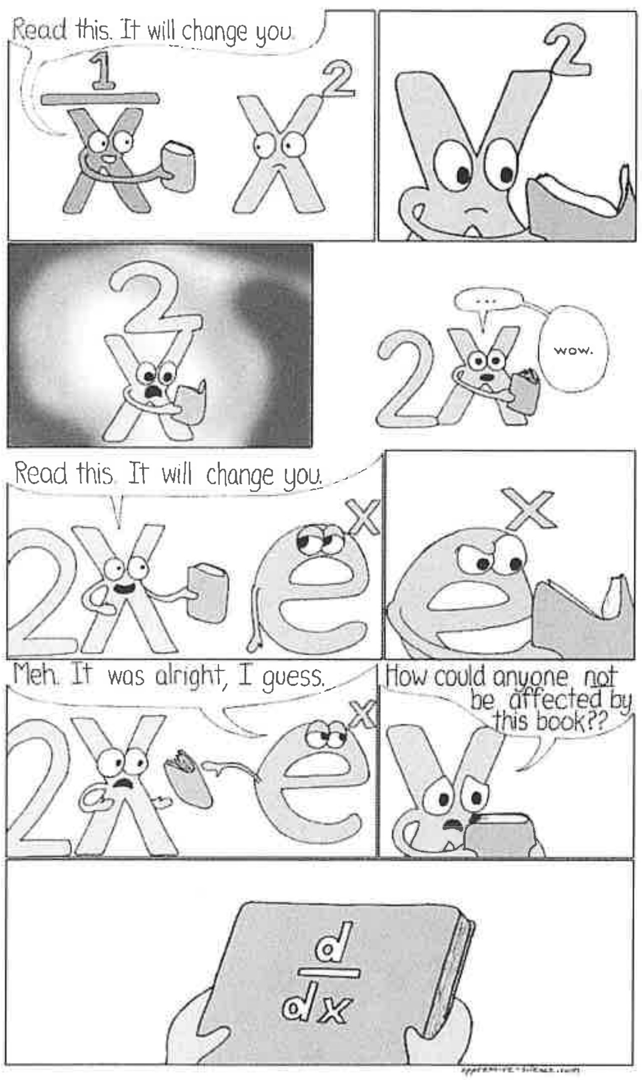
Name: SOLUTIONS

Unit 3 – Derivatives of Trig and Exponential Functions

WORKBOOK

MCV4U



W1 – Derivatives of Sine and Cosine Unit 3

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SOLUTIONS

1) Find the derivative with respect to x for each function.

a) $y = 4\sin x$

$y' = 4\cos x$

b) $f(x) = -3\cos x$

$f'(x) = -3(-\sin x)$

$f'(x) = 3\sin x$

c) $y = \cos x - \sin x$

$y' = -\sin x - \cos x$

d) $y = x^2 - 3\sin x$

$y' = 2x - 3\cos x$

e) $y = \cos x + 7\pi \sin x - 3x$

$y' = -\sin x + 7\pi\cos x - 3x$

f) $f(x) = \frac{\pi}{4}\cos x - \frac{\pi}{3}\sin x$

$f'(x) = \frac{\pi}{4}(-\sin x) - \frac{\pi}{3}\cos x$

$f'(x) = -\frac{\pi}{4}\sin x - \frac{\pi}{3}\cos x$

Find the equation of the line that is tangent to the function $y = \cos x$ and passes through the point $\left(\frac{\pi}{3}, \frac{1}{2}\right)$.

Slope:

$y' = -\sin x$

$y'\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right)$

$y'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$m = -\frac{\sqrt{3}}{2}$

Eq^n:

$y = mx + b$

$\frac{1}{2} = \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{3}\right) + b$

$\frac{1}{2} = -\frac{\sqrt{3}\pi}{6} + b$

$\frac{3}{6} + \frac{\sqrt{3}\pi}{6} = b$

$b = \frac{3 + \sqrt{3}\pi}{6}$

$$\boxed{y = \frac{-\sqrt{3}}{2}x + \frac{3 + \pi\sqrt{3}}{6}}$$

3) Find the equation of the line that is tangent to the function $y = -4\sin x$ at $x = \frac{\pi}{4}$.

Point:

$y\left(\frac{\pi}{4}\right) = -4\sin\left(\frac{\pi}{4}\right)$

$y\left(\frac{\pi}{4}\right) = -4\left(\frac{1}{\sqrt{2}}\right)$

$y\left(\frac{\pi}{4}\right) = -4\left(\frac{\sqrt{2}}{2}\right)$

$y\left(\frac{\pi}{4}\right) = -2\sqrt{2}$

$\left(\frac{\pi}{4}, -2\sqrt{2}\right)$

Slope:

$y' = -4\cos x$

$y'\left(\frac{\pi}{4}\right) = -4\cos\left(\frac{\pi}{4}\right)$

$y'\left(\frac{\pi}{4}\right) = -4\left(\frac{1}{\sqrt{2}}\right)$

$y'\left(\frac{\pi}{4}\right) = -2\sqrt{2}$

$m = -2\sqrt{2}$

Eq^n: $y = mx + b$

$-2\sqrt{2} = -2\sqrt{2}\left(\frac{\pi}{4}\right) + b$

$-2\sqrt{2} = -\frac{\pi\sqrt{2}}{2} + b$

$-2\sqrt{2} + \frac{\pi\sqrt{2}}{2} = b$

$$\boxed{y = -2\sqrt{2}x + \frac{\pi\sqrt{2}}{2} - 2\sqrt{2}}$$

4) Determine an equation for the tangent to the function $f(x) = \tan x$ at $x = \frac{\pi}{4}$.

Point:

$f(\frac{\pi}{4}) = \tan(\frac{\pi}{4})$

$f(\frac{\pi}{4}) = 1$

$(\frac{\pi}{4}, 1)$

Slope:

$f'(x) = \sec^2 x$

$f'(\frac{\pi}{4}) = \sec^2(\frac{\pi}{4})$

$f'(\frac{\pi}{4}) = \frac{1}{\cos^2(\frac{\pi}{4})}$

$f'(\frac{\pi}{4}) = \frac{1}{(\frac{1}{\sqrt{2}})^2}$

$m = 2$

Eq^n:

$y = mx + b$

$1 = 2(\frac{\pi}{4}) + b$

$1 - \frac{\pi}{2} = b$

$\boxed{y = 2x + 1 - \frac{\pi}{2}}$

5) Find an equation of a line that is tangent to $y = 2\sin x$ and whose slope is a max value.

$y = 2\sin x$ has a max slope when $y' = 2\cos x$ is a max.

$2 = 2\cos x$

$1 = \cos x$

$x = 0, 2\pi, 4\pi, 6\pi, \ldots$

max when $x = 2k\pi$, $k \in \mathbb{Z}$

(use any answer)

Point:

$y(0) = 2\sin(0)$

$y(0) = 0$

$(0, 0)$

Slope:

$y'(0) = 2\cos(0)$

$y'(0) = 2(1)$

$y'(0) = 2$

Eq^n: $y = mx + b$

$0 = 2(0) + b$

$b = 0$

$\boxed{y = 2x}$

Answers:

1)a) $\frac{dy}{dx} = 4\cos x$ **b)** $f'(x) = 3\sin x$ **c)** $\frac{dy}{dx} = -\sin x - \cos x$ **d)** $\frac{dy}{dx} = 2x - 3\cos x$ **e)** $\frac{dy}{dx} = -\sin x + 7\pi\cos x - 3$
f) $\frac{dy}{dx} = -\frac{\pi}{4}\sin x - \frac{\pi}{3}\cos x$

2) $y = -\frac{\sqrt{3}}{2}x + \frac{\pi\sqrt{3}+3}{6}$

3) $y = -2\sqrt{2}x + \frac{\sqrt{2}\pi}{2} - 2\sqrt{2}$

4) $y = 2x + 1 - \frac{\pi}{2}$

5) $y = 2x$; note: there are an infinite number of solutions. The slope is at a max value at any $x = 2k\pi$ where $k \in \mathbb{Z}$. Depending on which x value you choose, you will get a different y-int.

W2 – MORE Derivatives of Sine and Cosine Unit 3

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SOLUTIONS

1) Determine the derivative of each function.

a) $y = \sin(4x)$

$y' = \cos(4x)(4)$

$y' = 4\cos(4x)$

b) $f(x) = \sin(2x + \pi)$

$f'(x) = \cos(2x + \pi)(2)$

$f'(x) = 2\cos(2x + \pi)$

c) $y = -2\sin(3\theta)$

$y' = -2\cos(3\theta)(3)$

$y' = -6\cos(3\theta)$

d) $y = \sin^2 x$

$y = (\sin x)^2$

$y' = 2(\sin x)(\cos x)$

$y' = 2\sin x \cos x$

e) $f(x) = \cos^2 x - \sin^2 x$

$f(x) = (\cos x)^2 - (\sin x)^2$

$f'(x) = 2\cos x(-\sin x) - 2\sin x(\cos x)$

$f'(x) = -2\cos x \sin x - 2\sin x \cos x$

$f'(x) = -2(\cos x \sin x + \sin x \cos x)$

$f'(x) = -2(2\sin x \cos x)$

$f'(x) = -2\sin(2x)$

f) $y = 3\sin^2(2t - 4) - 2\cos^2(3t + 1)$

$y = 3[\sin(2t-4)]^2 - 2[\cos(3t+1)]^2$

$y' = 3(2)[\sin(2t-4)]\cos(2t-4)(2) - 2(2)[\cos(3t+1)][-\sin(3t+1)](3)$

$y' = 12\sin(2t-4)\cos(2t-4) + 12\cos(3t+1)\sin(3t+1)$

$y' = 6\sin[2(2t-4)] + 6\sin[2(3t+1)]$

$y' = 6\sin(4t-8) + 6\sin(6t+2)$

g) $f(t) = \sin^2(\cos t)$

$f(t) = [\sin(\cos t)]^2$

$f'(t) = 2[\sin(\cos t)][\cos(\cos t)](-\sin t)$

$f'(t) = -2\sin(t)\sin(\cos t)\cos(\cos t)$

h) $f(x) = -x^2\sin(3x - \pi)$

$f'(x) = -2x\sin(3x-\pi) + \cos(3x-\pi)(3)(-x^2)$

$f'(x) = -2x(3x-\pi) - 3x^2\cos(3x-\pi)$

i) $f(\theta) = \sin^2\theta\cos^2\theta$

$f(\theta) = (\sin\theta)^2(\cos\theta)^2$

$f'(\theta) = 2\sin\theta\cos\theta\cos^2\theta + 2\cos\theta(-\sin\theta)\sin^2\theta$

$f'(\theta) = 2\sin\theta\cos^3\theta - 2\cos\theta\sin^3\theta$

j) $y = x^{-1}\cos^2 x$

$y = x^{-1}(\cos x)^2$

$y' = -1x^{-2}(\cos^2 x) + 2\cos x(-\sin x)x^{-1}$

$y' = \dfrac{-\cos^2 x}{x^2} - \dfrac{2\cos x\sin x}{x}$

k) $y = 2\tan x - \tan(2x)$

$y' = 2\sec^2 x - \sec^2(2x)(2)$

$y' = 2\sec^2 x - 2\sec^2(2x)$

l) $y = (\tan x + \cos x)^2$

$y' = 2(\tan x + \cos x)(\sec^2 x - \sin x)$

2) Find the slope of the function $y = 2\cos x \sin(2x)$ at $x = \frac{\pi}{2}$.

$$y' = -2\sin x[\sin(2x)] + \cos(2x)(2)(2\cos x)$$

$$y' = -2\sin x[\sin(2x)] + 4\cos(2x)\cos x$$

$$y'(\tfrac{\pi}{2}) = -2\sin(\tfrac{\pi}{2})[\sin(\pi)] + 4\cos(\pi)\cos(\tfrac{\pi}{2})$$

$$y'(\tfrac{\pi}{2}) = -2(1)(0) + 4(-1)(0)$$

$$y'(\tfrac{\pi}{2}) = 0$$

$$\boxed{m = 0}$$

3) Find the equation of the line that is tangent to $y = x^2\sin(2x)$ at $x = -\pi$.

Slope:

$$y' = 2x\sin(2x) + \cos(2x)(2)x^2$$

$$y' = 2x\sin(2x) + 2x^2\cos(2x)$$

$$y'(-\pi) = 2(-\pi)\sin(-2\pi) + 2(-\pi)^2\cos(-2\pi)$$

$$y'(-\pi) = -2\pi(0) + 2\pi^2(1)$$

$$y'(-\pi) = 2\pi^2$$

Point:

$$y(-\pi) = (-\pi)^2\sin(-2\pi)$$

$$y(-\pi) = 0$$

$$(-\pi, 0)$$

Eqn: $y = mx + b$

$$0 = 2\pi^2(-\pi) + b$$

$$b = 2\pi^3$$

$$\boxed{y = 2\pi^2 x + 2\pi^3}$$

4) Determine $\frac{d^2y}{dx^2}$ for $y = x^2\cos x$.

$$\frac{dy}{dx} = 2x\cos x + (-\sin x)(x^2)$$

$$\frac{dy}{dx} = 2x\cos x - x^2\sin x$$

$$\frac{d^2y}{dx^2} = 2\cos x + (-\sin x)(2x) - [2x\sin x + \cos x(x^2)]$$

$$= 2\cos x - 2x\sin x - 2x\sin x - x^2\cos x$$

$$= 2\cos x - 4x\sin x - x^2\cos x$$

5)a) Write $y = \csc x$ in terms of $\sin x$ as a reciprocal function.

$y = \frac{1}{\sin x}$

b) Write the function in terms of a negative power of $\sin x$

$y = (\sin x)^{-1}$

c) Use the power rule and chain rule to find the derivative of $y = \csc x$

$y' = -1(\sin x)^{-2}(\cos x)$

$y' = \frac{-\cos x}{\sin^2 x}$

$y' = \frac{-\cos x}{(\sin x)(\sin x)}$

$y = -\cot x\left(\frac{1}{\sin x}\right)$

$\boxed{y = -\cot x \csc x}$

Answers:

1)a) $\frac{dy}{dx} = 4\cos(4x)$ **b)** $f'(x) = 2\cos(2x+\pi)$ **c)** $\frac{dy}{d\theta} = -6\cos(3\theta)$ **d)** $\frac{dy}{dx} = 2\sin x\cos x$

e) $f'(x) = -2\sin(2x)$ **f)** $\frac{dy}{dx} = 12\sin(2t-4)\cos(2t-4) + 12\cos(3t-1)\sin(3t-1)$

g) $f'(t) = -2\sin(\cos t)\cos(\cos t)\sin t$ **h)** $f'(x) = -3x^2\cos(3x-\pi) - 2x\sin(3x-\pi)$

i) $f'(\theta) = -2\sin^3\theta\cos^3\theta + 2\sin\theta\cos^3\theta$ **j)** $\frac{dy}{dx} = \frac{-2}{x}\cos x\sin x - \frac{\cos^2 x}{x^2}$

k) $y' = 2\sec^2 x - 2\sec(2x)$ **l)** $y' = 2(\tan x + \cos x)(\sec^2 x - \sin x)$

2) 0

3) $y = 2\pi^2 x + 2\pi^3$

4) $\frac{d^2y}{dx^2} = -x^2\cos x - 4x\sin x + 2\cos x$

5)a) $y = \frac{1}{\sin x}$ **b)** $y = (\sin x)^{-1}$ **c)** $\frac{dy}{dx} = -\csc x\cot x$

W3 – Derivatives of Exponential Functions Unit 3

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Solutions

1) Determine the derivative with respect to x for each function.

a) $g(x) = 4^x$

$g'(x) = 4^x(\ln 4)$

b) $f(x) = 11^x$

$f'(x) = 11^x[\ln(11)]$

c) $y = \left(\frac{1}{2}\right)^x$

$y' = \left(\frac{1}{2}\right)^x\left[\ln\left(\frac{1}{2}\right)\right]$

d) $N(x) = -3e^x$

$N'(x) = -3e^x$

e) $h(x) = e^x$

$h'(x) = e^x$

f) $y = \pi^x$

$y' = \pi^x[\ln(\pi)]$

2) Find the first, second, and third derivatives of the function $f(x) = e^x$

$f'(x) = f''(x) = f'''(x) = e^x$

3) Calculate the instantaneous rate of change of the function $y = 5^x$ when $x = 2$.

$y' = 5^x[\ln(5)]$

$y'(2) = 5^2[\ln(5)]$

$y'(2) \simeq 40.2$

4) Determine the slope of the graph of $y = \frac{1}{2}e^x$ at $x = 4$.

$y' = \frac{1}{2}e^x$

$y'(4) = \frac{1}{2}e^4$

$y'(4) \simeq 27.3$

5) Determine the equation of the line that is tangent to $y = 8^x$ at the point on the curve where $x = \frac{1}{2}$.

Point

$y(0.5) = 8^{1/2}$

$= 2\sqrt{2}$

$(0.5, 2\sqrt{2})$

Slope

$y' = 8^x[\ln(8)]$

$y'(0.5) = 8^{1/2}[\ln(8)]$

$y'(0.5) = 2\sqrt{2}[\ln(8)]$

$m = 2\sqrt{2}[\ln(8)]$

Eqn:

$y = mx + b$

$2\sqrt{2} = 2\sqrt{2}\ln(8)(\frac{1}{2}) + b$

$2\sqrt{2} = \sqrt{2}\ln(8) + b$

$b = 2\sqrt{2} - \sqrt{2}\ln(8)$

$b = \sqrt{2}[2 - \ln(8)]$

$y = 2\sqrt{2}\ln(8)x + \sqrt{2}[2 - \ln(8)]$

6) A fruit fly infestation is doubling every day. There are 10 flies when the infestation is first discovered.

a) Write an equation that relates the number of flies to time.

$A(t) = 10(2)^t$

b) Determine the number of flies present after 1 week.

$A(7) = 10(2)^7$

$A(7) = 1280$ flies

c) How fast is the fly population increasing after 1 week.

$A'(t) = 10(2)^t \ln(2)$

$A'(7) = 10(2)^7 \ln(2)$

$A'(7) = 887.2$ flies/day

d) How long will it take for the fly population to reach 500?

$500 = 10(2)^t$

$50 = 2^t$

$t = \log_2(50)$

$t \simeq 5.64$ days

e) How fast is the fly population increasing at this point?

$A'(5.64) = 10(2)^{5.64}\ln(2)$

$A'(5.64) \simeq 346$ flies/day

7) Refer to question 6. At which point is the fly population increasing at a rate of

i) 20 flies per day?

$$20 = 10(2)^t \ln(2)$$

$$\frac{2}{\ln(2)} = 2^t$$

$$\log_2\left[\frac{2}{\ln(2)}\right] = t$$

$t \simeq 1.53$ days

ii) 2000 flies per day?

$$2000 = 10(2)^t \ln(2)$$

$$\frac{200}{\ln(2)} = 2^t$$

$$t = \log_2\left[\frac{200}{\ln(2)}\right]$$

$t \simeq 8.17$ days

8) Determine the equation of the line perpendicular to the tangent line to the function $f(x) = \frac{1}{2}e^x$ at the point on the curve where $x = \ln 3$

Point:

$$f(\ln 3) = \frac{1}{2}e^{\ln 3}$$

$$= \frac{1}{2}(3)$$

$$= \frac{3}{2}$$

$(\ln 3, 1.5)$

Slope

$$f'(x) = \frac{1}{2}e^x$$

$$f'(\ln 3) = \frac{3}{2}$$

$$m = \frac{3}{2}$$

$$\perp m = \frac{-2}{3}$$

Eq^n

$$y = mx + b$$

$$\frac{3}{2} = \frac{-2}{3}\ln 3 + b$$

$$\frac{3}{2} + \frac{2}{3}\ln 3 = b$$

$$\boxed{y = \frac{-2}{3}x + \frac{3}{2} + \frac{2}{3}\ln 3}$$

Answers:

1)a) $g'(x) = 4^x(\ln 4)$ **b)** $f'(x) = 11^x(\ln 11)$ **c)** $y' = \left(\frac{1}{2}\right)^x\left(\ln\frac{1}{2}\right)$ **d)** $N'(x) = -3e^x$ **e)** $h'(x) = e^x$ **f)** $y' = \pi^x(\ln \pi)$
2) $f'(x) = f''(x) = f'''(x) = e^x$
3) 4.2
4) 27.3
5) $y = 6\sqrt{2}(\ln 2)x + \sqrt{2}(2 - 3\ln 2)$
6)a) $N(t) = 10(2)^t$ **b)** 1280 **c)** 887 flies/day **d)** 5.64 days **e)** 346 flies/day
7)i) 1.53 days **ii)** 8.17 days
8) $y = -\frac{2}{3}x + \frac{2}{3}\ln 3 + \frac{3}{2}$

W4 – Differentiation Rules for Exponential Functions Unit 3

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SOLUTIONS

1)a) Rewrite the function $y = b^x$ with base e.

$$y = e^{\ln(b^x)}$$

$$y = e^{x\ln(b)}$$

b) Find the derivative of your function in part a) and simplify.

$$y' = e^{x\ln(b)}[\ln(b)]$$

$$y' = e^{\ln(b^x)}[\ln(b)]$$

$$y' = b^x \ln(b)$$

2) Differentiate with respect to x.

a) $y = e^{-3x}$

$$y' = e^{-3x}(-3)$$

$$y' = -3e^{-3x}$$

b) $f(x) = e^{4x-5}$

$$f'(x) = e^{4x-5}(4)$$

$$f'(x) = 4e^{4x-5}$$

c) $y = e^{2x} - e^{-2x}$

$$y' = e^{2x}(2) - e^{-2x}(-2)$$

$$y' = 2e^{2x} + 2e^{-2x}$$

d) $y = 2^x + 3^x$

$$y' = 2^x\ln(2) + 3^x\ln(3)$$

e) $f(x) = 3e^{2x} - 2^{3x}$

$$f'(x) = 3e^{2x}(2) - 2^{3x}(3)$$

$$f'(x) = 6e^{2x} - 3(2)^{3x}$$

f) $y = 4xe^x$

$$y' = 4(e^x) + e^x(4x)$$

$$y' = 4e^x + 4xe^x$$

g) $y = 5^x e^{-x}$

$y' = 5^x(\ln 5)(e^{-x}) + (e^{-x})(-1)(5^x)$

$y' = 5^x(\ln 5)(e^{-x}) - e^{-x}(5^x)$

$y' = 5^x(e^{-x})(\ln 5 - 1)$

h) $f(x) = xe^{2x} + 2e^{-3x}$

$f'(x) = 1(e^{2x}) + (e^{2x})(2)x - 6e^{-3x}$

$f'(x) = e^{2x} + 2xe^{2x} - 6e^{-3x}$

$f'(x) = e^{2x}(1 + 2x - 6e^{-5x})$

3) Determine the derivative with respect to x for each function.

a) $y = e^{-x}\sin x$

$y' = -e^{-x}\sin x + \cos x(e^{-x})$

$y' = e^{-x}(-\sin x + \cos x)$

$y' = e^{-x}(\cos x - \sin x)$

b) $y = e^{\cos x}$

$y' = e^{\cos x}(-\sin x)$

$y' = -\sin x(e^{\cos x})$

c) $f(x) = e^{2x}(x^2 - 3x + 2)$

$f'(x) = 2e^{2x}(x^2 - 3x + 2) + (2x - 3)(e^{2x})$

$f'(x) = e^{2x}[2x^2 - 6x + 4 + 2x - 3]$

$f'(x) = e^{2x}(2x^2 - 4x + 1)$

d) $g(x) = 2x^2 e^{\cos(2x)}$

$g'(x) = 4x[e^{\cos(2x)}] + e^{\cos(2x)}[-\sin(2x)](2)(2x^2)$

$g'(x) = 4xe^{\cos(2x)}[1 - x\sin(2x)]$

4) Identify the coordinates of any local extrema of the function $y = e^x - e^{2x}$

$y' = e^x - 2e^{2x}$

$0 = e^x - 2e^{2x}$

$2e^{2x} = e^x$

$2e^x = 1$

$e^x = \frac{1}{2}$

$x = \ln(\frac{1}{2})$

$y[\ln(\frac{1}{2})] = e^{\ln(\frac{1}{2})} - e^{2[\ln(\frac{1}{2})]}$

$= \frac{1}{2} - \frac{1}{4}$

$= \frac{1}{4}$

2nd deriv. test:

$y'' = e^x - 4e^{2x}$

$y''[\ln(\frac{1}{2})] = \frac{1}{2} - 4(\frac{1}{4})$

$= -\frac{1}{2}$

$\therefore$ concave down

$\boxed{\text{Max at } [\ln(\frac{1}{2}), \frac{1}{4}]}$

5) Find an equation for the tangent to the curve $y = 2e^{2x} + 2x + 1$ when $x = 0$.

Point:

$y(0) = 2e^{2(0)} + 2(0) + 1$

$y(0) = 2 + 0 + 1$

$y(0) = 3$

$(0, 3)$

Slope:

$y' = 4e^{2x} + 2$

$y'(0) = 4e^{2(0)} + 2$

$y'(0) = 4 + 2$

$y'(0) = 6$

$m = 6$

Eqn

$y = mx + b$

$3 = 6(0) + b$

$b = 3$

$\boxed{y = 6x + 3}$

6) Find the equation of the tangent to $y = x \ln x$ that is parallel to $y = 3x + 7$.

$m = 3$

$y' = (1)\ln(x) + \frac{1}{x \ln e}(x)$

$y' = \ln(x) + 1$

$3 = \ln(x) + 1$

$2 = \ln(x)$

$x = e^2$

Point:

$y(e^2) = e^2 \ln(e^2)$

$y(e^2) = 2e^2$

$(e^2, 2e^2)$

Eqn:

$y = mx + b$

$2e^2 = 3(e^2) + b$

$2e^2 - 3e^2 = b$

$b = -e^2$

$\boxed{y = 3x - e^2}$

7) Find all local extrema for $y = \frac{1}{2}x(2)^{3x+1}$.

1st deriv. test:

	$-\infty$ -1	-0.48	0 ∞
y'	$-$		$+$
y	$\searrow$		$\nearrow$
		min	

$$y' = \frac{1}{2}(2)^{3x+1} + \ln(2)(2)^{3x+1}(3)\left(\frac{1}{2}x\right)$$

$$y' = \frac{1}{2}(2)^{3x+1}\left[1 + 3x\ln(2)\right]$$

$$0 = \frac{1}{2}(2)^{3x+1}\left[1 + 3x\ln(2)\right]$$

$0 = 2^{3x+1}$ $0 = 1 + 3x\ln(2)$

No solution $x = \dfrac{-1}{3\ln(2)}$

Min at $(-0.48, -0.18)$

$$x \simeq -0.48$$

$$y(-0.48) = -0.18$$

8) Continuous growth or decay follows the formula $A = ce^{kt}$, where c is the initial amount, and k is a rate factor. The mass of a radioactive substance is 1000 g on day 1, and only 100 g after 100 days. Find ...

a) k, then write the equation with c and k

$$100 = 1000e^{k(100)}$$

$$\frac{1}{10} = e^{100k}$$

$$\ln\left(\frac{1}{10}\right) = 100k$$

$$k = \frac{\ln\left(\frac{1}{10}\right)}{100}$$

$$k \simeq -0.023$$

$$A(t) = 1000e^{-0.023t}$$

b) the half-life,

$$500 = 1000e^{-0.023t}$$

$$\frac{1}{2} = e^{-0.023t}$$

$$\ln\left(\frac{1}{2}\right) = -0.023t$$

$$t \simeq 30.14 \text{ days}$$

b) the amount that remains after 300 days, and

$$A(300) = 1000e^{-0.023(300)}$$

$$A(300) \simeq 1 \text{ gram.}$$

c) the rate of decay after 50 days.

$$A'(t) = 1000e^{-0.023t}(-0.023)$$

$$A'(t) = -23e^{-0.023t}$$

$$A'(50) = -23e^{-0.023(50)}$$

$$A'(50) \simeq -7.28 \text{ g/day.}$$

Answers:

1)a) $y = e^{x\ln b}$ **b)** $\frac{dy}{dx} = \left(e^{x\ln b}\right)\ln b$

2)a) $y' = -3e^{-3x}$ **b)** $f'(x) = 4e^{4x-5}$ **c)** $y' = 2(e^{2x} + e^{-2x})$ **d)** $y' = 2^x(\ln 2) + 3^x(\ln 3)$

e) $f'(x) = 6e^{2x} - 3(2^{3x})\ln 2$ **f)** $y' = 4xe^x + 4e^x$ **g)** $y' = -(5^x)(e^{-x})(1 - \ln 5)$ **h)** $f'(x) = e^{2x}(2x + 1 - 6e^{-5x})$

3)a) $y' = e^{-x}(\cos x - \sin x)$ **b)** $y' = -\sin x\,(e^{\cos x})$ **c)** $f'(x) = e^{2x}(2x^2 - 4x + 1)$ **d)** $g'(x) = -4xe^{\cos(2x)}[x\sin(2x) - 1]$

4) local max of $y = 0.25$ when $x = \ln(0.5)$

5) $y = 6x + 3$

6) $y = 3x - e^2$

7) CN ~ -0.48, so the point (-0.48,-0.18) is a local minimum

8)a) k ~ -0.023, so the formula is $A = 1000e^{-0.023t}$

b) t ~ 30 days

c) $A(300) \approx 1g$

d) $A'(50) \approx -7.3g/day$

W5 – Implicit Differentiation and Derivatives of Log Functions
MCV4U
Jensen

Unit 3

SOLUTIONS

1) For each problem, use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y.

a) $2x^3 = 2y^2 + 5$

$6x^2 = 4y\frac{dy}{dx}$

$\frac{6x^2}{4y} = \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{3x^2}{2y}$

b) $5y^2 = 2x^3 - 5y$

$10y\frac{dy}{dx} = 6x^2 - 5\frac{dy}{dx}$

$10y\frac{dy}{dx} + 5\frac{dy}{dx} = 6x^2$

$\frac{dy}{dx}(10y+5) = 6x^2$

$\frac{dy}{dx} = \frac{6x^2}{10y+5}$

c) $5x^3 = -3xy + 2$

$15x^2 = -3y + \frac{dy}{dx}(-3x)$

$\frac{15x^2+3y}{-3x} = \frac{dy}{dx}$

$\frac{3(5x^2+y)}{-3x} = \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-5x^2-y}{x}$

d) $2x^3 = (3xy+1)^2$

$6x^2 = 2(3xy+1)\left[3y + \frac{dy}{dx}(3x)\right]$

$6x^2 = (6xy+2)\left[3y + \frac{dy}{dx}(3x)\right]$

$6x^2 = 18xy^2 + 18x^2y\frac{dy}{dx} + 6y + 6x\frac{dy}{dx}$

$6x^2 - 18xy^2 - 6y = \frac{dy}{dx}(18x^2y + 6x)$

$\frac{\cancel{6}(x^2-3xy^2-y)}{\cancel{6}(3x^2y+x)} = \frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = \frac{x^2-3xy^2-y}{3x^2y+x}}$

e) $x^3 - 3x^2y + 4xy^2 = 12$

$3x^2 - 6xy + \frac{dy}{dx}(-3x^2) + 4y^2 + 2y\frac{dy}{dx}(4x) = 0$

$\frac{dy}{dx}(8xy-3x^2) = -3x^2 + 6xy - 4y^2$

$\frac{dy}{dx} = \frac{-3x^2+6xy-4y^2}{8xy-3x^2}$

f) $4\sin(2y)\cos x = 2$

$4\cos(2y)\left(2\frac{dy}{dx}\right)\cos x + (-\sin x)\left[4\sin(2y)\right] = 0$

$8\cos(2y)\cos x\frac{dy}{dx} = 4\sin x\sin(2y)$

$\frac{dy}{dx} = \frac{4\sin x\sin(2y)}{8\cos x\cos(2y)}$

$\frac{dy}{dx} = \frac{\sin x\sin(2y)}{2\cos x\cos(2y)}$

g) $y^2 = \frac{x^2-4}{x^2+4}$

$$2y\frac{dy}{dx} = \frac{2x(x^2+4) - 2x(x^2-4)}{(x^2+4)^2}$$

$$2y\frac{dy}{dx} = \frac{2x(x^2+4-x^2+4)}{(x^2+4)^2}$$

$$2y\frac{dy}{dx} = \frac{16x}{(x^2+4)^2}$$

$$\frac{dy}{dx} = \frac{16x}{2y(x^2+4)^2}$$

$$\frac{dy}{dx} = \frac{8x}{y(x^2+4)^2}$$

2) Find the equation of the tangent line to $(x+y)^3 = x^3 + y^3$ at the point $(-1,1)$.

<u>Slope</u>:

$$3(x+y)^2\left(1+\frac{dy}{dx}\right) = 3x^2 + 3y^2\frac{dy}{dx}$$

$$3(x^2+2xy+y^2)\left(1+\frac{dy}{dx}\right) = 3x^2 + 3y^2\frac{dy}{dx}$$

$$(3x^2+6xy+3y^2)\left(1+\frac{dy}{dx}\right) = 3x^2 + 3y^2\frac{dy}{dx}$$

$$3x^2 + 6xy + 3y^2 + 3x^2\frac{dy}{dx} + 6xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 3x^2 + 3y^2\frac{dy}{dx}$$

$$3x^2\frac{dy}{dx} + 6xy\frac{dy}{dx} = -6xy - 3y^2$$

$$\frac{dy}{dx}(3x^2+6xy) = -3y(2x+y)$$

$$\frac{dy}{dx} = \frac{-3y(2x+y)}{3x(x+2y)}$$

$$\frac{dy}{dx} = \frac{-y(2x+y)}{x(x+2y)}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=-1\\y=1}} = \frac{-1[2(-1)+1]}{-1[-1+2(1)]}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=-1\\y=1}} = \frac{1}{-1} = -1$$

$y = mx + b$

$1 = -1(-1) + b$

$b = 0$

$\boxed{y = -1x}$

3) Differentiate each of the following with respect to x.

$y = \frac{\ln x}{2x+3}$ **b)** $f(x) = e^{x^7}$ **c)** $f(x) = \ln\left(\frac{x^2+1}{x^3-x}\right)$

$y' = \frac{\frac{1}{x}(2x+3) - 2\ln x}{(2x+3)^2} \cdot \frac{x}{x}$

$y' = \frac{2x+3 - 2x\ln x}{x(2x+3)^2}$

$f'(x) = e^{x^7}(7x^6)$

$f'(x) = 7x^6(e)^{x^7}$

$f'(x) = \frac{1}{\left(\frac{x^2+1}{x^3-x}\right)} \cdot \frac{2x(x^3-x) - (3x^2-1)(x^2+1)}{(x^3-x)^2}$

$f'(x) = \frac{x^3-x}{x^2+1} \cdot \frac{2x^4 - 2x^2 - 3x^4 - 3x^2 + x^2 + 1}{(x^3-x)^2}$

$f'(x) = \frac{1}{x^2+1} \cdot \frac{-x^4 - 4x^2 + 1}{(x^3-x)^2}$

$f'(x) = \frac{-x^4 - 4x^2 + 1}{(x^2+1)(x^3-x)}$

d) $y = \log_2(4x^2)$

$y' = \frac{1}{4x^2 \ln(2)} \cdot 8x$

$y' = \frac{2}{x\ln(2)}$

Answers:

1)a) $\frac{dy}{dx} = \frac{3x^2}{2y}$ **b)** $\frac{dy}{dx} = \frac{6x^2}{10y+5}$ **c)** $\frac{dy}{dx} = \frac{-y-5x^2}{x}$ **d)** $\frac{dy}{dx} = \frac{-3y^2x-y+x^2}{3x^2y+x}$ **e)** $\frac{dy}{dx} = \frac{6xy-3x^2-4y^2}{8xy-3x^2}$ **f)** $\frac{dy}{dx} = \frac{\sin(2y)\sin x}{2\cos(2y)\cos x}$

g) $\frac{dy}{dx} = \frac{8x}{y(x^2+4)^2}$

2) $y = -x$

3)a) $y' = \frac{2x+3-2x\ln x}{x(2x+3)^2}$ **b)** $f'(x) = \frac{3}{x}$ **c)** $f'(x) = \frac{2x}{x^2+1} - \frac{3x^2-1}{x^3-x}$ **d)** $y' = \frac{2}{x\ln 2}$

W6 – Connections with Exponential Models Unit 3

MCV4U

Jensen

1) A 100-mg sample of thorium-233 (Th-233) is placed into a nuclear reactor. After 10 min, the sample has decayed to 73 mg. Use the equation $N(t) = N_0 e^{-\lambda t}$ to answer the following questions:

a) Determine the disintegration constant λ for Th-233.

$$73 = 100e^{-10\lambda}$$
$$0.73 = e^{-10\lambda}$$
$$\ln(0.73) = -10\lambda$$
$$\lambda \simeq 0.031$$

b) Determine the half-life of Th-233

$$50 = 100e^{0.031t}$$
$$0.5 = e^{0.031t}$$
$$\ln(0.5) = 0.031t$$
$$t \simeq 22 \text{ minutes}$$

Alternate method:

$$73 = 100\left(\tfrac{1}{2}\right)^{10/H}$$
$$0.73 = \left(\tfrac{1}{2}\right)^{10/H}$$
$$\log_{0.5}(0.73) = \frac{10}{H}$$
$$H \simeq 22 \text{ minutes}$$

c) Write the equation that gives the amount of Th-233 remaining as a function of time, in terms of half-life.

$$N(t) = 100\left(\tfrac{1}{2}\right)^{t/22}$$

d) How fast is the sample decaying after 5 min?

$$N'(t) = 100\left(\tfrac{1}{2}\right)^{t/22} \ln\left(\tfrac{1}{2}\right)\left(\tfrac{1}{22}\right)$$
$$N'(5) \simeq -2.7 \text{ mg/min}$$

2) Radon-222 (Rn-222) is a radioactive element that spontaneously decays into polonium-218 (Po-218) with a half-life of 3.8 days. The atoms of these two substances have approximately the same mass. Suppose that the initial sample of radon has a mass of 100 mg.

The mass of radon, in milligrams, as a function of time is given by the function $M_{Rn}(t) = M_0 \left(\frac{1}{2}\right)^{\frac{t}{3.8}}$, where M_0 is the initial mass of radon and M_{Rn} is the mass of radon after time t, in days.

a) How much radon will remain after

i) 1 day?

$$M_{Rn}(1) = 100\left(\frac{1}{2}\right)^{1/3.8}$$

$$\simeq 83.3 \text{ mg}$$

ii) 1 week?

$$M_{Rn}(7) = 100\left(\frac{1}{2}\right)^{7/3.8}$$

$$\simeq 27.9 \text{ mg}$$

b) At what rate is the radon decaying at each of these times?

$$M(t) = 100\left(\frac{1}{2}\right)^{t/3.8}$$

$$M'(t) = 100\left(\frac{1}{2}\right)^{t/3.8} \ln\left(\frac{1}{2}\right)\left(\frac{1}{3.8}\right)$$

$$M'(1) \simeq -15.2 \text{ mg/day} \qquad M'(7) \simeq -5.1 \text{ mg/day}$$

c) How long will it take for a sample of radon to decay to 25% of its initial mass?

$$25 = 100\left(\frac{1}{2}\right)^{t/3.8}$$

$$0.25 = \left(\frac{1}{2}\right)^{t/3.8}$$

$$\log_{0.5}(0.25) = \frac{t}{3.8}$$

$$t \simeq 7.6 \text{ days}$$

3) Consider a car shock absorber modelled by the equation $h(t) = e^{-0.5t}\sin t$, where $h(t)$ represents the vertical displacement, in meters, as a function of time, t, in seconds. Determine when the maximum vertical velocity, in m/s, occurs and its value, given that $v(t) = h'(t)$.

$v(t) = h'(t) = -0.5e^{-0.5t}\sin t + \cos t\,(e^{-0.5t})$

$= -0.5e^{-0.5t}(\sin t - 2\cos t)$

$a(t) = h''(t) = 0.25e^{-0.5t}(\sin t - 2\cos t) + (\cos t + 2\sin t)(-0.5e^{-0.5t})$

$0 = 0.25e^{-0.5t}[\sin t - 2\cos t - 2(\cos t + 2\sin t)]$

$0 = 0.25e^{-0.5t}(-3\sin t - 4\cos t)$

$0 = -3\sin t - 4\cos t$

$4\cos t = -3\sin t$

$-\frac{4}{3} = \tan t$

$t \simeq -0.927, 2.214, 5.356, \ldots$

Test critical #'s AND

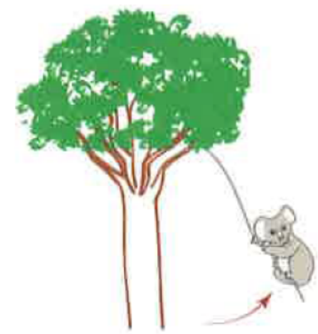
$v(0) = 1$ m/s

$v(2.214) \simeq 0.0687$ m/s

$v(5.356) = -0.33$ m/s

Max velocity of 1 m/s at 0 seconds.

4) Rocco and Biff are two koala bears that are foraging for food together in a eucalyptus tree. Suddenly, a gust of wind causes Rocco to lose his grip and begin to fall. He quickly grabs a nearby vine and begins to swing away from the tree. Rocco's horizontal displacement as a function of time is given by the equation $x(t) = 5\cos\left(\frac{\pi t}{2}\right)e^{-0.1t}$ where x is Rocco's horizontal displacement from the bottom of his swing arc, in meters, at time t, in seconds.



a) Biff can grab Rocco if Rocco swings back to within 1 meter from where he started falling. Will Biff be able to rescue Rocco? Explain, using mathematical reasoning.

Initial Position: $x(0) = 5\cos(0) = 5$m

Period $= \frac{2\pi}{\left(\frac{\pi}{2}\right)} = 4$ seconds

Position after 4 seconds: $x(4) = 5\cos(2\pi)e^{-0.4} \simeq 3.35$m

He is 1.65 m away from where he started. Biff will NOT be able to rescue Rocco.

b) The other option Rocco has is to let go of the vine at the bottom of one of the swing arcs and drop to the ground. But Rocco will only feel safe doing this if his horizontal velocity at the bottom of the swing is less than 2 m/s. Assuming that Biff is unable to save his friend, how many times must Rocco swing back and forth on the vine before he can safely drop to the ground?

$$x(t) = 5e^{-0.1t}\cos\left(\frac{\pi t}{2}\right)$$

$$v(t) = x'(t) = -0.5e^{-0.1t}\cos\left(\frac{\pi t}{2}\right) - \frac{\pi}{2}\sin\left(\frac{\pi t}{2}\right)(5)e^{-0.1t}$$

$v(1) =$
$v(3) =$
$v(5) =$
$v(7) =$
$v(9) =$

$v(11) =$
$v(13) =$
$v(15) = 1.75$ m/s

\# of swings $= \frac{15}{4} = 3.75$ times

when is he at the bottom?

$0 = \cos\left(\frac{\pi t}{2}\right)$

$\frac{\pi t}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$t = 1, 3, 5, 7, \ldots$

$t = 1+2k \; \{k \in \mathbb{Z} \mid k \geq 0\}$

5) The voltage signal from a standard North American wall socket can be described by the equation $V(t) = 170\sin(120\pi t)$, where t is time, in seconds, and V is the voltage, in volts, at time t.

a) Find the max and min voltage levels and the times at which they occur.

max $= 0 + 170 = 170$ V

$170 = 170\sin(120\pi t)$
$1 = \sin(120\pi t)$
$120\pi t = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$
$t = \frac{1}{240}, \frac{5}{240}, \frac{9}{240}, \ldots$

A max voltage of 170 V when $t = \frac{1+4k}{240} \; \{k \in \mathbb{Z} \mid k \geq 0\}$

min $= 0 - 170 = -170$ V

$-170 = 170\sin(120\pi t)$
$-1 = \sin(120\pi t)$
$120\pi t = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \ldots$
$t = \frac{3}{240}, \frac{7}{240}, \frac{11}{240}, \ldots$

A min voltage of -170 V when $t = \frac{3+4k}{240} \; \{k \in \mathbb{Z} \mid k \geq 0\}$

b) For the given signal, determine

i) the period, T, in seconds

period $= \frac{2\pi}{|k|} = \frac{2\pi}{120\pi} = \frac{1}{60}$ seconds

ii) the frequency, f, in hertz

frequency $= \frac{1}{\text{period}} = 60$ Hz

iii) the amplitude, A, in volts

$A = |a| = 170$ V

6) Consider a simple pendulum that has a length of 50 cm and a max horizontal displacement of 8 cm.

a) Find the period of the pendulum.

$$\text{period} = T = 2\pi\sqrt{\frac{0.5}{9.8}} \approx 1.42 \text{ seconds}$$

b) Determine a function that gives the horizontal position of the bob as a function of time.

$$h(t) = 8\cos\left(\frac{2\pi t}{2\pi\sqrt{\frac{0.5}{9.8}}}\right) \doteq 8\cos(4.43t)$$

c) Determine a function that gives the velocity of the bob as a function of time.

$$v(t) = h'(t) = 8\left[-\sin(4.43t)\right](4.43)$$
$$= -35.44\sin(4.43t)$$

d) Determine a function that gives the acceleration of the bob as a function of time.

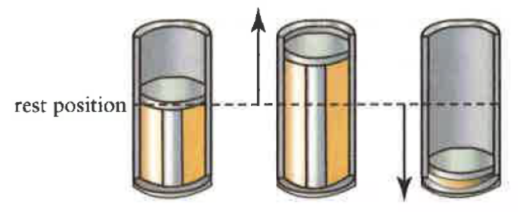
$$a(t) = v'(t) = -35.44\left[\cos(4.43t)\right](4.43)$$
$$= -157\cos(4.43t)$$

e) Find the max velocity of the bob and the time at which it first occurs.

$$\max = 0 + 35.44 = 35.44 \text{ cm/s}$$

$$35.44 = -35.44\sin(4.43t)$$
$$-1 = \sin(4.43t)$$
$$4.43t = \frac{3\pi}{2}$$
$$t \approx 1.06 \text{ seconds}$$

7) A piston in an engine oscillates up and down from a rest position as shown. The motion of this piston can be approximated by the function $h(t) = 0.05\cos(13t)$, where t is time, in seconds, and h is the displacement of the piston head from rest position, in meters, at time t.



a) Determine an equation for the velocity of the piston head as a function of time.

$$v(t) = h'(t) = 0.05\left[-\sin(13t)\right](13)$$
$$= -0.65\sin(13t)$$

b) Find the max and min velocities and the times at which they occur.

max $= 0 + 0.65 = 0.65$ m/s

$0.65 = -0.65\sin(13t)$

$-1 = \sin(13t)$

$13t = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \ldots$

$t = \frac{3\pi}{26}, \frac{7\pi}{26}, \frac{11\pi}{26}, \ldots$

max velocity of 0.65 m/s

when $t = \frac{3\pi + 4k\pi}{26}$ $\{k \in \mathbb{Z} \mid k \geq 0\}$

min $= 0 - 0.65 = -0.65$ m/s

$-0.65 = -0.65\sin(13t)$

$1 = \sin(13t)$

$13t = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$

$t = \frac{\pi}{26}, \frac{5\pi}{26}, \frac{9\pi}{26}, \ldots$

min velocity of -0.65 m/s

when $t = \frac{\pi + 4k\pi}{26}$ $\{k \in \mathbb{Z} \mid k \geq 0\}$

Answers:

1)a) 0.031/min **b)** 22 min **c)** $N(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{22}}$ **d)** -2.65 mg/min

2)a)i) 83.3 mg **ii)** 27.9 mg **b)i)** -15.2 mg/day **ii)** -5.1 mg/day **c)** 7.6 days

3) $t = 0$ s, $v = 1$ m/s

4)a) He will NOT be able to rescue Rocco. **b)** He must swing back and forth 3.75 times before he can safely drop to the ground.

5) **a)** max voltage: 170 V at times t, in seconds, $t = \left\{\frac{4k+1}{240}, k \in \mathbb{Z}, k \geq 0\right\}$

min voltage: -170 V at times t, in seconds, $t = \left\{\frac{4k+3}{240}, k \in \mathbb{Z}, k \geq 0\right\}$

b)i) $T = \frac{1}{60}$ s **ii)** $f = 60$ Hz **iii)** $A = 170$ V

6)a) 1.42 s **b)** $h(t) = 8\cos(1.41\pi t)$ **c)** $v(t) = -11.28\pi\sin(1.41\pi t)$ **d)** $a(t) = -15.9\pi^2\cos(1.41\pi t)$

e) max velocity: 35.4 cm/s at time $t = 1.06$ s [redacted]

7) a) $v(t) = -0.65\sin(13t)$ **b)** max velocity: 0.65 m/s at $t = \left\{\frac{(4k+3)\pi}{26}, k \in \mathbb{Z}, k \geq 0\right\}$; min velocity: -0.65 m/s at $t = \left\{\frac{(4k+1)\pi}{26}, k \in \mathbb{Z}, k \geq 0\right\}$