or min, e.g.
extrema — minimum	slope = 0		= 0	- on L + on R	+ at x_{min}	⌢ or ⌣
inflection pt.	Curvature changes: ⌢ ↔ ⌣				= 0 potential inflection point	Check f''(x) on either side of a potential inflection point.
concave up	⌍	⌎	-	+	+	
concave down	⌌	⌏	+	-	-	

Unit 2 Outline

Unit Goal: Make connections, graphically and algebraically, between the key features of a function and its first and second derivatives, and use the connections in curve sketching.

Section	Subject	Learning Goals	Curriculum Expectations
L1	Increasing and Decreasing	- use the first derivative to determine when a function is increasing or decreasing	A2.1
L2	Max and Min	- use the first derivative test to see if a critical point is a max, min, or neither	A2.1
L3	Second Derivative and Concavity	- Use the second derivative to find intervals of concavity and points of inflection	B1.1,1.2,B1.4
L4	Rational Functions	- Sketch the graph of rational functions using critical points and knowledge of asymptotes	B1.3
L5	Curve Sketching	- Sketch the graph of various polynomial and rational functions using the algorithm for curve sketching	B1.5
L6	Optimization	- solve optimization problems using mathematical models and derivatives	B2.4, B2.5

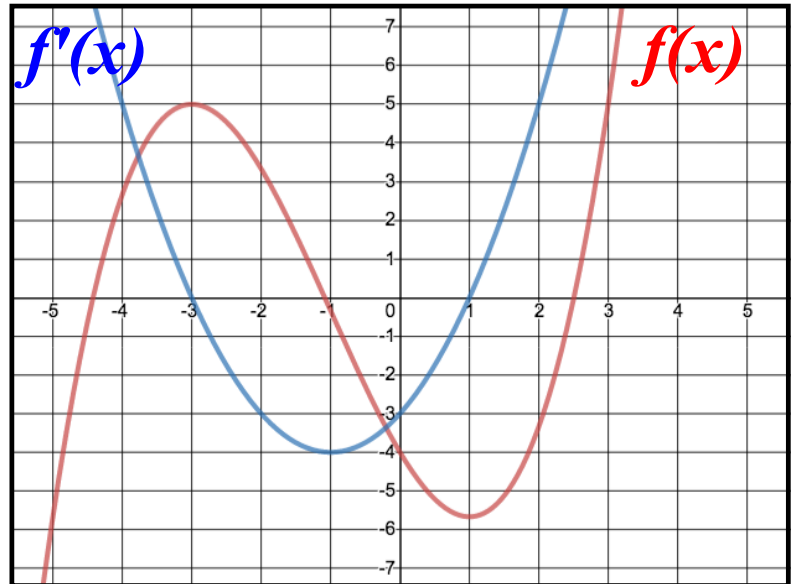| Assessments	F/A/O	Ministry Code	P/O/C	KTAC
Note Completion	A		P	
Practice Worksheet Completion	F/A		P	
Quiz – Curve Sketching	F		P	
PreTest Review	F/A		P	
Test – Curve Sketching	O	A2.1, B1.1-B1.5, B2.4, B2.5	P	K(25%), T(25%), A(25%), C(25%)

Increasing: As x-values increase, y-values are increasing

Decreasing: As x-values increase, y-values are decreasing

Part 1: Discovery

$$f(x) = \frac{1}{3}x^3 + x^2 - 3x - 4$$

$$f'(x) = x^2 + 2x - 3$$

a) Over which values of x is $f(x)$ increasing?

$x < -3$ and $x > 1$

b) Over which values of x is $f(x)$ decreasing?

$-3 < x < 1$

c) What is true about the graph of $f'(x)$ when $f(x)$ is increasing?

$f'(x) > 0$; it is above the x-axis

d) What is true about the graph of $f'(x)$ when $f(x)$ is decreasing?

$f'(x) < 0$; it is below the x-axis

Effects of $f'(x)$ on $f(x)$: When the graph of $f'(x)$ is positive, or above the x-axis, on an interval, then the function $f(x)$ **increases** over that interval. Similarly, when the graph of $f'(x)$ is negative, or below the x-axis, on an interval, then the function $f(x)$ **decreases** over that interval.
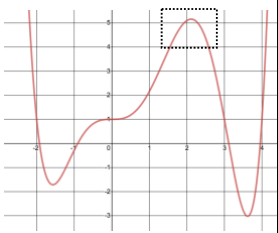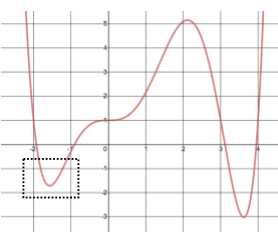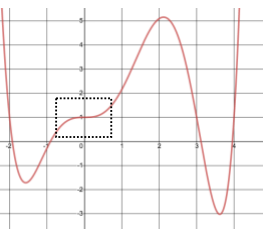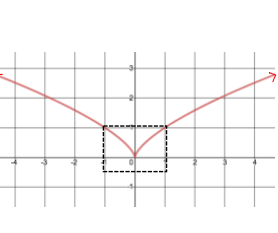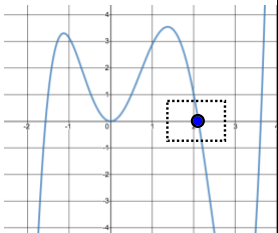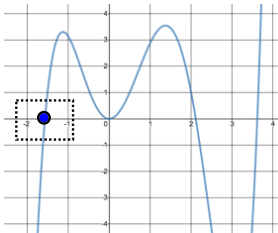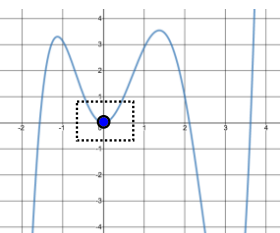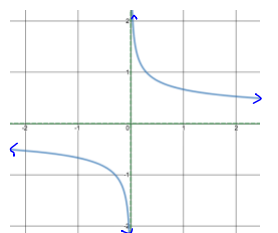
If $f'(x) > 0$ on an interval, $f(x)$ is increasing on that interval

If $f'(x) < 0$ on an interval, $f(x)$ is decreasing on that interval

Part 2: Properties of graphs of $f(x)$ and $f'(x)$

A critical number is a value $'a'$ in the domain where $f'(a) = 0$ or $f'(a)$ does not exist.

A critical number could yield…

	A local max	A local min	Neither	Max/Min at Cusp
$f(x)$				
$f'(x)$				
	$f(x)$ has a local max $f'(x) = 0$ and changes from + to -	$f(x)$ has a local min $f'(x) = 0$ and changes from - to +	No local max/min $f'(x) = 0$ but does not change signs	$f(x)$ has a local min $f'(x)$ does not exist but changes from − to +

Conclusion:

Local extrema occur when the sign of the derivative CHANGES. If the sign of the derivative does not change, you do not have a local extrema.

A critical number of a function is a value of a in the domain of the function where either $f'(a) = 0$ or $f'(a)$ does not exist. If a is a critical number, $(a, f(a))$ is a critical point. Critical points could be local extrema but not necessarily. You must test around the critical points to see if the derivative changes sign. This is called the **'First Derivative Test'**.

Example 1: Determine all local extrema for the function below using critical numbers and the first derivative test. State when the function is increasing or decreasing.

$f(x) = 2x^3 - 9x^2 - 24x - 10$

$f'(x) = 6x^2 - 18x - 24$

$0 = 6(x^2 - 3x - 4)$
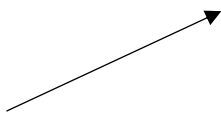
$0 = 6(x - 4)(x + 1)$

$x_1 = 4 \quad x_2 = -1$
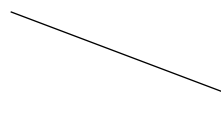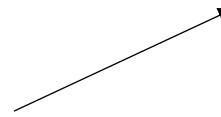
Critical Numbers: $x_1 = 4 \quad x_2 = -1$

Critical Points: $(4, -122)$ and $(-1, 3)$

$f(4) = 2(4)^3 - 9(4)^2 - 24(4) - 10 = -122$

$f(-1) = 2(-1)^3 - 9(-1)^2 - 24(-1) - 10 = 3$

Sign Chart:

	$-\infty$	-1	4	∞
Test value for x		-2	0	5
$f'(x)$		$+$	$-$	$+$
$f(x)$		Increasing	Decreasing	Increasing
			Local max at $(-1, 3)$	Local min at $(4, -122)$

Increasing: $x < -1$ or $x > 4$
Decreasing: $-1 < x < 4$

Notice how we could use the graph of the derivative to verify our solution:
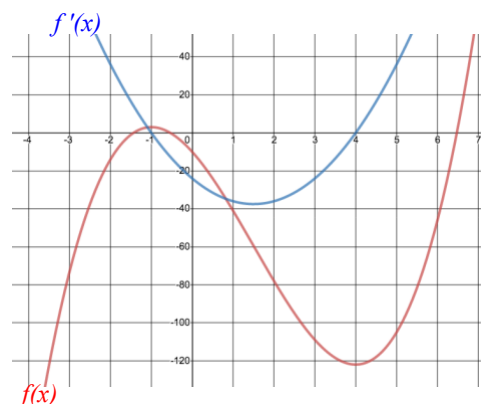
Example 2: For each function, use the graph of $f'(x)$ to sketch a possible function $f(x)$.

a)

$f'(x)$ is the constant function $y = -2$.

Therefore, $f(x)$ must be a linear function with a slope of -2

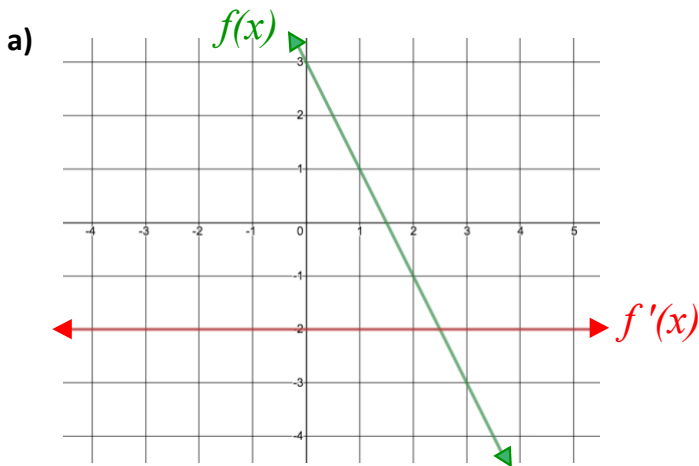The y-intercept could be anywhere.

b)
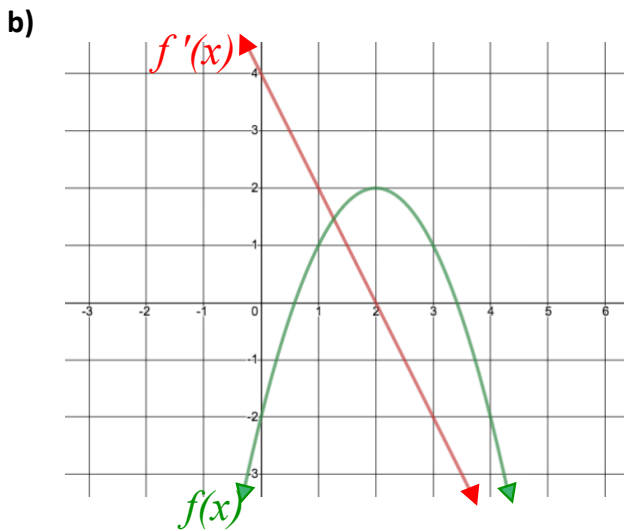
$f'(x)$ is a linear function

Therefore, $f(x)$ must be a quadratic function

$f'(x) > 0$ when $x < 2$ and $f'(x) < 0$ when $x > 2$

Therefore, $f(x)$ is increasing when $x < 2$ and decreasing when $x > 2$

$f'(x)$ switching from + to − at $x = 2$

There must be a local max at $f(2)$.

c)

$f'(x)$ is a quadratic function

Therefore, $f(x)$ must be a cubic function

$f'(x) > 0$ when $x < 2$ when $x > 2$

Therefore, $f(x)$ is increasing when $x < 2$ and when $x > 2$

$f'(x)$ does not switch signs on either side of $x = 2$

Therefore, there is no local min or max at $f(2)$ but the tangent line would be horizontal at that point.

d)
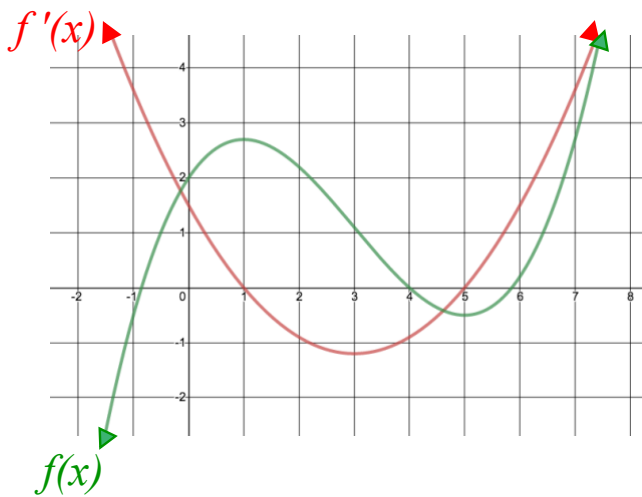
$f'(x)$ is a quadratic function

Therefore, $f(x)$ must be a cubic function

$f'(x) > 0$ when $x < 1$ when $x > 5$

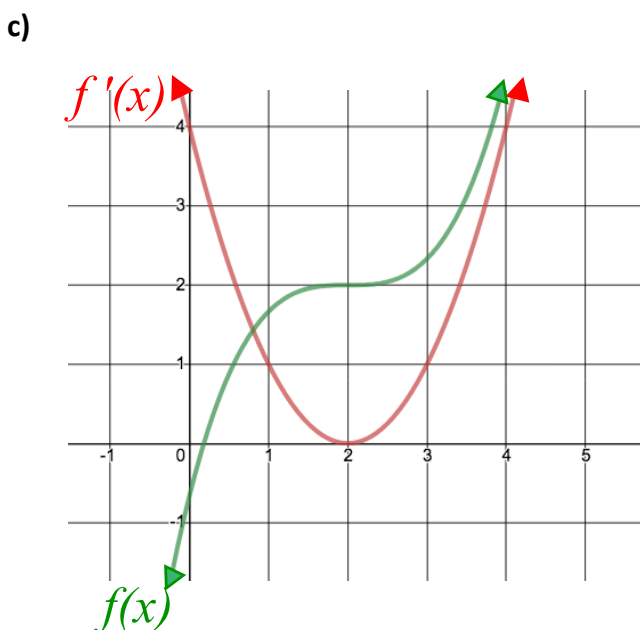Therefore, $f(x)$ is increasing when $x < 1$ and when $x > 5$

$f'(x) < 0$ when $1 < x < 5$

Therefore, $f(x)$ is decreasing when $1 < x < 5$

$f'(x)$ switches signs on either side of $x = 1$ (from + to -) and $x = 5$ (from – to +)

Therefore, there is a max at $f(1)$ and a local min at $f(5)$

Example 3: Sketch a continuous function for each set of conditions

a) $f'(x) > 0$ when $x < 0$, $f'(x) < 0$ when $x > 0$, $f(0) = 4$
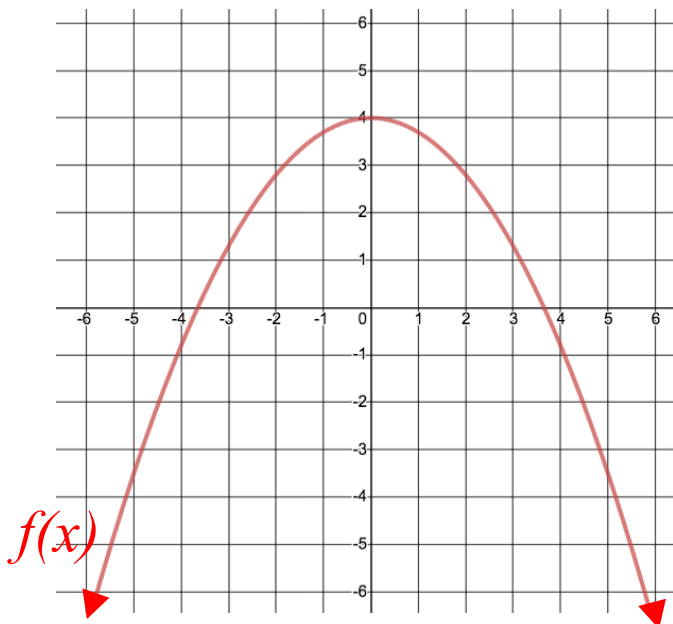
$f(x)$ is increasing when $x < 0$ and decreasing when $x > 0$.

Therefore, there must be a local max at $x = 0$.

$f(0) = 4$

b) $f'(x) > 0$ when $x < -1$ and when $x > 2$, $f'(x) < 0$ when $-1 < x < 2$, $f(0) = 0$
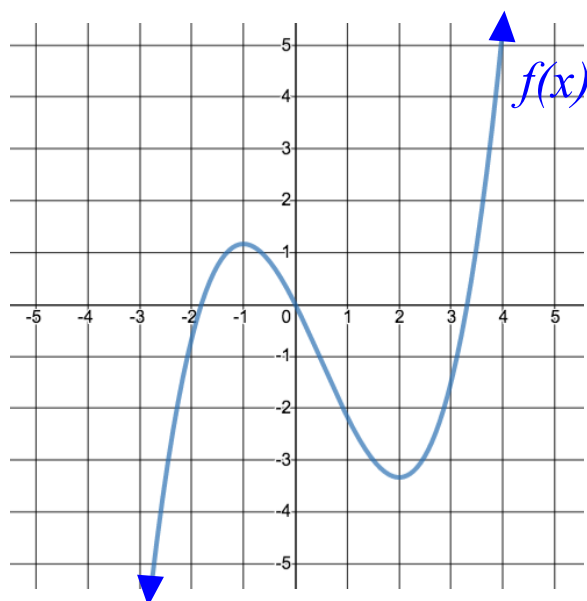
$f(x)$ is increasing when $x < -1$ and when $x > 2$.

$f(x)$ is decreasing when $-1 < x < 2$.

Therefore, there must be a local max at $x = -1$ and a local min at $x = 2$.

$f(0) = 0$

Example 4: The temperature of a person with a certain strain of flu can be approximated by the function $T(d) = -\frac{5}{18}d^2 + \frac{15}{9}d + 37$, where $0 < d < 6$; T represents the person's temperature, in degrees Celsius and d is the number of days after the person first shows symptoms. During what interval will the person's temperature be increasing?

$$T'(d) = -\frac{5}{9}d + \frac{15}{9}$$
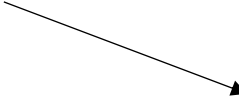
Find critical numbers:

$$0 = -\frac{5}{9}d + \frac{15}{9}$$

$$\frac{5}{9}d = \frac{15}{9}$$

$$5d = 15$$

$d = 3$ is a critical number

Test value for x	0 ... 3 ... 6	
	1	4
$T'(d)$	$+$	$-$
	Increasing	Decreasing
$T(d)$	↗	↘

Therefore, a person's temperature will be increasing over the first 3 days.

Part 1: Review

Determine all local extrema for the function below using critical numbers and the first derivative test. State when the function is increasing and decreasing.
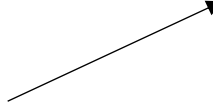
$f(x) = 2x^3 + 3x^2 - 36x + 5$

Remember: Local extrema occur when the sign of the derivative CHANGES. If the sign of the derivative does not change, you have neither local extrema.

$f'(x) = 6x^2 + 6x - 36$

$0 = 6(x^2 + x - 6)$

$0 = 6(x + 3)(x - 2)$

$x = -3$ and $x = 2$ are critical numbers

Critical points:

$f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3) + 5 = 86$ $(-3, 86)$

$f(2) = 2(2)^3 + 3(2)^2 - 36(2) + 5 = -39$ $(2, -39)$

Test value for x	$-\infty$	-4	-3	0	2	3	∞
$f'(x)$		+		−		+	
$f(x)$		Increasing		Decreasing		Increasing	

Local max at $(-3, 86)$

Local min at $(2, -39)$

Interval of increasing: $(-\infty, -3) \cup (2, \infty)$

Interval of decreasing: $(-3, 2)$

Part 2: Local vs Absolute Extrema

Local max or min values of a function are also called local extrema, or turning points.

Local max: If the y-coordinate of all points in the vicinity are less than the y-coordinate of the point. The sign of the derivative would change from positive before the point, to zero at the point, to negative after.

Local min: If the y-coordinate of all points in the vicinity are greater than the y-coordinate of the point. The sign of the derivative would change from negative before the point, to zero at the point, to positive after.

Absolute max/min: A function $f(x)$ has an ABSOLUTE max or min at point a if $f(a)$ is the biggest or smallest value of $f(x)$ for ALL x in the domain.

Example 1: Consider the graph of a function on the interval $[0, 10]$.

a) Identify the local maximum points.

B and D

b) Identify the local minimum points.

C

c) What do all the points identified in parts a) and b) have in common?

They each would have horizontal tangent lines.

d) Identify the absolute max and min points in the interval [0,10]

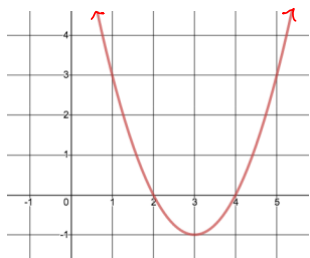Absolute max is at D

Absolute min is at A
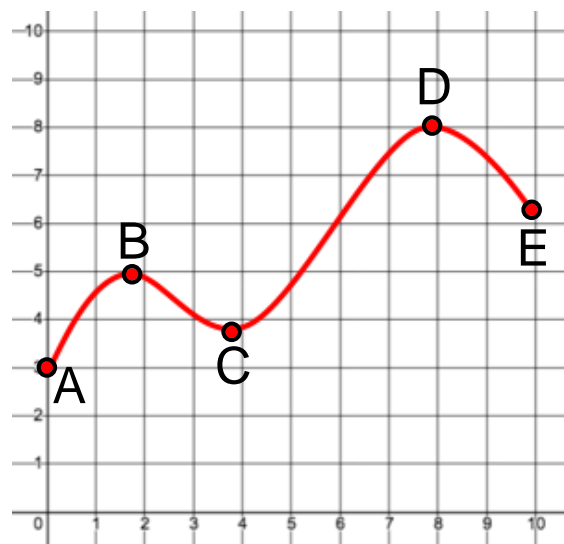
Reminder: A critical number of a function is a value of a **in the domain of the function** where either $f'(a) = 0$ or $f'(a)$ does not exist. If a is a critical number, $(a, f(a))$ is a critical point.

Scenarios for critical numbers:

1) $f'(a) = 0$
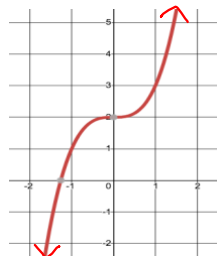Local extrema at $(a, f(a))$
$f(x) = x^2 - 6x + 8$

2) $f'(a) = 0$
No local extrema at $(a, f(a))$
$f(x) = x^3 + 2$

3) $f'(a)$ does not exist
local extrema at $(a, f(a))$ (cusp)
$f(x) = x^{\frac{2}{3}}$
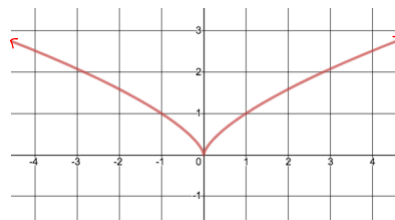
4) $f'(a)$ does not exist
No local extrema at $(a, f(a))$
$f(x) = x^{\frac{1}{3}}$
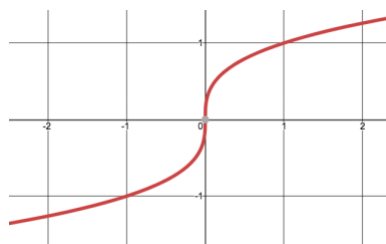
5) $\boldsymbol{f'(a)}$ **does not exist**
No local extrema

$f(a)$ does not exist either, therefore a is NOT a critical number

$f(x) = \frac{1}{x}$
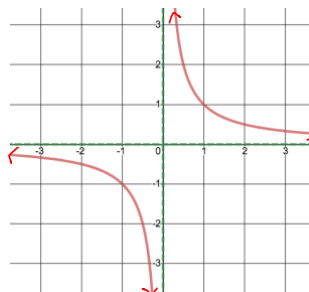
To determine the absolute extrema values of a function on an interval, find the critical numbers, then substitute the critical numbers and also the x-coordinates of the endpoints of the interval into the function.

Example 2: Find the absolute max and min of the function $f(x) = x^3 - 12x - 3$ on the interval $-3 \le x \le 4$.

$f'(x) = 3x^2 - 12$

$0 = 3(x^2 - 4)$

$0 = 3(x - 2)(x + 2)$

$x = 2$ and $x = -2$ are critical numbers

$f(-3) = (-3)^3 - 12(-3) - 3 = 6$

$f(-2) = 13$
$f(2) = -19$
$f(4) = 13$

> Test critical numbers AND endpoints of interval.

Absolute min: $(2, -19)$

Absolute max: $(-2, 13)$ and $(4, 13)$

Example 3: The surface area of a cylindrical container is to be 100 cm^2. Its volume is given by the function $V(r) = 50r - \pi r^3$, where r is the radius of the cylinder in cm. Find the max volume of the cylinder if the radius cannot exceed 3 cm.

$V'(r) = 50 - 3\pi r^2$

Find any critical numbers:

$0 = 50 - 3\pi r^2$

$3\pi r^2 = 50$

$r = \sqrt{\dfrac{50}{3\pi}}$ is a critical number

Test endpoints of interval and critical number to find absolute max

$V(0) = 0$ cm³

$V\left(\sqrt{\dfrac{50}{3\pi}}\right) = 76.8$ cm³

$V(3) = 65.2$ cm³

Therefore, the max volume of the cylinder is about 76.8 cm³ when the radius is about 2.3 cm.

The **second derivative** is the derivative of the first derivative. It is the rate of change of the slope of the tangent.

Part 1: Discovery

Example 1:

a) Given the graph of $f(x) = x^4 - 2x^3 - 5$

$f'(x) = 4x^3 - 6x^2$

$f''(x) = 12x^2 - 12x$

When is $f''(x) = 0$?

$0 = 12x^2 - 12x$

$0 = 12x(x - 1)$

$x = 0$ or $x = 1$
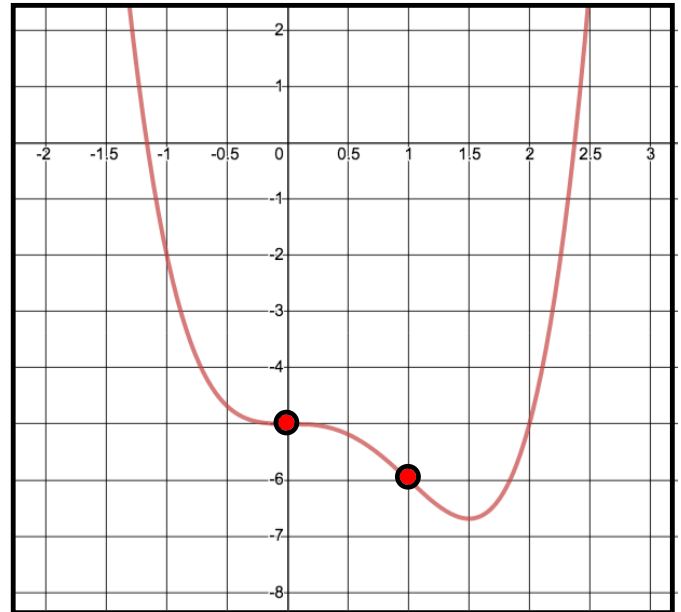
b) Use your pencil to simulate a tangent line to the function when $x = -1$. Drag the pencil slowly to the right, keeping it tangent to the curve, approaching $x = 0$. What is happening to the slope of the tangent? Is it above or below the curve? What is the value of $f''(-0.5)$?

- The curve is above the tangent line.
- The tangent line slopes are increasing.
- $f''(-0.5) = 9$; it's positive

c) Drag the pencil slowly to the right, keeping it tangent to the curve, moving through $x = 0$. What is happening to the slope of the tangent as it moves through $x = 0$? What is the value of $f''(0.5)$?

- The slope of the tangent stops increasing and starts decreasing.
- The curve goes from above the tangent line to below the tangent line.
- $f''(0.5) = -3$; it's negative

d) What happens to the slope of the tangent as it moves through $x = 1$?

- The slope of the tangent stops decreasing and starts increasing.
- The curve goes from below the tangent line to above the tangent line.

Summary of findings:

How $f''(x)$ effects $f(x)$:

The graph of a function is concave up over an interval if the curve is above all of the tangents on the interval. The slopes of the tangent lines are increasing, therefore $f''(x) > 0$ over this interval.

The graph of a function is concave down over an interval if the curve is below all of the tangents on the interval. The slopes of the tangent lines are decreasing, therefore $f''(x) < 0$ over this interval.

$f(x)$ is concave **UP** on an interval if $f''(x) > 0$ over that interval (tangent line slopes are increasing)

$f(x)$ is concave **DOWN** on an interval if $f''(x) < 0$ over that interval (tangent line slopes are decreasing)

A **POINT OF INFLECTION** is a point in the domain of the function at which the graph changes from being concave up to concave down or vice versa. The second derivative, $f''(x)$, is equal to zero at this point (or is undefined) and changes sign on either side. The tangent lines change from increasing to decreasing OR from decreasing to increasing.
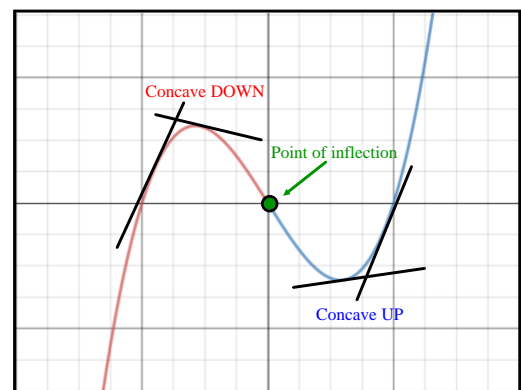
However, just like that not every critical point is a local max / min, not every zero or restriction of the second derivative is an inflection point either. They are just the pool of points you need to check in order to find the inflection point(s) of a curve.

$f(x) = x^4$
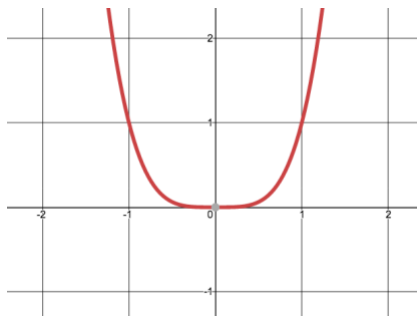
$f''(x) = 12x^2$

$f''(0) = 0$

But $x = 0$ is not a point of inflection; the function has no change in concavity. Tangent slopes are always increasing.

Note: It often happens that a graph has different concavity on the two sides of a vertical asymptote. However, because a curve is not continuous at a vertical asymptote, it can never have an inflection point there. We will look at these types of functions next lesson (rational functions).
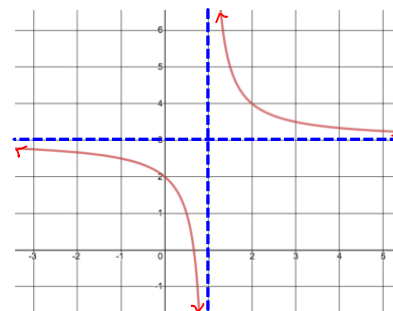
The **second derivative test** can also be used to help check for local min/max points.

In the second derivative test we check the critical points themselves (those where $f'(x) = 0$), by evaluating $f''(x)$ AT each critical point.

If $f'(c) = 0$ and $f''(c) > 0$, then f has a local **minimum** at c.

If $f'(c) = 0$ and $f''(c) < 0$, then f has a local **maximum** at c.

Note: Even though it is often easier to use than the first derivative test, the second derivative test can fail at some points (eg. $y = x^4$). If the second derivative test fails, then the first derivative test must be used to classify the point in question.

Relationship between $f(x)$, $f'(x)$, and $f''(x)$

$f(x) = 0$	Zeros (x-intercepts) of the function	
$f(x) > 0$	Function is positive (above x-axis)	
$f(x) < 0$	Function is negative (below x-axis)	

$f'(x) = 0$	Horizontal tangent; possible local extrema (turning point)
$f'(x) > 0$	$f(x)$ is increasing
$f'(x) < 0$	$f(x)$ is decreasing

$f''(x) = 0$	Possible point of inflection (change in concavity)
$f''(x) > 0$	$f(x)$ is concave up
$f''(x) < 0$	$f(x)$ is concave down

Tests of Critical Numbers:

Absolute Extrema on an Interval $[a, b]$	1. Find CN $x = c$, at $f'(x) = 0$ or undefined 2. Check endpoints and critical numbers; $f(a), f(c), f(b)$ 3. Choose the minimum and maximum values

Local Extrema – First Derivative Test of Critical Numbers	1. Find CN $x = c$, at $f'(x) = 0$ or undefined
	2. Make a sign chart for $f'(x)$. Use test values.
	3. Draw conclusions about $f(x)$
	    - If $f(x)$ changes from increasing to decreasing, $(c, f(c))$ is a local max
	    - If $f(x)$ changes from decreasing to increasing, $(c, f(c))$ is a local min

Local Extrema – Second Derivative Test of Critical Numbers	1. Find CN $x = c$, at $f'(x) = 0$ or undefined
	2. Calculate the second derivative $f''(x)$
	3. Test the critical numbers in $f''(x)$
	    - if $f''(c) > 0$, $f(x)$ is concave up and $(c, f(c))$ is a local min
	    - if $f''(c) < 0$, $f(x)$ is concave down and $(c, f(c))$ is a local max
	    - if $f''(c) = 0$, the test fails and you must use the First Derivative Test

Example 2: For the function $f(x) = x^4 - 6x^2 - 5$, find all points of inflection (POI) and the intervals of concavity.

$f'(x) = 4x^3 - 12x$

$f''(x) = 12x^2 - 12$

$0 = 12x^2 - 12$

$0 = 12(x^2 - 1)$

$0 = 12(x - 1)(x + 1)$

$x_1 = 1 \quad x_2 = -1$
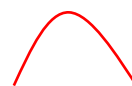
Possible points of inflection:

$f(1) = (1)^4 - 6(1)^2 - 5 = -10$

$f(-1) = (-1)^4 - 6(-1)^2 - 5 = -10$

Use sign chart to check if there are changes of concavity on either side of these points.

Sign Chart:

Test value for x	$-\infty$ $\quad$ -2	-1 $\quad$ 0	1 $\quad$ 2 $\quad$ ∞
$f''(x)$	$+$	$-$	$+$
$f(x)$	Concave UP	Concave DOWN	Concave UP

POI at $(-1, -10)$ $\qquad$ POI at $(1, -10)$

Concave up: $(-\infty, -1) \cup (1, \infty)$

Concave down: $(-1, 1)$

Example 3: For the function below, find the critical points. Then, classify them using the second derivative test.

$g(x) = x^3 - 3x^2 + 2$ Critical points: $(0, 2)$ and $(2, -2)$

$g'(x) = 3x^2 - 6x$ $g(0) = 2$

$0 = 3x^2 - 6x$ $g(2) = -2$

$0 = 3x(x - 2)$

$x = 0$ and $x = 2$ are critical numbers

Second derivative test:

$g''(x) = 6x - 6$

	$x = 0$	$x = 2$
$g''(x)$	$-$	$+$
$g(x)$	Concave DOWN	Concave UP
	$(0, 2)$ is a local MAX	$(2, -2)$ is a local MIN

Example 4: Sketch a graph of a function that satisfies each set of conditions.

a) $f''(x) = -2$ for all x, $f'(-3) = 0$, $f(-3) = 9$
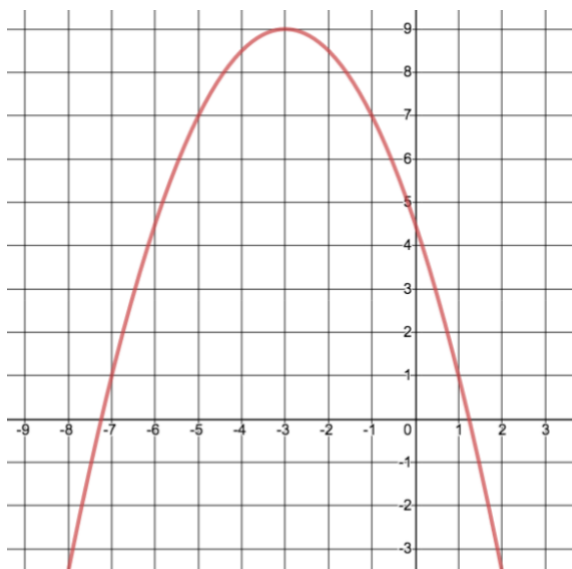
$f''(x) = -2$ for all x tells us that the function is always concave down.

$f'(-3) = 0$ means indicates there is a local max at $(-3, 9)$

b) $f''(x) < 0$ when $x < -1$, $f''(x) > 0$ when $x > -1$, $f'(-3) = 0$, $f'(1) = 0$

$f(x)$ is concave down when $x < -1$

$f(x)$ is concave up when $x > -1$

Local max at $x = -3$

Local min at $x = 1$

Part 1: Warm-Up

Find the intervals of concavity and the coordinates of any points of inflection for $y = \frac{1}{3}x^3 - 12x^2 + 5$

$y' = x^2 - 24x$

$y'' = 2x - 24$

$0 = 2(x - 12)$

$x = 12$

Possible point of inflection:

> **Remember:**
>
> $f''(x) = 0$ or undefined is a possible POI
>
> If $f''(x) < 0$, $f(x)$ is concave DOWN
>
> If $f''(x) > 0$, $f(x)$ is concave UP

$y = \frac{1}{3}(12)^3 - 12(12)^2 + 5 = -1147$

$(12, -1147)$ *is a possible point of inflection*

	$-\infty$	12	∞
Test value for x	11		13
y''	$-$		$+$
y	Concave DOWN		Concave UP

POI at
$(12, -1147)$

Concave up: $(12, \infty)$

Concave down: $(-\infty, 12)$

Part 2: Reminder of some simple rational functions

Degree of denominator > degree of numerator:

$$y = \frac{1}{x-2}$$
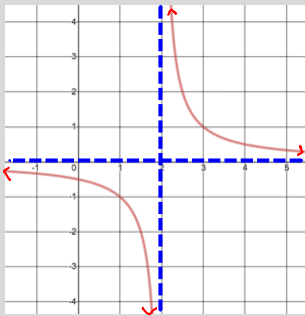
$$y = \frac{1}{x^2-4}$$
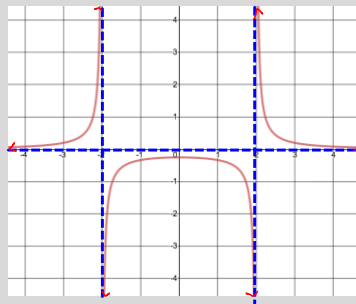
$$y = \frac{1}{(x-1)^2}$$
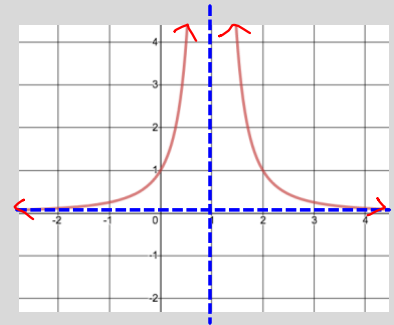
Notice: Horizontal asymptotes all are at $y = 0$

Vertical asymptotes are at zeros of the denominator

Degree of denominator = degree of numerator:

$$y = \frac{3x-2}{x-1}$$
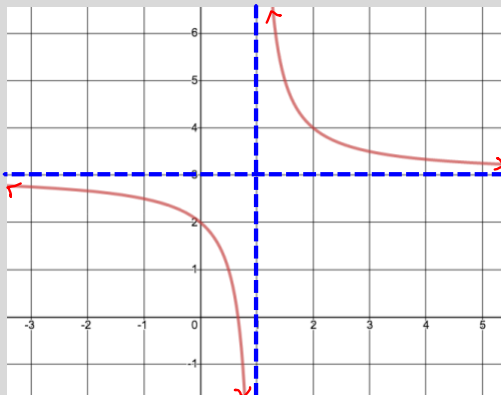
Notice: HA at quotient of leading coefficients

VA at zero of the denominator

Degree of denominator < degree of numerator:

$$y = \frac{x^2-1}{x+3}$$
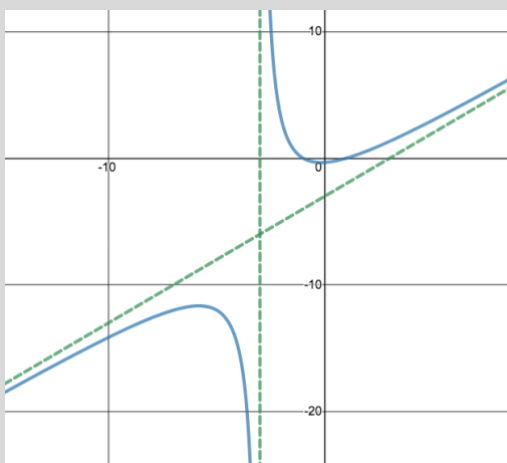
Notice: Oblique asymptote at quotient of numerator and denominator; VA at zero of the denominator

Vertical Asymptote vs. Hole in Graph

$f(x) = \frac{(x-2)}{(x-1)(x-2)}$
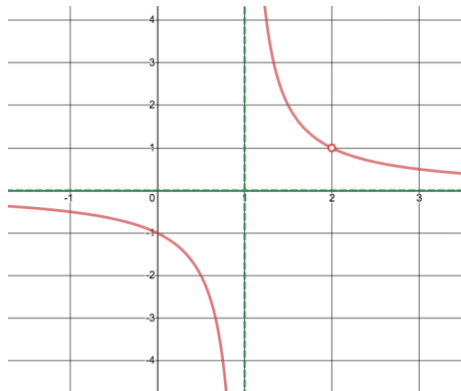
Notice: VA at $x = 1$; $f(1) = \frac{-1}{0}$

Hole at $(2, 1)$; $f(2) = \frac{0}{0}$

(remove discontinuity to find y-value of hole)

Conclusion: If $f(a) = \frac{\#}{0}$, $x = a$ is a VA

If $f(a) = \frac{0}{0}$, there is a hole in the graph when $x = a$

Limit Definition of Asymptotes:

For the rational function $y = \frac{f(x)}{g(x)}$

There is a Vertical Asymptote at $x = a$ when $g(a) = 0$ and $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \pm\infty$

There is a Horizontal Asymptote at $y = L$ when $\lim\limits_{x \to \pm\infty} \frac{f(x)}{g(x)} = L$

Note: Horizontal asymptote only exists if the degree of the numerator is **less than or equal to** the degree of the denominator.

Part 3: Apply What You Know to Graph Rational Functions

Example 1: State the Horizontal Asymptotes of the following functions:

a) $y = \frac{3x^2+2}{6x^2-4x-1}$

HA: $y = \frac{3}{6} = \frac{1}{2}$

b) $y = \frac{3x^2+2}{6x^3-4x-1}$

HA: $y = 0$

Example 2: Consider the function $f(x) = \frac{1}{(x+2)(x-3)}$

a) Find the asymptotes

HA: $y = 0$

VA: $x = -2$ and $x = 3$

b) Find the one-sided limits as the x-values approach the vertical asymptotes (sub values very close to the limit for x, and find what the value of the function is approaching)

$$\lim_{x \to -2^-} \frac{1}{(x+2)(x-3)} = \infty$$

Test: $f(-2.00001) = 19999.96$; therefore going towards $+\infty$

$$\lim_{x \to -2^+} \frac{1}{(x+2)(x-3)} = -\infty$$

Test: $f(-1.9999) = -2000.04$; therefore going towards $-\infty$

$$\lim_{x \to 3^-} \frac{1}{(x+2)(x-3)} = -\infty$$

Test: $f(2.9999) = -2000.04$; therefore going towards $-\infty$

$$\lim_{x \to 3^+} \frac{1}{(x+2)(x-3)} = \infty$$

Test: $f(3.00001) = 19999.96$; therefore going towards $+\infty$

c) Sketch the graph
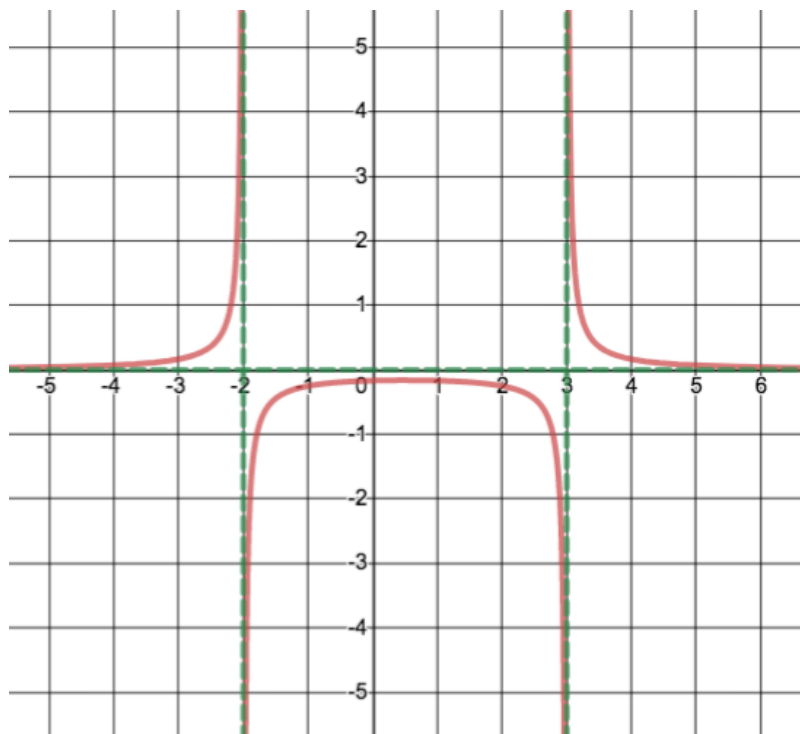
Example 3: Consider the function $f(x) = \frac{1}{x^2+1}$

a) Where are the vertical and horizontal asymptotes?

HA: $y = 0$
VA: none $x^2 + 1 \neq 0$

b) Find any local max/min points and the intervals of increase/decrease

$$f'(x) = \frac{0(x^2 + 1) - 2x(1)}{(x^2 + 1)^2}$$

$$f'(x) = \frac{-2x}{(x^2 + 1)^2}$$

Find Critical Numbers:

$$0 = \frac{-2x}{(x^2 + 1)^2}$$

$$0 = -2x$$

$$x = 0$$

$$x = 0$$

Critical Point: $(0,1)$

$$f(0) = 1$$

	$-\infty$		0		∞
Test value for x		-1		1	
$f'(x)$		$+$		$-$	
		Increasing		Decreasing	
$f(x)$		↗		↘	
			Local max at $(0,1)$		

Increasing: $(-\infty, 0)$

Decreasing: $(0, \infty)$

c) Find the points of inflection

$$f'(x) = \frac{-2x}{(x^2+1)^2}$$

$$f''(x) = \frac{-2(x^2+1)^2 - 2(x^2+1)(2x)(-2x)}{(x^2+1)^4}$$

$$f''(x) = \frac{-2(x^2+1) - 2(2x)(-2x)}{(x^2+1)^3}$$

$$f''(x) = \frac{-2x^2 - 2 + 8x^2}{(x^2+1)^3}$$

$$f''(x) = \frac{6x^2 - 2}{(x^2+1)^3}$$

Find Possible POI's:
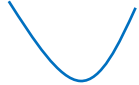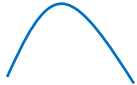
$$0 = \frac{6x^2 - 2}{(x^2+1)^3}$$

$$0 = 6x^2 - 2$$

$$x^2 = \frac{2}{6}$$

$$x = \pm\frac{1}{\sqrt{3}} \cong 0.577$$

$(0.577, 0.75)$ and $(-0.577, 0.75)$

	$-\infty$		-0.577		0.577		∞
Test value for x		-1		0		1	
$f''(x)$		$+$		$-$		$+$	
$f(x)$		Concave UP		Concave DOWN		Concave UP	
			POI at $(-0.577,0.75)$		POI at $(0.577,0.75)$		

d) Sketch a graph of the function
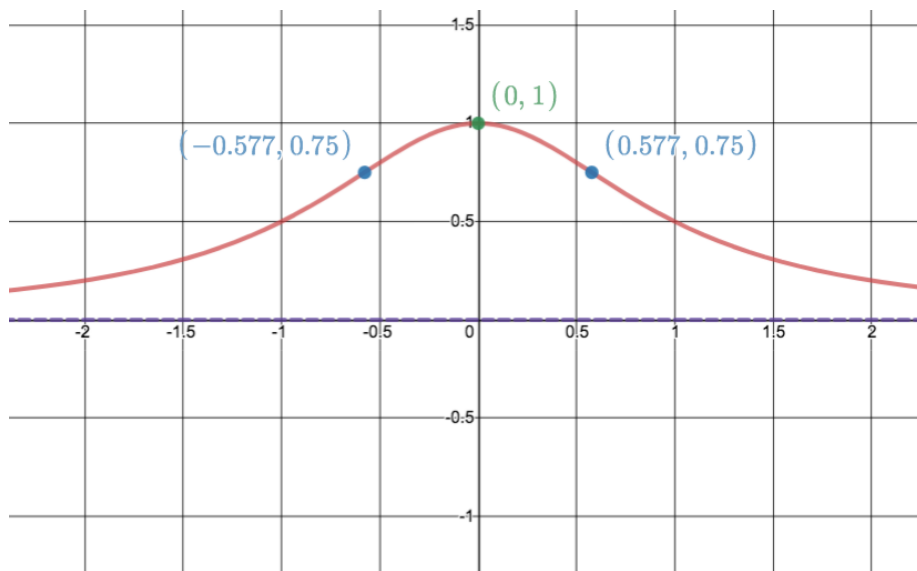
Algorithm for Curve Sketching

1. Determine any restrictions in the domain. State any horizontal and vertical asymptotes or holes in the graph.
2. Determine the intercepts of the graph
3. Determine the critical numbers of the function (where is $f'(x) = 0$ or undefined)
4. Determine the possible points of inflection (where is $f''(x)$=0 or undefined)
5. Create a sign chart that uses the critical numbers and possible points of inflection as dividing points.
6. Use the sign chart to find intervals of increase/decrease and the intervals of concavity. Use all critical numbers, possible points of inflection, and vertical asymptotes as dividing points.
7. Identify local extrema and points of inflection
8. Sketch the function

Example 1: Use the algorithm for curve sketching to analyze the key features of each of the following functions and to sketch a graph of them.

a) $g(x) = x^3 + 6x^2 + 9x$

1. No restrictions on the domain; no asymptotes

2. x-intercepts:

$0 = x^3 + 6x^2 + 9x$

$0 = x(x^2 + 6x + 9)$

$0 = x(x + 3)^2$

$x = 0$ or $x = -3$

$(0,0)$ and $(-3,0)$ are x-intercepts

y-intercept:

$g(0) = 0$

$(0,0)$ is the y-intercept

3. $g'(x) = 3x^2 + 12x + 9$

$0 = 3(x^2 + 4x + 3)$

$0 = 3(x + 3)(x + 1)$

Critical Numbers: $x = -3$ and $x = -1$

Critical Points: $(-3,0)$ and $(-1,-4)$

4. $g''(x) = 6x + 12$

$0 = 6(x + 2)$

$x = -2$

$(-2, -2)$ is a possible point of inflection

5/6/7.

Test value for x	$-\infty$ -4	-3 -2.5	-2 -1.5	-1 0 ∞
$g'(x)$	$+$	$-$	$-$	$+$
$g''(x)$	$-$	$-$	$+$	$+$
$g(x)$	Concave Down Increasing	Concave Down Decreasing	Concave UP Decreasing	Concave UP Increasing

Local max at $(-3, 0)$

POI at $(-2, -2)$

Local min at $(-1, -4)$

8.
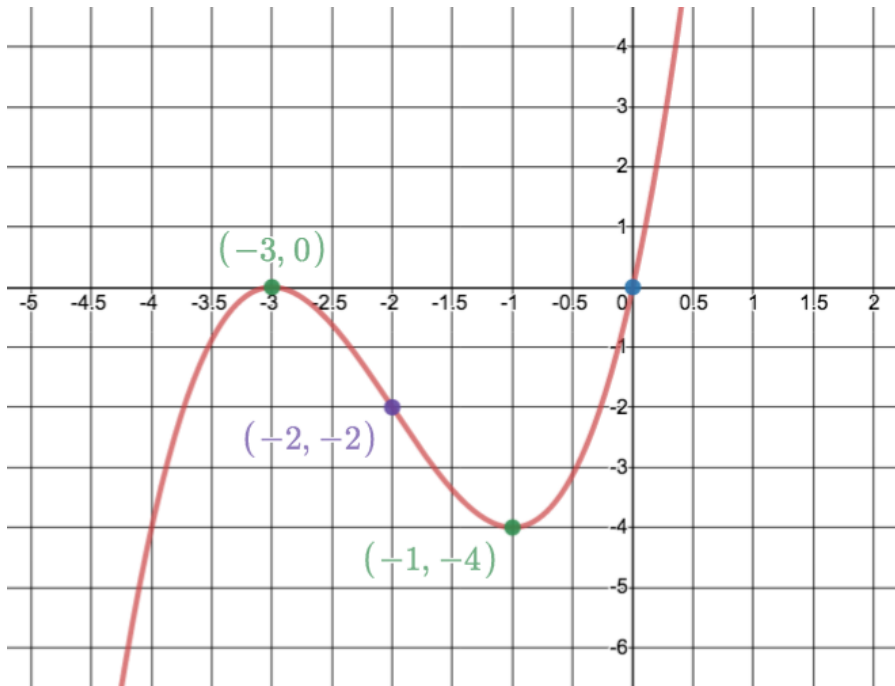
b) $f(x) = \dfrac{1}{(x+1)(x-4)} = \dfrac{1}{x^2-3x-4}$

1. VA: $x = -1$ and $x = 4$ (from restrictions on denominator)

HA: $y = 0$ (degree in denominator is higher than degree in numerator)

2. x-int: NONE

y-int: $f(0) = \dfrac{1}{(0+1)(0-4)} = -\dfrac{1}{4}$ $(0, -0.25)$

3. $f'(x) = \dfrac{0(x+1)(x-4) - (2x-3)(1)}{(x^2-3x-4)^2}$

$f'(x) = \dfrac{-2x+3}{(x^2-3x-4)^2}$

Check for zeros:

$0 = -2x + 3$

$x = \dfrac{3}{2} = 1.5$

Check for restrictions:

$0 = (x^2 - 3x - 4)^2$

$0 = [(x-4)(x+1)]^2$

$x = 4$ and $x = -1$

However, these are not in the domain of $f(x)$, therefore are NOT critical numbers

Critical Number: $x = \frac{3}{2} = 1.5$

Critical Point: $f(1.5) = -0.16$ $(1.5, -0.16)$

4. $f''(x) = \dfrac{-2(x^2-3x-4)^2 - 2(x^2-3x-4)(2x-3)(-2x+3)}{(x^2-3x-4)^4}$

$$f''(x) = \frac{(x^2 - 3x - 4)[-2(x^2 - 3x - 4) - 2(2x - 3)(-2x + 3)]}{(x^2 - 3x - 4)^4}$$

$$f''(x) = \frac{-2(x^2 - 3x - 4) - 2(2x - 3)(-2x + 3)}{(x^2 - 3x - 4)^3}$$

$$f''(x) = \frac{-2x^2 + 6x + 8 - 2(-4x^2 + 12x - 9)}{(x^2 - 3x - 4)^3}$$

$$f''(x) = \frac{-2x^2 + 6x + 8 + 8x^2 - 24x + 18}{(x^2 - 3x - 4)^3}$$

$$f''(x) = \frac{6x^2 - 18x + 26}{(x^2 - 3x - 4)^3}$$

Check for zeros:	Check for restrictions:
$0 = 6x^2 - 18x + 26$	$0 = (x^2 - 3x - 4)^3$
$b^2 - 4ac = (-18)^2 - 4(6)(26) = -300$	$0 = [(x - 4)(x + 1)]^3$
$b^2 - 4ac < 0$ therefore no real solutions	$x = 4$ and $x = -1$
$f(x)$ has not points of inflection	

Possible changes in concavity at $x = -1$ and $x = 4$ but $f(x)$ is not defined at these values either, therefore they can NOT be considered points of inflection.

5/6/7.

	$-\infty$	-1	1.5	4	∞
Test value for x		-2	0	2	5
$f'(x)$		$+$	$+$	$-$	$-$
$f''(x)$		$+$	$-$	$-$	$+$
$f(x)$		Concave UP Increasing	Concave DOWN Increasing	Concave DOWN Decreasing	Concave UP Decreasing

Local Max at $(1.5, -0.16)$

8.

(1.5, −0.16)
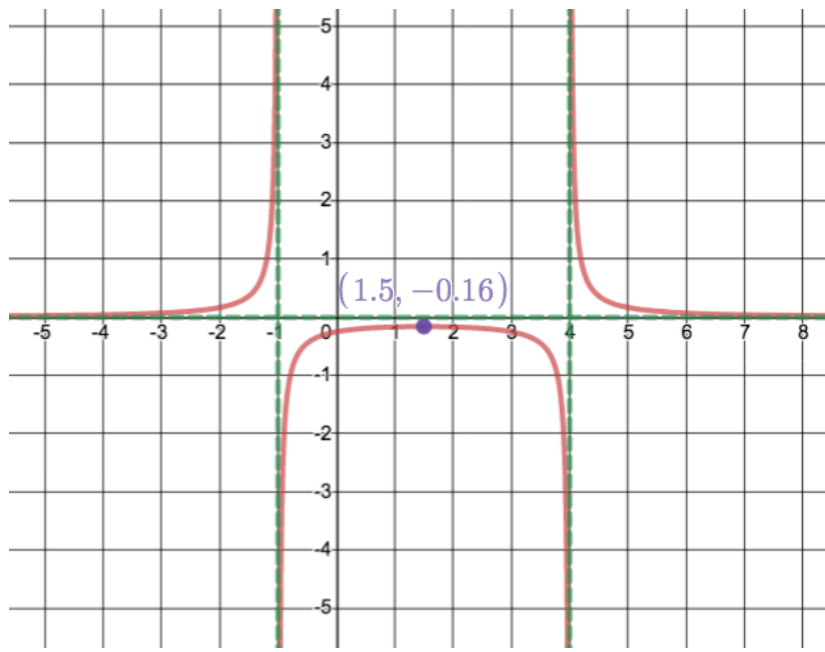
c) $h(x) = x^4 - 5x^3 + x^2 + 21x - 18$

1. No restrictions; no asymptotes

2. x-int:

$0 = x^4 - 5x^3 + x^2 + 21x - 18$

$0 = (x - 1)(x^3 - 4x^2 - 3x + 18)$

$0 = x^4 - 5x^3 + x^2 + 21x - 18$

$0 = (x - 1)(x^3 - 4x^2 - 3x + 18)$

$0 = (x - 1)(x + 2)(x^2 - 6x + 9)$

$0 = (x - 1)(x + 2)(x - 3)^2$

$x = 1 \quad x = -2 \quad x = 3$

$(1, 0), (-2, 0)$ and $(3, 0)$

y-int: $h(0) = -18 \quad (0, -18)$

Factor: $x^4 - 5x^3 + x^2 + 21x - 18$

Possible zeros: $\pm 1, 2, 3, 4, 6, 9, 18$

$h(1) = 0; x - 1$ is a factor

1	1	−5	1	21	−18
		1	−4	−3	18
	1	−4	−3	18	0

Factor: $x^3 - 4x^2 - 3x + 18$

Possible zeros: $\pm 1, 2, 3, 4, 6, 9, 18$

$h(-2) = 0; x + 2$ is a factor

−2	1	−4	−3	18
		−2	12	−18
	1	−6	9	0

3. $h'(x) = 4x^3 - 15x^2 + 2x + 21$

$0 = 4x^3 - 15x^2 + 2x + 21$

$0 = (x + 1)(4x^2 - 19x + 21)$

$0 = (x + 1)[4x^2 - 7x - 12x + 21]$

$0 = (x + 1)[x(4x - 7) - 3(4x - 7)]$

$0 = (x + 1)(4x - 7)(x - 3)$

Critical Numbers: $x = -1, \frac{7}{4}, 3$

Critical Points: $(-1, -32)$, $(1.75, 4.39)$, and $(3, 0)$

Factor: $4x^3 - 15x^2 + 2x + 21$

Possible zeros: $\pm 1, \frac{1}{2}, \frac{1}{4}, 3, \frac{3}{2}, \frac{3}{4}, 7, \frac{7}{2}, \frac{7}{4}, 21, \frac{21}{2}, \frac{21}{4}$

$h'(-1) = 0$; $x + 1$ is a factor

$$
\begin{array}{c|cccc}
-1 & 4 & -15 & 2 & 21 \\
 & & -4 & 19 & -21 \\
\hline
 & 4 & -19 & 21 & 0
\end{array}
$$

4. $h''(x) = 12x^2 - 30x + 2$

$0 = 2(6x^2 - 15x + 1)$

$0 = 6x^2 - 15x + 1$

$x = \dfrac{15 \pm \sqrt{(-15)^2 - 4(6)(1)}}{2(6)}$

$x = \dfrac{15 \pm \sqrt{201}}{12}$

$x \cong 2.43$ and $x \cong 0.07$

Possible points of inflection: $(2.43, 2.05)$ and $(0.07, -16.56)$

5/6/7.

Test value for x	$-\infty$ -2	-1 0	0.07 1	1.75 2	2.43 2.5	3 4	∞
$h'(x)$	$-$	$+$	$+$	$-$	$-$	$+$	
$h''(x)$	$+$	$+$	$-$	$-$	$+$	$+$	
$h(x)$	Decreasing Concave UP	Increasing Concave UP	Increasing Concave DOWN	Decreasing Concave DOWN	Decreasing Concave UP	Increasing Concave UP	

Local min at $(-1, -32)$

POI at $(0.07, -16.56)$

Local max at $(1.75, 4.39)$

POI at $(2.43, 2.05)$

Local min at $(3, 0)$
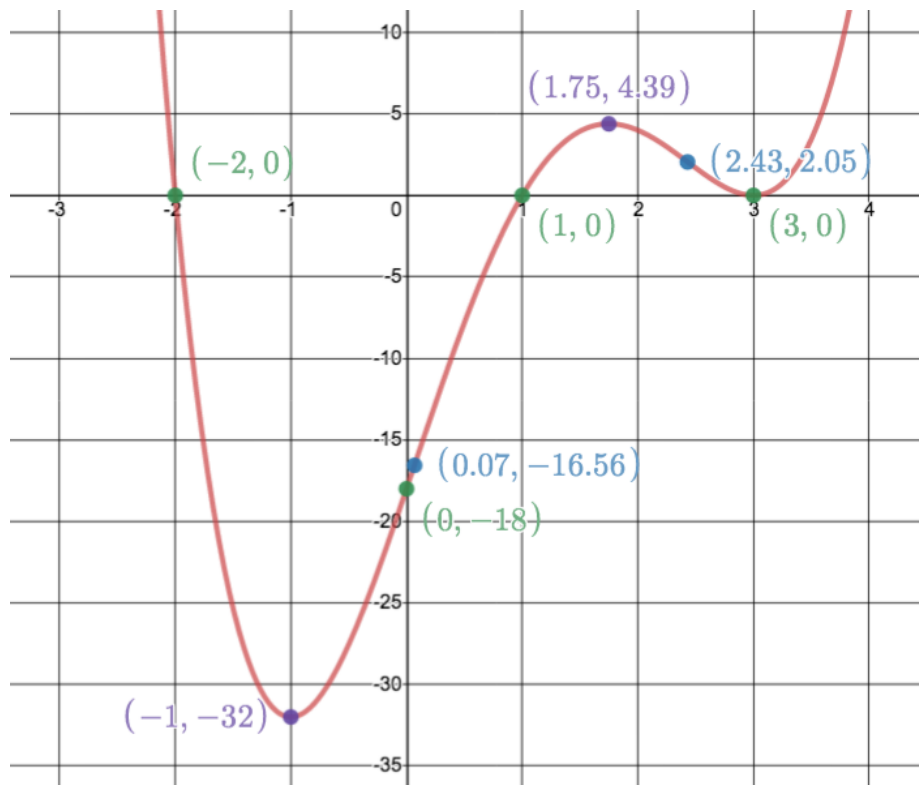
Tips for Optimization Problems:

- Diagrams can be helpful
- Identify the independent variable and express all other variables in terms of it
- Define a function in terms of the independent variable
- Identify any restriction on the variable
- Solve for $f'(x) = 0$ to identify critical points
- Check critical points and endpoints

Components of a Calculus Problem

- Setting Up Equation
- Algebra
- Solving Equation
- MAGIC
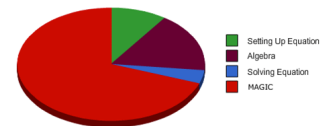
Optimization Warm Up:

A lifeguard has 200 meters of rope and some buoys with which she intends to enclose a rectangular area at a lake for swimming. The beach will form one side of the rectangle, with the rope forming the other 3 sides. Find the dimensions that will produce the maximum enclosed area.

$width = x$
$length = 200 - 2x$

$A = (length)(width)$
$A = x(200 - 2x)$
$A = 200x - 2x^2$

Note: The domain of this function is restricted to values $0 < x < 100$ because there is only 200 meters of rope to use.

To determine the max area, test endpoints of the interval as well as any critical numbers.

Critical Number(s):

$A'(x) = 200 - 4x$
$0 = 200 - 4x$
$4x = 200$
$x = 50$ is a critical number

Tests:

$A(0) = 0[200 - 2(0)]$ $A(50) = 50[200 - 2(50)]$ $A(100) = 100[200 - 2(100)]$
$A(0) = 0$ m² $A(50) = 5000$ m² $A(100) = 0$ m²

Therefore, the max area of 5000 m² occurs when the width is 50m and the length is 100m.

Example 1: A cardboard box with a square base is to have a volume of 8 Liters (1 L = 1000 cm^3)
Find the dimensions that will minimize the amount of cardboard to be used. What is the minimum surface area?

$SA = 2x^2 + 4xh$

Express h in terms of x using the volume equation
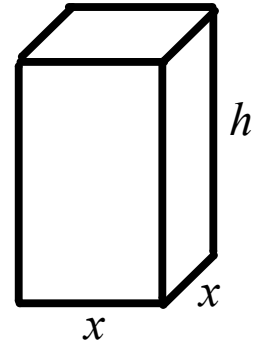
$SA = 2x^2 + 4x\left(\dfrac{8000}{x^2}\right)$

$SA = 2x^2 + 32000x^{-1}$

From Volume Equation:

$V = (area\ of\ base)(height)$
$8000 = x^2h$
$h = \dfrac{8000}{x^2}$

Find min value by finding zero(s) of the first derivative:

$SA'(x) = 4x - 32000x^{-2}$

$0 = 4x - 32000x^{-2}$

$32000x^{-2} = 4x$

$32000 = 4x^3$

$8000 = x^3$

$x = 20$ cm

Verify it is a min using the second derivative test:

$SA''(x) = 4 + 64000x^{-3}$

$SA''(20) = 4 + 64000(20)^{-3}$

$SA''(20) = 12$, therefore SA is concave up at $x = 20$, and there is a local min point.

Solve for h when $x = 20$:

$h = \dfrac{8000}{(20)^2}$

$h = 20$

Find The minimum surface area:

$SA(20) = 2(20)^2 + 32000(20)^{-1}$

$SA(20) = 2400$ cm³

Therefore, a minimum surface area of 2400 cm³ can be obtained when the dimensions of the box are 20 by 20 by 20 cm.

Example 2: A soup can of volume 500 cm³ is to be constructed. The material for the top costs 0.4¢/cm² while the material for the bottom and sides costs 0.2¢/cm². Find the dimensions that will minimize the cost of producing the can. What is the min cost?

$SA = 2\pi r^2 + 2\pi rh$

$SA = \pi r^2 + \pi r^2 + 2\pi rh$

$SA(r) = \pi r^2 + \pi r^2 + 2\pi r \left(\dfrac{500}{\pi r^2}\right)$

$SA(r) = \pi r^2 + \pi r^2 + \dfrac{1000}{r}$

From Volume Equation:

$V = \pi r^2 h$
$500 = \pi r^2 h$
$h = \dfrac{500}{\pi r^2}$

$C(r) = 0.4(\pi r^2) + 0.2(\pi r^2) + 0.2\left(\dfrac{1000}{r}\right)$

$C(r) = 0.6\pi r^2 + \dfrac{200}{r}$

$C'(r) = 1.2\pi r - \dfrac{200}{r^2}$

$0 = 1.2\pi r - \dfrac{200}{r^2}$

$\dfrac{200}{r^2} = 1.2\pi r$

$\dfrac{200}{1.2\pi} = r^3$

$r = \sqrt[3]{\dfrac{200}{1.2\pi}}$

$r = 3.76 \; cm$

2ⁿᵈ Derivative Test

$C''(r) = 1.2\pi + \dfrac{400}{r^3}$

$C''(3.76) = 11.29$; therefore at the point $\left(3.76, C(3.76)\right)$, $C(r)$ is concave up and the point is a MIN point.

$C(3.76) = 79.84$ cents

A min cost of 79.84 cents can be obtained by using dimensions $r = 3.76$ cm and $h = \dfrac{500}{\pi(3.76)^2} = 11.26$ cm

Example 3: Ian and Ada are both training for a marathon. Ian's house is located 20 km north of Ada's house. At 9:00 am one Saturday, Ian leaves his house and jogs south at 8 km/h. At the same time, Ada leaves her house and jogs east at 6 km/h. When are Ian and Ada closest together, given that they both run for 2.5 hours?

$$s^2 = JA^2 + AB^2$$

$$s = \sqrt{JA^2 + AB^2}$$

Express s in terms of time (t)

$$JA = 20 - 8t$$

$$AB = 6t$$

$$s = \sqrt{(20 - 8t)^2 + (6t)^2}$$

$$s = [400 - 320t + 64t^2 + 36t^2]^{\frac{1}{2}}$$

$$s = (100t^2 - 320t + 400)^{\frac{1}{2}}$$

Find Critical Number(s):

$$s'(t) = \frac{1}{2}(100t^2 - 320t + 400)^{-\frac{1}{2}}(200t - 320)$$

$$s'(t) = \frac{200t - 320}{2\sqrt{100t^2 - 320t + 400}}$$

$$s'(t) = \frac{100t - 160}{\sqrt{100t^2 - 320t + 400}}$$

$$0 = \frac{100t - 160}{\sqrt{100t^2 - 320t + 400}}$$

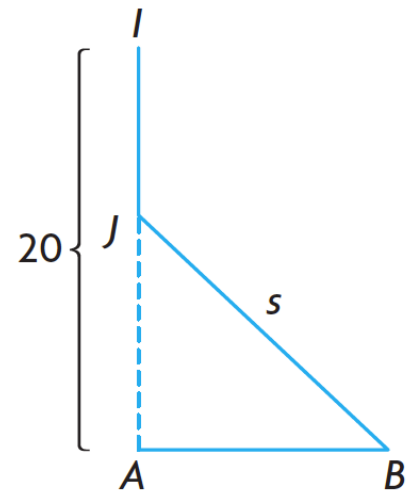$$0 = 100t - 160$$

$$160 = 100t$$

$$t = 1.6$$

Check endpoints of interval and critical number to determine minimum value:

$$s(0) = \sqrt{[20 - 8(0)]^2 + [6(0)]^2} = 20$$

$$s(1.6) = \sqrt{[20 - 8(1.6)]^2 + [6(1.6)]^2} = 12$$

$$s(2.5) = \sqrt{[20 - 8(2.5)]^2 + [6(2.5)]^2} = 15$$

Therefore, the minimum distance between Ada and Ian occurs after 1.6 hours (10:36 am).