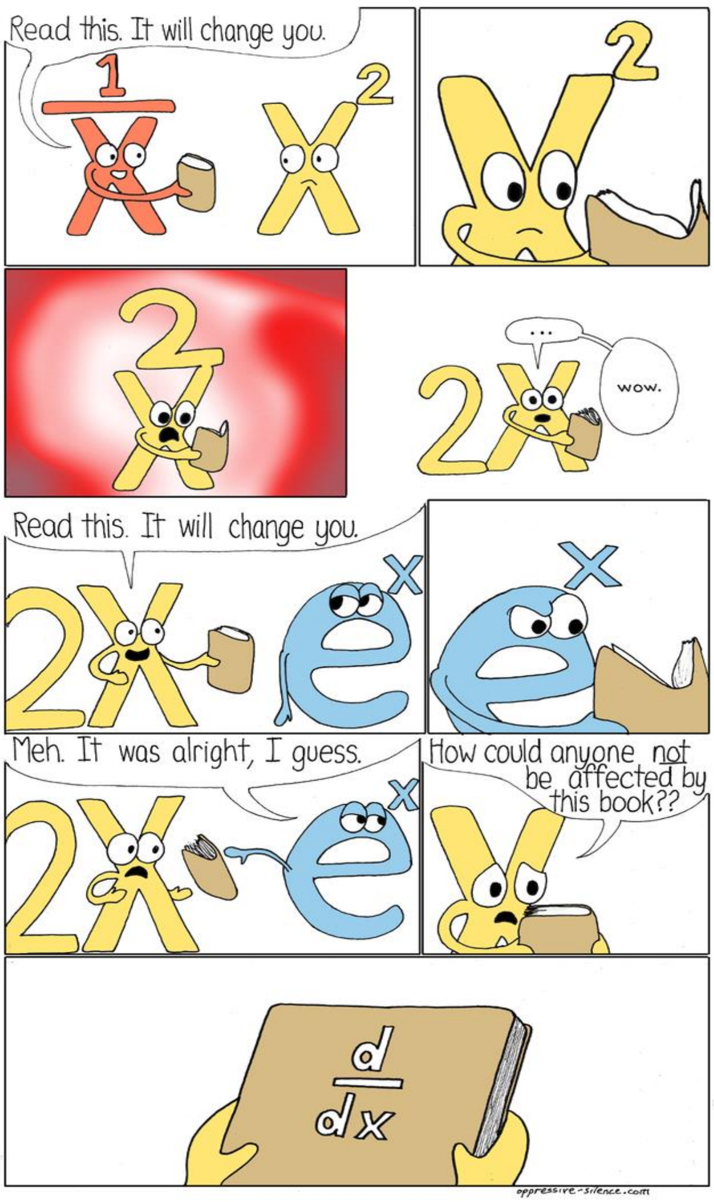
Unit 3 – Derivatives of Trig and Exponential Functions

Lesson Package

MCV4U

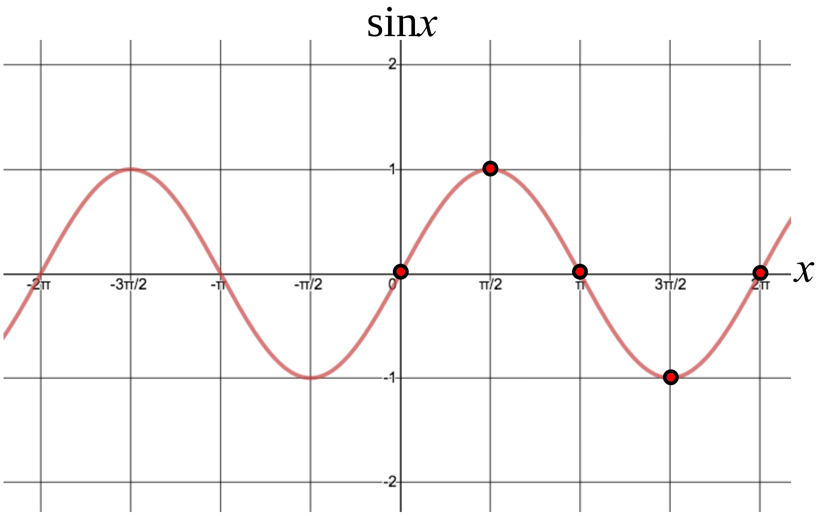
Name:



L1 – Derivatives of Sine and Cosine Unit 3
MCV4U
Jensen

Part 1: Investigation

Example 1: Find the derivative of $\sin x$



x	$\sin x$	$\frac{d}{dx}\sin x$
0		
$\frac{\pi}{2}$		
π		
$\frac{3\pi}{2}$		
2π		

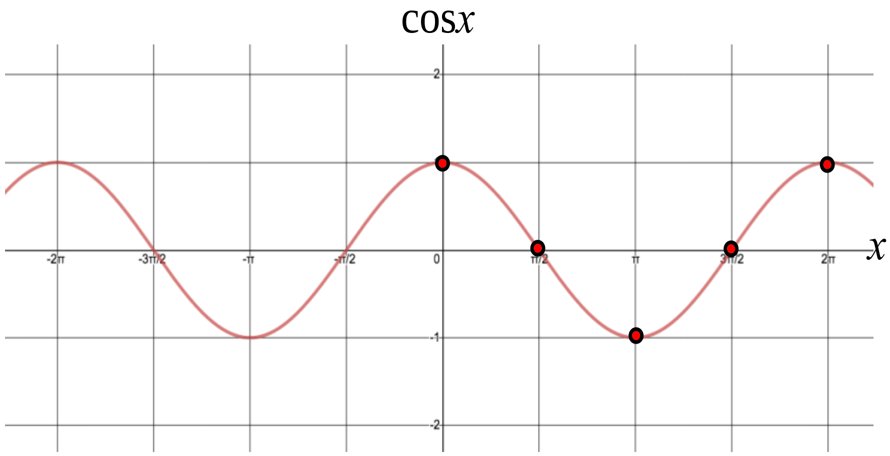
a) Complete the $\sin x$ column in the table.

b) Estimate the instantaneous rate of change (tangent slope) of $\sin x$ at $x = 0$ using a secant line. Use the interval $\left[\frac{0\pi}{100}, \frac{1\pi}{100}\right]$.

c) Complete the instantaneous rate of change column.

d) What do you notice about the values of $\frac{d}{dx}\sin x$? Plot the values and graph the derivative of $\sin x$. What is the derivative of $\sin x$?

Example 2: Repeat the process to find the derivative of $\cos x$



x	$\cos x$	$\frac{d}{dx}\cos x$
0		
$\frac{\pi}{2}$		
π		
$\frac{3\pi}{2}$		
2π		

a) Complete the $\cos x$ column in the table.

b) Estimate the instantaneous rate of change (tangent slope) of $\cos x$ at $x = \frac{\pi}{2}$ using a secant line. Use the interval $\left[\frac{50\pi}{100}, \frac{51\pi}{100}\right]$.

c) Complete the instantaneous rate of change column.

d) What do you notice about the values of $\frac{d}{dx}\cos x$? Plot the values and graph the derivative of $\cos x$. What is the derivative of $\cos x$?

Example 3: Find the derivative of $\tan x$.

Derivatives of Trig Functions:

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Part 2: Differentiating equations involving trig functions

The rules for differentiation apply to sinusoidal functions. A reminder of these rules is below:

Rule	Derivative
Power Rule If $f(x) = x^n$	$f'(x) = nx^{n-1}$
Constant Multiple Rule If $f(x) = c \cdot g(x)$ where c is a constant	$f'(x) = c \cdot g'(x)$
Sum Rule If $h(x) = f(x) + g(x)$	$h'(x) = f'(x) + g'(x)$
Difference Rule If $h(x) = f(x) - g(x)$	$h'(x) = f'(x) - g'(x)$
Product Rule If $h(x) = f(x)g(x)$	$h'(x) = f'(x)g(x) + f(x)g'(x)$
Quotient Rule If $h(x) = f(x) \div g(x)$	$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$
Power of a Function Rule If $h(x) = \left(f(x)\right)^n$	$h'(x) = n[f(x)]^{n-1} \times f'(x)$
Chain Rule If $h(x) = f(g(x))$	$h'(x) = f'[g(x)] \times g'(x)$

Example 4: Differentiate each of the following

a) $y = 2\sin x$ **b)** $f(x) = -3\cos x$ **c)** $y = 4\tan x$

Example 5: Differentiate with respect to x

a) $y = \sin x + \cos x$

b) $y = 2\cos x - 4\sin x$

Example 6: Find the slope of the tangent line to the graph of $f(x) = 3\sin x$ at the point where $x = \frac{\pi}{4}$

Remember special triangles:

Example 7: Find the equation of the tangent line to the curve $f(x) = -2\sin x$ at the point where $x = \frac{\pi}{6}$.

L2 – MORE Derivatives of Sine and Cosine Unit 3
MCV4U
Jensen

Reminder of rules:

Rule	Derivative
Power Rule If $f(x) = x^n$	$f'(x) = nx^{n-1}$
Constant Multiple Rule If $f(x) = c \cdot g(x)$ where c is a constant	$f'(x) = c \cdot g'(x)$
Sum Rule If $h(x) = f(x) + g(x)$	$h'(x) = f'(x) + g'(x)$
Difference Rule If $h(x) = f(x) - g(x)$	$h'(x) = f'(x) - g'(x)$
Product Rule If $h(x) = f(x)g(x)$	$h'(x) = f'(x)g(x) + f(x)g'(x)$
Quotient Rule If $h(x) = f(x) \div g(x)$	$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$
Power of a Function Rule If $h(x) = (f(x))^n$	$h'(x) = n[f(x)]^{n-1} \times f'(x)$
Chain Rule If $h(x) = f(g(x))$	$h'(x) = f'[g(x)] \times g'(x)$

Example 1: Determine the derivative with respect to x

a) $y = \sin(2x)$

b) $y = \sin^2 x$

c) $y = \sin(x^2)$

d) $x^2 \sin x$

Example 2: Find the derivative with respect to x for each function.

a) $y = \cos(3x)$

b) $f(x) = 2\sin(\pi x)$

c) $g(x) = \tan(x^2 + 3x)$

Example 3: Differentiate with respect to x.

a) $y = \cos^3 x$

b) $f(x) = 2\sin^3 x - 4\cos^2 x$

Example 4: Find each derivative with respect to t.

a) $y = t^3 \cos t$

b) $h(t) = \sin(4t)\cos^2 t$

Example 5: Find the derivative of $y = x\tan(2x - 1)$

L3 – Derivatives of Exponential Functions Unit 3
MCV4U
Jensen

Part 1: Review of e and $\ln x$

Properties of e:

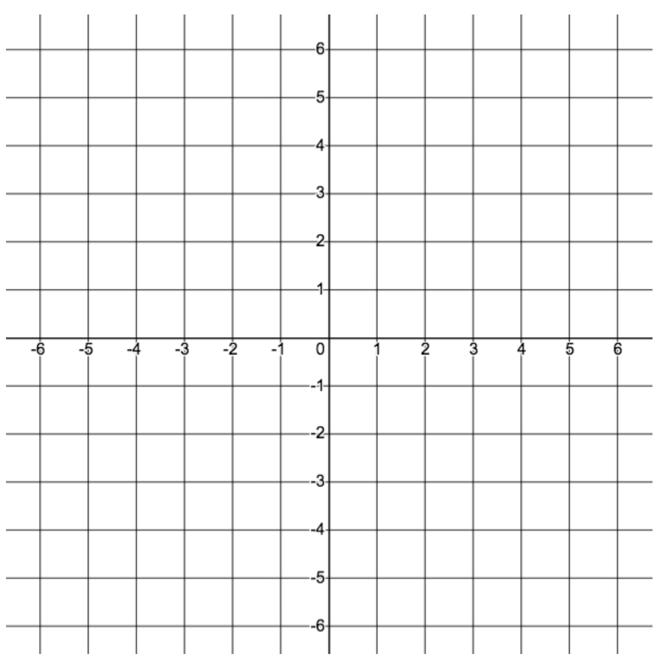
- $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx$
- e is an ____________________ number, similar to π. They are non-terminating and non-repeating.
- $\log_e x$ is known as the ____________________ and can be written as __________.
- Many naturally occurring phenomena can be modelled using base-e exponential and logarithmic functions.
- $\log_e e = \ln e =$

Example 1: Graph the functions $y = e^x$ and $y = \ln x$

$y = e^x$	
x	y

$y = \ln x$	
x	y

Note: $y = \ln x$ is the inverse of $y = e^x$



Example 2: The population of a bacterial culture as a function of time is given by the equation $P(t) = 200e^{0.094t}$, where P is the population after t days.

a) What is the initial population of the bacterial culture?

b) Estimate the population after 3 days.

c) How long will the bacterial culture take to double its population?

d) Re-write this function as an exponential function having base 2.

Use the exponential growth formula:

$$A(t) = A_0(2)^{\frac{t}{D}}$$

where D is the doubling period

Part 2: Derivatives of Exponential Functions

Rule: If $f(x) = b^x$, $f'(x) = b^x \ln b$

Example 3: Determine the derivative of each function

a) $y = 2^x$

b) $y = e^x$

c) $y = 3(2)^x$

d) $y = 3^x + 1$

Notice that $\frac{d}{dx} e^x = e^x$

Example 4: Find the equation of the line that is tangent to the curve $y = 2e^x$ at $x = \ln 3$.

Example 5: A biologist is studying the increase in the population of a particular insect in a provincial park. The population triples every week. Assume the population continues to increase at this rate. Initially, there are 100 insects.

a) Determine the number of insects present after 4 weeks.

b) How fast is the number of insects increasing

i) when they are initially discovered?

ii) at the end of 4 weeks?

L4 – Differentiation Rules for Exponential Functions Unit 3
MCV4U
Jensen

Part 1: Derivatives of Exponential Functions

Chain Rule:

If $h(x) = f(g(x))$

$h'(x) = f'[g(x)] \times g'(x)$

Apply to exponential functions:

If $h(x) = b^{g(x)}$

$h'(x) = b^{g(x)} \times \ln b \times g'(x)$

Example 1: Find the derivative of each function.

a) $y = xe^x$

b) $y = e^{2x+1}$

c) $y = e^x - e^{-x}$

d) $y = 2e^x \cos x$

e) $y = x^2 10^x$

Example 2: Identify the local extrema of the function $f(x) = x^2 e^x$.

Example 3: The effectiveness of studying for an exam depends on how many hours a student studies. Some experiments show that if the effectiveness, E, is put on a scale of 0 to 10, then $E(t) = 0.5\left[10 + te^{-\frac{t}{20}}\right]$, where t is the number of hours spent studying for an examination. If a student has up to 30 hours for studying, how many hours are needed for maximum effectiveness.

L5 – Implicit Differentiation and Derivatives of Log Functions Unit 3
MCV4U
Jensen

Part 1: Chain Rule Using Leibniz Notation

If y is a function of u and u is a function of x, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

Example 1: Suppose we wish to differentiate $y = (5 + 2x)^{10}$ in order to calculate $\frac{dy}{dx}$. We make a substitution and let $u = 5 + 2x$ so that $y = u^{10}$

Part 2: Implicit Differentiation

Up until now, most functions were written in the form in which y was defined explicitly as a function of x, such as $y = x^3 - 4x$. In that equation, y is isolated and is expressed ESPLICITLY as a function of x.

Functions can also be defined implicitly by relations, such as a circle $x^2 + y^2 = 16$. In this case, y is not isolated or explicitly defined in terms of x. You could rearrange to isolate for y but often this is very difficult or impossible. It will be necessary to use the chain rule to implicitly differentiate when it is in implicit form.

$$\frac{d}{dx}f(y) = \frac{d}{dy}f(y) \times \frac{dy}{dx}$$

Example 2: Differentiate each of the following using implicit differentiation

a) $x^2 + y^2 = 16$

b) $y^2 + x^3 - y^3 + 6 = 3y$

Part 3: Derivative of Logarithms

Proof of the derivative of $y = \log_a x$

Start by writing in inverse form:

Now use implicit differentiation to differentiate with respect to x

Rule:

$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

If you need chain rule:

$$\frac{d}{dx}\log_a[f(x)] = \frac{1}{f(x)\ln a}f'(x) = \frac{f'(x)}{f(x)\ln a}$$

Example 3: Differentiate each of the following with respect to x.

a) $y = 2\ln(1 + x^2)$

b) $f(x) = 1 - \log_4(2x - 1)$

L6 – Applications of Exponential & Trigonometric Derivatives Unit 3
MCV4U
Jensen

Part 1: Review of Differentiating Exponential Functions

Example 1: Differentiate $y = 3^x e^{\sin x}$ with respect to x.

Rule	Derivative
Exponential Functions If $h(x) = b^{g(x)}$	$h'(x) = b^{g(x)} \times \ln b \times g'(x)$
Trig Functions If $f(x) = \sin x$ $g(x) = \cos x$ $h(x) = \tan x$	$f'(x) = \cos x$ $g'(x) = -\sin x$ $h'(x) = \sec^2 x$
Log Functions If $g(x) = \log_a[f(x)]$	$g'(x) = \frac{f'(x)}{f(x)\ln a}$

Example 2: A radioactive isotope of gold, Au-198, is used in the diagnosis and treatment of liver disease. Suppose that a 6.0 mg sample of Au-198 is injected into a liver, and that this sample decays to 4.6 mg after 1 day. Assume the amount of Au-198 reaming after t days is given by $N(t) = N_0 e^{-\lambda t}$.

a) Determine the disintegration constant for Au-198

The Greek letter λ, lambda, indicates the disintegration constant, which is related to how fast a radioactive substance decays.

b) Determine the half-life of Au-198

c) Write the equation that gives the amount of Au-198 remaining as a function of time in terms of its half-life.

d) How fast is the sample decaying after 3 days?

Example 3: A power supply delivers a voltage signal that consists of an alternating current (AC) component and a direct current (DC) component. Where t is the time, in seconds, the voltage, in volts, at time t is given by the function $V(t) = 5\sin t + 12$.

a) What are the max and min voltages? At which times do these values occur?

b) Determine the period, T, in seconds, frequency, f, in hertz, and amplitude, A, in volts, for this signal.

Example 4: For small amplitudes, and ignoring the effects of friction, a pendulum is an example of simple harmonic motion. Simple harmonic motion is motion that can be modelled by a sinusoidal function, and the graph of a function modelling simple harmonic motion has a constant amplitude.

The period of a simple pendulum depends only on its length and can be found using the relation $T = 2\pi\sqrt{\frac{l}{g}}$, where T is the period, in seconds, l is the length of the pendulum, in meters, and g is the acceleration due to gravity. On or near the surface of Earth, g has a constant value of 9.8 m/s^2.

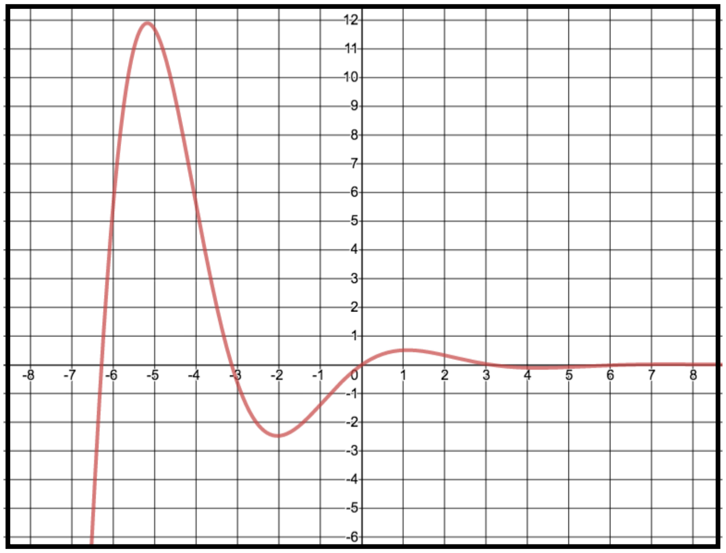
Under these conditions, the horizontal position of the bob as a function of time can be described by the function $h(t) = A\cos\left(\frac{2\pi t}{T}\right)$, where A is the amplitude of the pendulum, t is time, in seconds, and T is the period of the pendulum, in seconds.

Find the max speed of the bob and the time at which that speed first occurs for a pendulum having a length of 1.0 meters and an amplitude of 5 cm.

Example 5: The vertical displacement of a SUV's body after passing over a bump is modelled by the function $h(t) = e^{-0.5t} \sin t$, where h is the vertical displacement, in meters, at time t, in seconds.

a) Use technology to generate a rough sketch of the graph of the function.

Notice that the amplitude diminishes over time. This motion is called DAMPED HARMONIC MOTION.



b) Determine when the max displacement of the sport utility vehicle's body occurs.

c) Determine the maximum displacement.