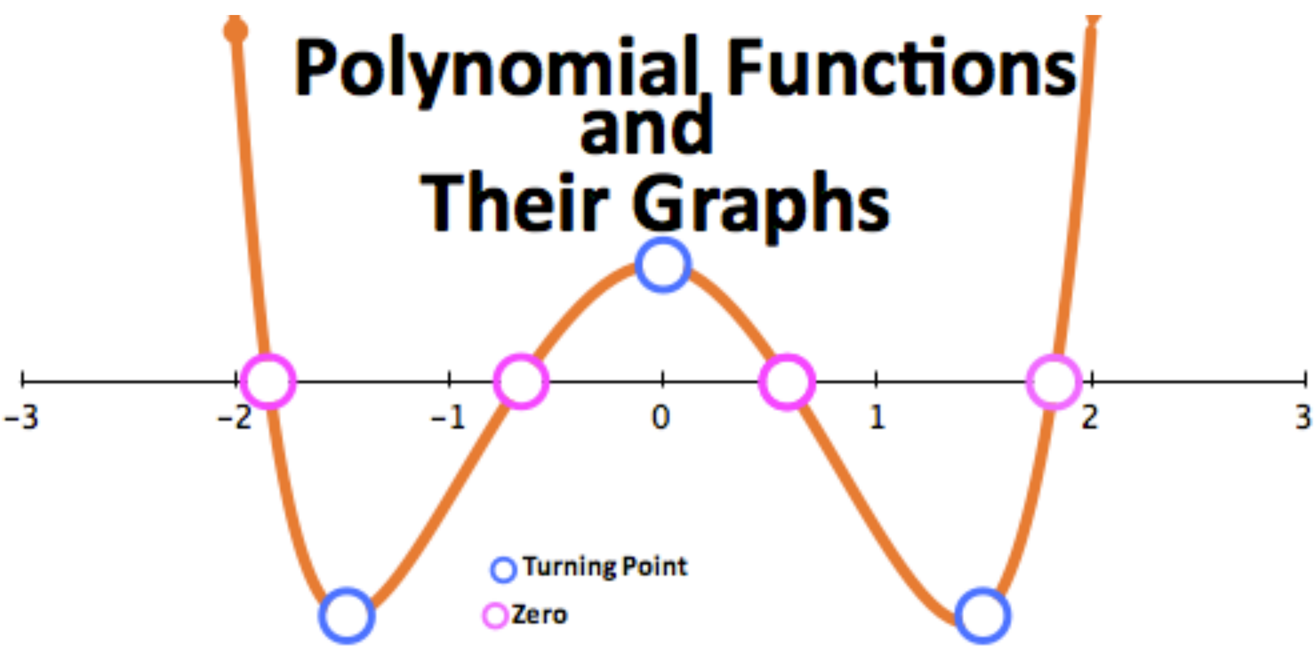
UNIT 1

Chapter 1- Polynomial Functions

Lesson Package

MHF4U



Chapter 1 Outline

Unit Goal: By the end of this unit, you will be able to identify and describe some key features of polynomial functions, and make connections between the numeric, graphical, and algebraic representations of polynomial functions.

Section	Subject	Learning Goals	Curriculum Expectations
L1	Power Functions	- describe key features of graphs of power functions - learn interval notation - be able to describe end behaviour	C1.1, 1.2, 1.3
L2	Characteristics of Polynomial Functions	- describe characteristics of equations and graphs of polynomial functions - learn how degree related to turning points and x-intercepts	C1.1, 1.2, 1.3, 1.4
L3	Factored Form Polynomial Functions	- connect how factored form equation related to x-intercepts of graph of polynomial function - given graph, determine equation in factored form	C1.5, 1.7, 1.8
L4	Transformations of Polynomial Functions	- understand how the parameters a, k, d, and c transform power functions	C1.6
L5	Symmetry in Polynomial Functions	- understand the properties of even and odd polynomial functions	C1.9

Assessments	**F/A/O**	**Ministry Code**	**P/O/C**	**KTAC**
Note Completion	A		P	
Practice Worksheet Completion	F/A		P	
Quiz – Properties of Polynomial Functions	F		P	
PreTest Review	F/A		P	
Test - Functions	O	C1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9	P	K(21%), T(34%), A(10%), C(34%)

L1 – 1.1 – Power Functions Lesson
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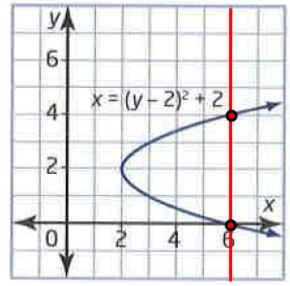
Things to Remember About Functions

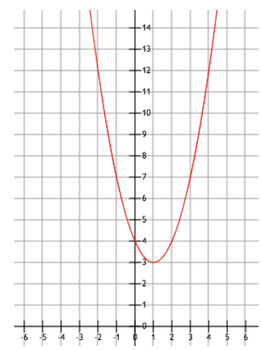


- A relation is a function if for every x-value there is only 1 corresponding y-value. The graph of a relation represents a function if it passes the **vertical line test**, that is, if a vertical line drawn anywhere along the graph intersects that graph at no more than one point.

- The **DOMAIN** of a function is the complete set of all possible values of the independent variable (x)
  - Set of all possible x-vales that will output real y-values

- The **RANGE** of a function is the complete set of all possible resulting values of the dependent variable (y)
  - Set of all possible y-values we get after substituting all possible x-values

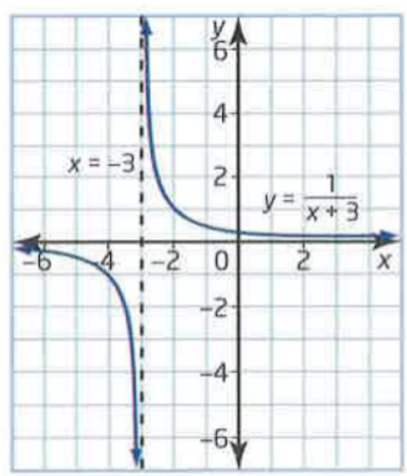
- For the function $f(x) = (x-1)^2 + 3$
  - D: $\{X \in \mathbb{R}\}$
  - R: $\{Y \in \mathbb{R} | y \geq 3\}$



- The degree of a function is the highest exponent in the expression
  - $f(x) = 6x^3 - 3x^2 + 4x - 9$ has a degree of **3**

- An **ASYMPTOTE** is a line that a curve approaches more and more closely but never touches.

The function $\boldsymbol{y = \frac{1}{x+3}}$ has two asymptotes:



Vertical Asymptote: Division by zero is undefined. Therefore the expression in the denominator of the function can not be zero. Therefore $x \neq -3$. This is why the vertical line $x = -3$ is an asymptote for this function.

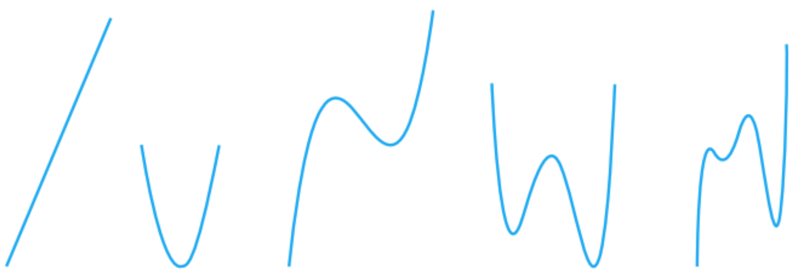
Horizontal Asymptote: For the range, there can never be a situation where the result of the division is zero. Therefore the line $y = 0$ is a horizontal asymptote. For all functions where the denominator is a higher degree than the numerator, there will by a horizontal asymptote at $y = 0$.

Polynomial Functions

A ***polynomial function*** has the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x^1 + a_0$$

- n Is a whole number
- x Is a variable
- the **coefficients** $a_0, a_1, \ldots, a_n$ are real numbers
- the **degree** of the function is n, the exponent of the greatest power of x
- a_n, the coefficient of the greatest power of x, is the **leading coefficient**
- a_0, the term without a variable, is the **constant term**
- The domain of a polynomial function is the set of real numbers D: $\{X \in \mathbb{R}\}$
- The range of a polynomial function may be all real numbers, or it may have a lower bound or an upper bound (but not both)
- The graph of polynomial functions do not have horizontal or vertical asymptotes
- The graphs of polynomial functions of degree 0 are **horizontal lines**. The shapes of other graphs depends on the degree of the function. Five typical shapes are shown for various degrees:



A **power function** is the simplest type of polynomial function and has the form:

$$f(x) = ax^n$$

- a is a real number
- x is a variable
- n is a whole number

Example 1: Determine which functions are polynomials. State the degree and the leading coefficient of each polynomial function.

a) $g(x) = \sin x$

This is a trigonometric function, not a polynomial function.

b) $f(x) = 2x^4$

This is a polynomial function of degree 4.
The leading coefficient is 2

c) $y = x^3 - 5x^2 + 6x - 8$

This is a polynomial function of degree 3.
The leading coefficient is 1.

d) $g(x) = 3^x$

This is not a polynomial function but an exponential function, since the base is a number and the exponent is a variable.

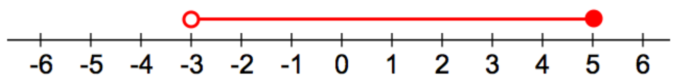
Interval Notation

In this course, you will often describe the features of the graphs of a variety of types of functions in relation to real-number values. Sets of real numbers may be described in a variety of ways:

1) as an inequality $-3 < x \leq 5$

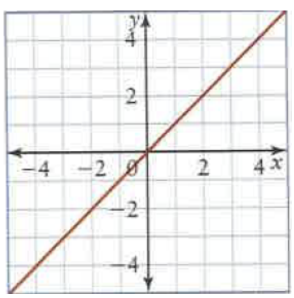
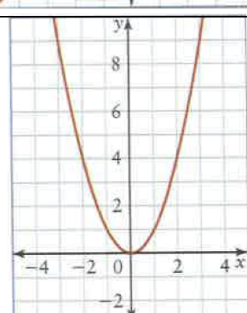
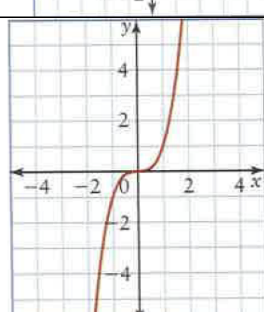
2) interval (or bracket) notation $(-3, 5]$

3) graphically on a number line



Note:
- Intervals that are infinite are expressed using ∞ (infinity) or $-\infty$ (negative infinity)
- **Square brackets** indicate that the end value is included in the interval
- **Round brackets** indicate that the end value is NOT included in the interval
- A **round** bracket is always used at infinity and negative infinity

Example 2: Below are the graphs of common power functions. Use the graph to complete the table.

Power Function	Special Name	Graph	Domain	Range	End Behaviour as $x \to -\infty$	End Behaviour as $x \to \infty$
$y = x$	Linear		$(-\infty, \infty)$	$(-\infty, \infty)$	$y \to -\infty$ Starts in quadrant 3	$y \to \infty$ Ends in quadrant 1
$y = x^2$	Quadratic		$(-\infty, \infty)$	$[0, \infty)$	$y \to \infty$ Starts in quadrant 2	$y \to \infty$ Ends in quadrant 1
$y = x^3$	Cubic		$(-\infty, \infty)$	$(-\infty, \infty)$	$y \to -\infty$ Starts in quadrant 3	$y \to \infty$ Ends in quadrant 1

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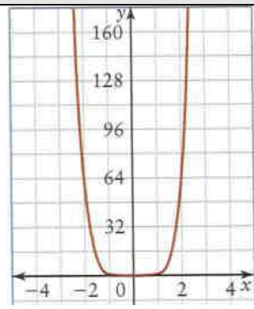
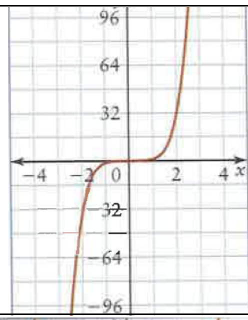
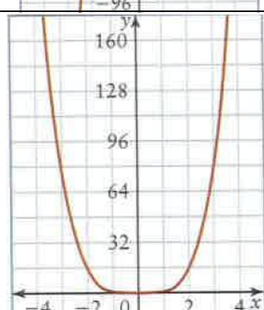
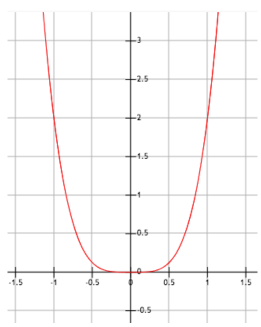
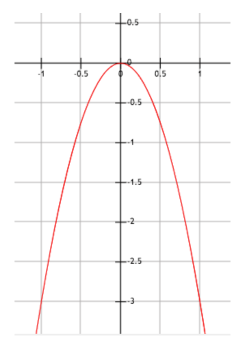
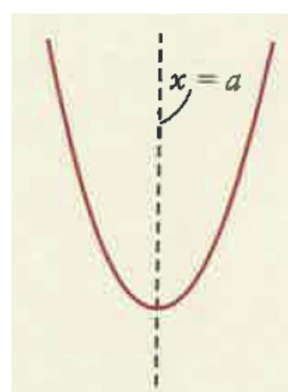
Power Function	Special Name	Graph	Domain	Range	End Behaviour as $x \to -\infty$	End Behaviour as $x \to \infty$
$y = x^4$	Quartic		$(-\infty, \infty)$	$[0, \infty)$	$y \to \infty$ Starts in quadrant 2	$y \to \infty$ Ends in quadrant 1
$y = x^5$	Quintic		$(-\infty, \infty)$	$[-\infty, \infty)$	$y \to -\infty$ Starts in quadrant 3	$y \to \infty$ Ends in quadrant 1
$y = x^6$	Sextic		$(-\infty, \infty)$	$[0, \infty)$	$y \to \infty$ Starts in quadrant 2	$y \to \infty$ Ends in quadrant 1
[redacted]						

Key Features of EVEN Degree Power Functions

When the leading coefficient (a) is positive		When the leading coefficient (a) is negative	
End behaviour	as $x \to -\infty, y \to \infty$ and as $x \to \infty, y \to \infty$ Q2 to Q1	**End behaviour**	as $x \to -\infty, y \to -\infty$ and as $x \to \infty, y \to -\infty$ Q3 to Q4
Domain	$(-\infty, \infty)$	**Domain**	$(-\infty, \infty)$
Range	$[0, \infty)$	**Range**	$[0, -\infty)$
Example: $f(x) = 2x^4$		**Example:** $f(x) = -3x^2$	

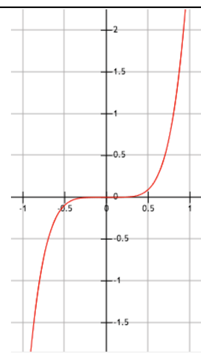
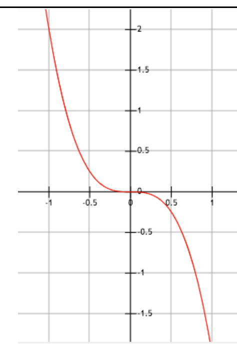
Line Symmetry

A graph has line symmetry if there is a vertical line $x = a$ that divides the graph into two parts such that each part is a reflection of the other.



Note: The graphs of even degree power functions have line symmetry about the vertical line $x = 0$ (the y-axis).

Key Features of ODD Degree Power Functions

When the leading coefficient (a) is positive		When the leading coefficient (a) is negative	
End behaviour	as $x \to -\infty, y \to -\infty$ and as $x \to \infty, y \to \infty$ Q3 to Q1	**End behaviour**	as $x \to -\infty, y \to \infty$ and as $x \to \infty, y \to -\infty$ Q2 to Q4
Domain	$(-\infty, \infty)$	**Domain**	$(-\infty, \infty)$
Range	$(-\infty, \infty)$	**Range**	$(-\infty, \infty)$
Example: $f(x) = 3x^5$		**Example:** $f(x) = -2x^3$	

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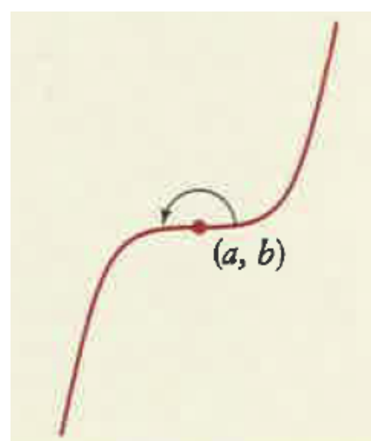
Point Symmetry



A graph has point point symmetry about a point (a, b) if each part of the graph on one side of (a, b) can be rotated 180° to coincide with part of the graph on the other side of (a, b).

Note: The graph of odd degree power functions have point symmetry about the origin (0, 0).

Example 3: Write each function in the appropriate row of the second column of the table. Give reasons for your choices.

$y = 2x$ $\quad$ $y = 5x^6$ $\quad$ $y = -3x^2$ $\quad$ $y = x^7$

$y = -\frac{2}{5}x^9$ $\quad$ $y = -4x^5$ $\quad$ $y = x^{10}$ $\quad$ $y = -0.5x^8$

End Behaviour	Functions	Reasons
Q3 to Q1	$y = 2x$ $y = x^7$	Odd exponent Positive leading coefficient
Q2 to Q4	$y = -\frac{2}{5}x^9$ $y = -4x^5$	Odd exponent Negative leading coefficient
Q2 to Q1	$y = 5x^6$ $y = x^{10}$	Even exponent Positive leading coefficient
Q3 to Q4	$y = -3x^2$ $y = -0.5x^8$	Even exponent Negative leading coefficient
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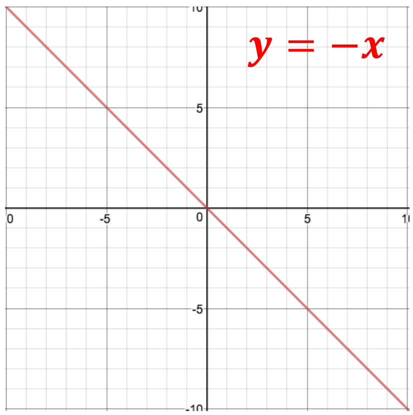
Example 4: For each of the following functions

i) State the domain and range
ii) Describe the end behavior
iii) Identify any symmetry

a)

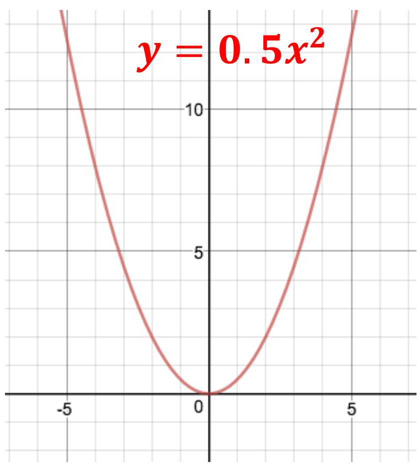


i) Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$

ii) As $x \to -\infty, y \to \infty$ and as $x \to \infty, y \to -\infty$
The graph extends from quadrant 2 to 4

iii) Point symmetry about the origin (0, 0)

b)

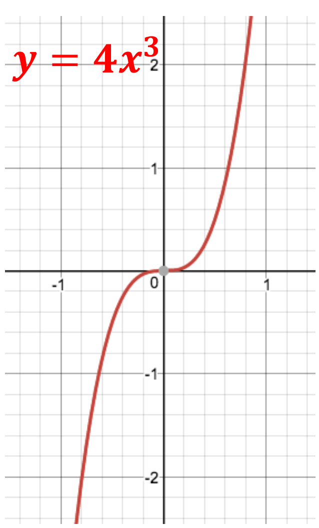


i) Domain: $(-\infty, \infty)$ Range: $[0, \infty)$

ii) As $x \to -\infty, y \to \infty$ and as $x \to \infty, y \to \infty$
The graph extends from quadrant 2 to 1

iii) Line symmetry about the line $x = 0$ (the y-axis)

c)



i) Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$

ii) As $x \to -\infty, y \to -\infty$ and as $x \to \infty, y \to \infty$
The graph extends from quadrant 3 to 1

iii) Point symmetry about the origin (0, 0)

L2 – 1.2 – Characteristics of Polynomial Functions Lesson
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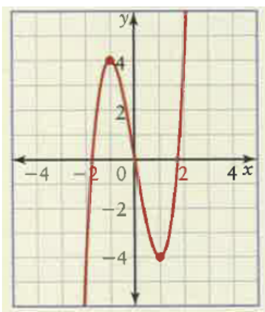
In section 1.1 we looked at power functions, which are single-term polynomial functions. Many polynomial functions are made up of two or more terms. In this section we will look at the characteristics of the graphs and equations of polynomial functions.

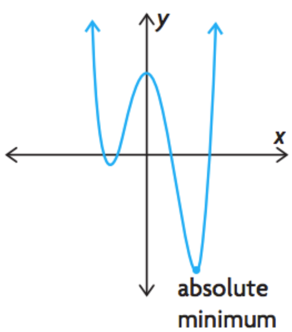
New Terminology – Local Min/Max vs. Absolute Min/Max

Local Min or Max Point – Points that are minimum or maximum points on some interval around that point.

Absolute Max or Min – The greatest/least value attained by a function for ALL values in its domain.



In this graph, (-1, 4) is a **local max** and (1, -4) is a **local min**. These are not absolute min and max points because there are other points on the graph of the function that are smaller and greater. Sometimes local min and max points are called **turning points**.



On the graph of this function…

There are **3** local min/max points. **2** are local min and **1** is a local max.

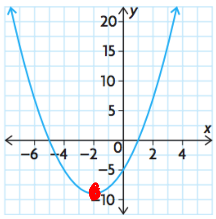
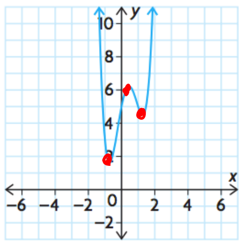
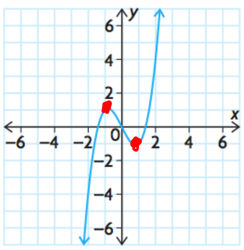
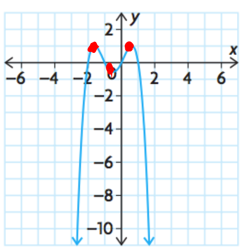
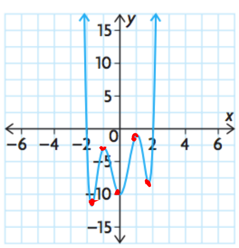
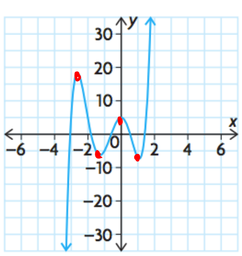
One of the local min points is also an absolute min (it is labeled) .

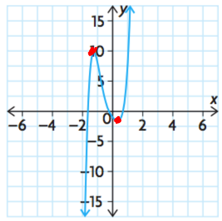
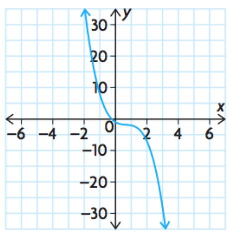
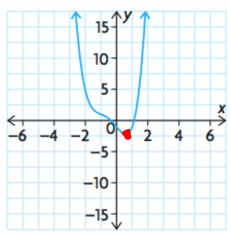
Investigation: Graphs of Polynomial Functions

The degree and the leading coefficient in the equation of a polynomial function indicate the end behaviours of the graph.

The degree of a polynomial function provides information about the shape, turning points (local min/max), and zeros (x-intercepts) of the graph.

Complete the following table using the equation and graphs given:

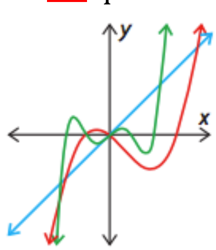
Equation and Graph	Degree	Even or Odd Degree?	Leading Coefficient	End Behaviour	Number of turning points	Number of x-intercepts
$f(x) = x^2 + 4x - 5$ 	2	Even	+1	as $x \to -\infty$, $y \to \infty$ as $x \to \infty$, $y \to \infty$ Q2 to Q1	1	2
$f(x) = 3x^4 - 4x^3 - 4x^2 + 5x + 5$ 	4	Even	+3	as $x \to -\infty$, $y \to \infty$ as $x \to \infty$, $y \to \infty$ Q2 to Q1	3	0
$f(x) = x^3 - 2x$ 	3	Odd	+1	as $x \to -\infty$, $y \to -\infty$ as $x \to \infty$, $y \to \infty$ Q3 to Q1	2	3
$f(x) = -x^4 - 2x^3 + x^2 + 2x$ 	4	Even	-1	as $x \to -\infty$, $y \to -\infty$ as $x \to \infty$, $y \to -\infty$ Q3 to Q4	3	4
$f(x) = 2x^6 - 12x^4 + 18x^2 + x - 10$ 	6	Even	+2	as $x \to -\infty$, $y \to \infty$ as $x \to \infty$, $y \to \infty$ Q2 to Q1	5	2
$f(x) = 2x^5 + 7x^4 - 3x^3 - 18x^2 + 5$ 	5	Odd	+2	as $x \to -\infty$, $y \to -\infty$ as $x \to \infty$, $y \to \infty$ Q3 to Q1	4	5

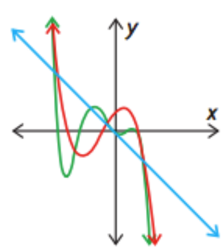
Equation and Graph	Degree	Even or Odd Degree?	Leading Coefficient	End Behaviour	Number of turning points	Number of x-intercepts
$f(x) = 5x^5 + 5x^4 - 2x^3 + 4x^2 - 3x$ 	5	Odd	+5	as $x \to -\infty$, $y \to -\infty$ as $x \to \infty$, $y \to \infty$ Q3 to Q1	2	3
$f(x) = -2x^3 + 4x^2 - 3x - 1$ 	3	Odd	-2	as $x \to -\infty$, $y \to \infty$ as $x \to \infty$, $y \to -\infty$ Q2 to Q4	0	1
$f(x) = x^4 + 2x^3 - 3x - 1$ 	4	Even	+ 1	as $x \to -\infty$, $y \to \infty$ as $x \to \infty$, $y \to \infty$ Q2 to Q1	1	2

Summary of Findings:

- A polynomial function of degree n has at most **$n - 1$** local max/min points (turning points)
- A polynomial function of degree n may have up to **n** distinct zeros (x-intercepts)
- If a polynomial function is **odd** degree, it must have at least one x-intercept, and an even number of turning points
- If a polynomial function is **even** degree, it may have no x-intercepts, and an odd number of turning points

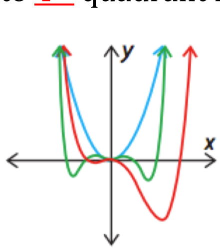
- An odd degree polynomial function extends from…
  - **3rd** quadrant to **1st** quadrant if it has a positive leading coefficient
  - **2nd** quadrant to **4th** quadrant if it has a negative leading coefficient

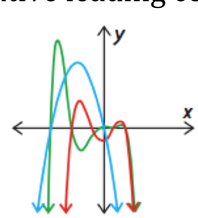




***Note:** Odd degree polynomials have OPPOSITE end behaviours*

- An even degree polynomial function extends from…
  - **2nd** quadrant to **1st** quadrant if it has a positive leading coefficient
  - **3rd** quadrant to **4th** quadrant if is has a negative leading coefficient





***Note:** Even degree polynomials have THE SAME end behaviour.*

Example 1: Describe the end behaviours of each function, the possible number of turning points, and the possible number of zeros. Use these characteristics to sketch possible graphs of the function

a) $f(x) = -3x^5 + 4x^3 - 8x^2 + 7x - 5$

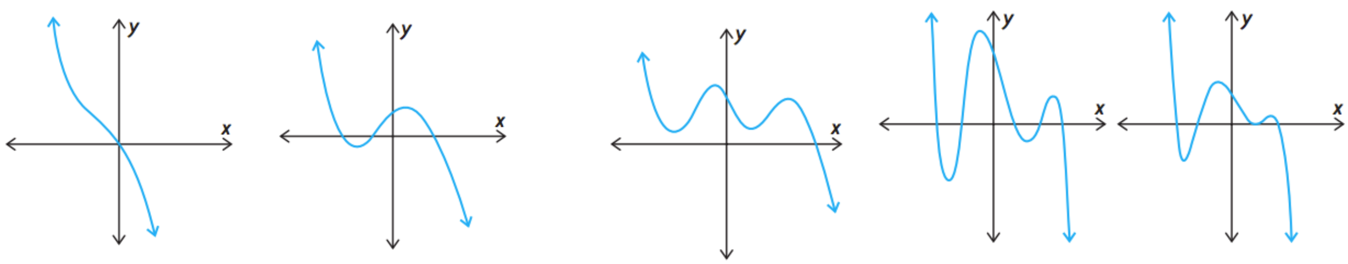
The degree is odd and the leading coefficient is negative so the function must extend from Q2 to Q4

As $x \to -\infty, y \to \infty$
As $x \to \infty, y \to -\infty$

The function can have at most 5 x-intercepts (1, 2, 3, 4, or 5)
The function can have at most 4 turning points (0, 2, or 4)

> ***Note:*** *Odd degree functions must have an even number of turning points.*

Possible graphs of 5th degree polynomial functions with a negative leading coefficient:



b) $g(x) = 2x^4 + x^2 + 2$

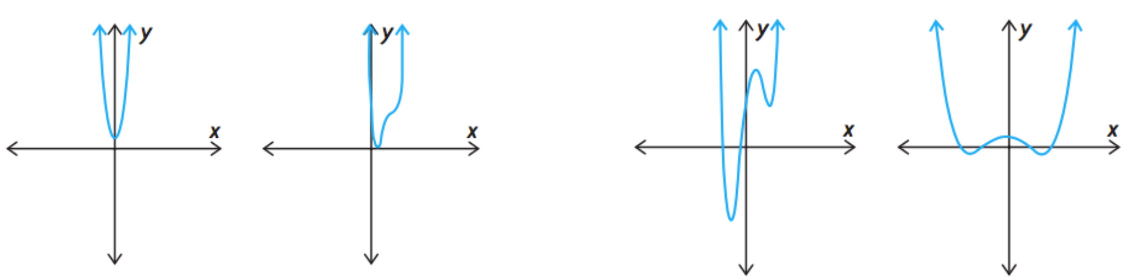
The degree is even and the leading coefficient is positive so the function must extend from the second quadrant to the first quadrant.

As $x \to -\infty, y \to \infty$
As $x \to \infty, y \to \infty$

The function can have at most 4 x-intercepts (0,1, 2, 3, or 4)
The function can have at most 3 turning points (1, or 3)

> ***Note:*** *Even degree functions must have an odd number of turning points.*

Possible graphs of 4th degree polynomial functions with a positive leading coefficient:

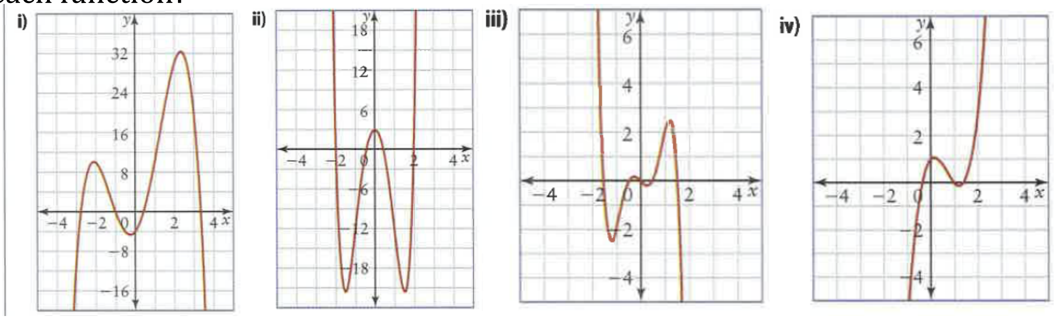


Example 2: Fill out the following chart

Degree	Possible # of x-intercepts	Possible # of turning points
1	1	0
2	0, 1, 2	1
3	1, 2, 3	0, 2
4	0, 1, 2, 3, 4	1, 3
5	1, 2, 3, 4, 5	0, 2, 4

Example 3: Determine the key features of the graph of each polynomial function. Use these features to match each function with its graph. State the number of x-intercepts, the number of local max/min points, and the number of absolute max/min points for the graph of each function. How are these features related to the degree of each function?

a) $f(x) = 2x^3 - 4x^2 + x + 1$
b) $g(x) = -x^4 + 10x^2 + 5x - 4$
c) $h(x) = -2x^5 + 5x^3 - x$
d) $p(x) = x^6 - 16x^2 + 3$



a) The function is cubic with a positive leading coefficient. The graph extends from Q3 to Q1. The y-intercept is 1. Graph iv) corresponds to this equation.

There are 3 x-intercepts and the degree is 3. The function has one local max and one local min, which is a total of two turning points, which is one less than the degree. There is no absolute max or min point.

b) The function is quartic with a negative leading coefficient. The graph extends from quadrant 3 to 4. The y-intercept is -4. Graph i) corresponds to this equation.

There are 4 x-intercepts and the degree is 4. The function has two local max and one local min, which is a total of 3 turning points, which is one less than the degree. The function has one absolute max point.

c) The function is quintic with a negative leading coefficient. The graph extends from quadrant 2 to 4. The y-intercept is 0. Graph iii) corresponds to this equation.

There are 5 x-intercepts and the degree is 5. The function has two local max and two local min, which is a total of 4, which is one less than the degree. The function has no absolute max or min points.

d) The function is degree 6 with a positive leading coefficient. The graph extends from quadrant 2 to 1. The y-intercept is 3. Graph ii) corresponds to this equation.

There are 4 x-intercepts and the degree is 6. The function has two local max and one local min, which is a total of 3, which is three less than the degree. The function has two absolute min points.

Finite Differences

For a polynomial function of degree n, where n is a positive integer, the n^{th} differences…

- are equal
- have the same sign as the leading coefficient
- are equal to $a \cdot n!$, where a is the leading coefficient

Note:

$n!$ is read as n factorial.

$n! = n\times(n-1)\times(n-2)\times \ldots \times 2\times 1$

$5! = 5\times4\times3\times2\times1 = 120$

Example 4: The table of values represents a polynomial function. Use finite differences to determine

- **a)** the degree of the polynomial function
- **b)** the sign of the leading coefficient
- **c)** the value of the leading coefficient

x	y	First differences	Second differences	Third differences
-3	-36			
-2	-12	$-12-(-36)=24$		
-1	-2	$-2-(-12)=10$	$10-24=-14$	
0	0	$0-(-2)=2$	$2-10=-8$	$-8-(-14)=6$
1	0	$0-0=0$	$0-2=-2$	$-2-(-8)=6$
2	4	$4-0=4$	$4-0=4$	$4-(-2)=6$
3	18	$18-4=14$	$14-4=10$	$10-4=6$
4	48	$48-18=30$	$30-14=16$	$16-10=6$

a) The third differences are constant. So, the table of values represents a cubic function. The degree of the function is 3.

b) The leading coefficient is positive since 6 is positive.

c)
$$6 = a \cdot n!$$
$$6 = a \cdot 3!$$
$$6 = a \cdot 6$$
$$\frac{6}{6} = a$$
$$a = 1$$

Therefore, the leading coefficient of the polynomial function is 1.

Example 5: For the function $2x^4 - 4x^2 + x + 1$ what is the value of the constant finite differences?

$$Finite\ differences = a \cdot n!$$
$$= 2 \cdot 4!$$
$$= 2 \cdot 24$$
$$= 48$$

Therefore, the fourth differences would all be 48 for this function.

L3 – 1.3 – Factored Form Polynomial Functions Lesson
MHF4U
Jensen

In this section, you will investigate the relationship between the factored form of a polynomial function and the x-intercepts of the corresponding graph, and you will examine the effect of repeated factor on the graph of a polynomial function.

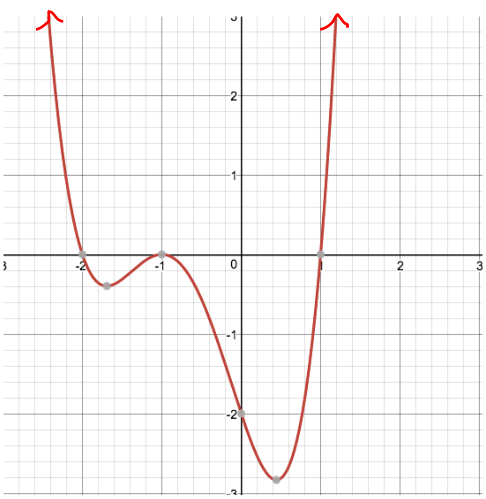
Factored Form Investigation

If we want to graph the polynomial function $f(x) = x^4 + 3x^3 + x^2 - 3x - 2$ accurately, it would be most useful to look at the factored form version of the function:

$f(x) = (x + 1)^2(x + 2)(x - 1)$

Lets start by looking at the graph of the function and making connections to the factored form equation.

Graph of $f(x)$:



From the graph, answer the following questions…

a) What is the degree of the function?

The highest degree term is x^4, therefore the function is degree 4 (quartic)

b) What is the sign of the leading coefficient?

The leading coefficient is 1, therefore the leading coefficient is POSITIVE

c) What are the x-intercepts?

The x-intercepts are $(-2, 0)$ of order 1, $(-1, 0)$ of order 2, and $(1, 0)$ of order 1

d) What is the y-intercept?

The y-intercept is the point $(0, -2)$

e) The x-intercepts divide the graph in to into four intervals. Write the intervals in the first row of the table. In the second row, choose a test point within the interval. In the third row, indicate whether the function is positive (above the x-axis) or negative (below the y-axis).

Interval	$(-\infty, -2)$	$(-2, -1)$	$(-1, 1)$	$(1, \infty)$
Test Point	$f(-3)$ $= (-3+1)^2(-3+2)(-3-1)$ $= (-2)^2(-1)(-4)$ $= 16$	$f(-1.5)$ $= (-1.5+1)^2(-1.5+2)(-1.5-1)$ $= (-0.5)^2(0.5)(-2.5)$ $= -0.3125$	$f(0)$ $= (0+1)^2(0+2)(0-1)$ $= (1)^2(2)(-1)$ $= -2$	$f(3)$ $= (3+1)^2(3+2)(3-1)$ $= (4)^2(5)(2)$ $= 160$
Sign of $f(x)$	$+$	$-$	$-$	$+$

f) What happens to the sign of the of $f(x)$ near each x-intercept?

At (-2, 0) which is order 1, it changes signs
At (-1, 0) which is order 2, the sign does NOT change
At (1, 0) which is order 1, it changes signs

Conclusions from investigation:

The x-intercepts of the graph of the function correspond to the roots (zeros) of the corresponding equation. For example, the function $f(x) = (x-2)(x+1)$ has x-intercepts at **2** and **-1**. These are the roots of the equation $(x-2)(x+1) = 0$.

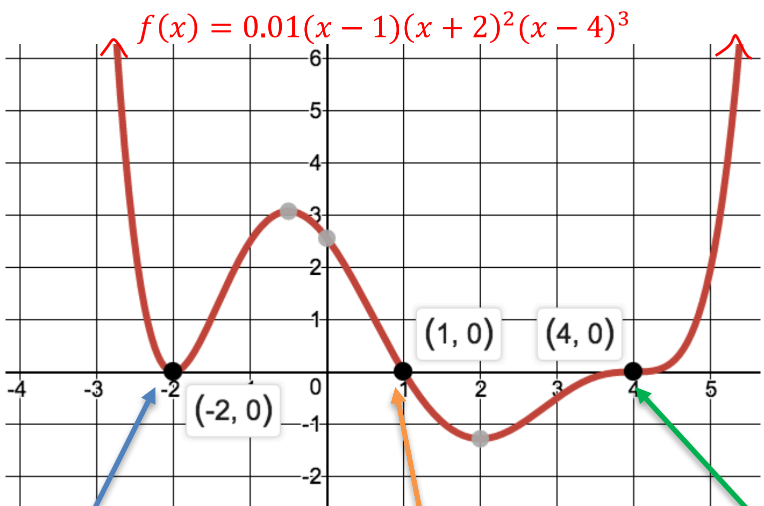
If a polynomial function has a factor $(x-a)$ that is repeated n times, then $x = a$ is a zero of **ORDER** n.

Order – the exponent to which each factor in an algebraic expression is raised.

For example, the function $f(x) = (x-3)^2(x-1)$ has a zero of order **two** at $x = 3$ and a zero of order **one** at $x = 1$.

The graph of a polynomial function changes sign at zeros of **odd** order but does not change sign at zeros of **even** order.

Shapes based on order of zero:



ORDER 2

(-2, 0) is an x-intercept of order 2. Therefore, it doesn't change sign.

"Bounces off" x-axis.

Parabolic shape.

ORDER 1

(1, 0) is an x-intercept of order 1. Therefore, it changes sign.

"Goes straight through" x-axis.

Linear Shape

ORDER 3

(4, 0) is an x-intercept of order 3. Therefore, it changes sign.

"S-shape" through x-axis.

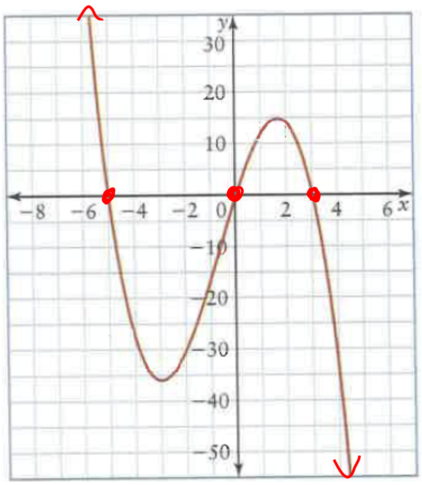
Cubic shape.

Example 1: Analyzing Graphs of Polynomial Functions

For each graph,

i) the least possible degree and the sign of the leading coefficient
ii) the x-intercepts and the factors of the function
iii) the intervals where the function is positive/negative

a)



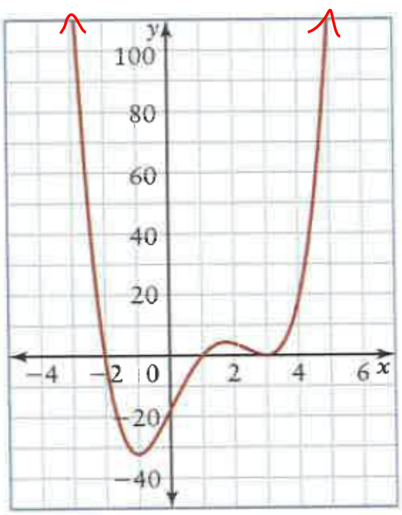
i) Three x-intercepts of order 1, so the least possible degree is 3. The graph goes from Q2 to Q4 so the leading coefficient is negative.

ii) The x-intercepts are -5, 0, and 3.
The factors are $(x+5)$, x, and $(x-3)$

iii)

Interval	$(-\infty,-5)$	$(-5,0)$	$(0,3)$	$(3,\infty)$
Sign of $f(x)$	+	−	+	−

b)



i) Two x-intercepts of order 1, and one x-intercept of order 2, so the least possible degree is 4. The graph goes from Q2 to Q1 so the leading coefficient is positive.

ii) The x-intercepts are -2, 1, and 3.
The factors are $(x+2)$, $(x-1)$, and $(x-3)^2$

iii)

Interval	$(-\infty,-2)$	$(-2,1)$	$(1,3)$	$(3,\infty)$
Sign of $f(x)$	+	−	+	+

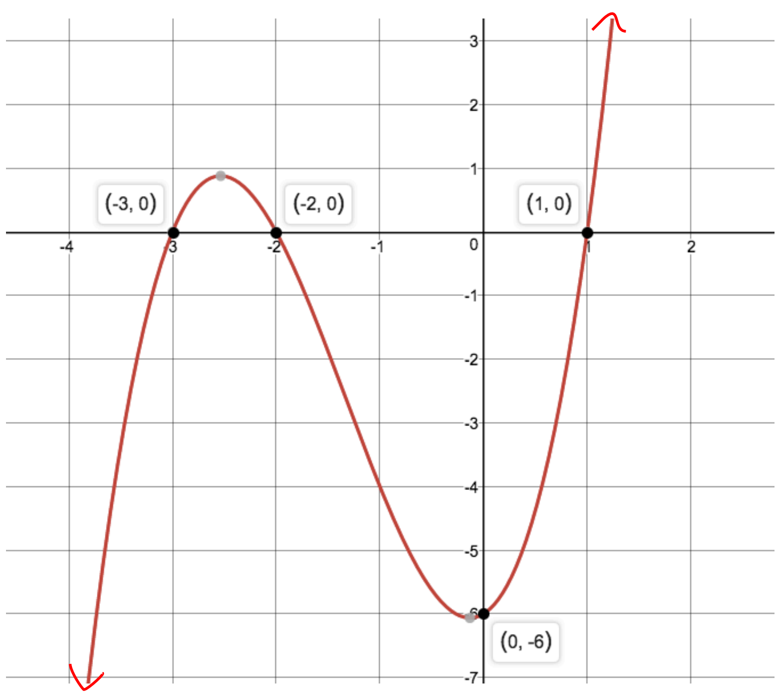
Example 2: Analyze Factored Form Equations to Sketch Graphs

Degree	Leading Coefficient	End Behaviour	x-intercepts	y-intercept
The exponent on x when all factors of x are multiplied together OR Add the exponents on the factors that include an x.	The product of all the x coefficients	Use degree and sign of leading coefficient to determine this	Set each factor equal to zero and solve for x	Set $x = 0$ and solve for y

Sketch a graph of each polynomial function:

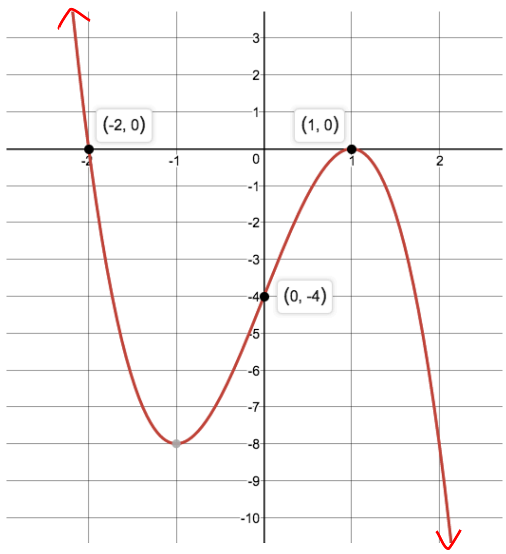
a) $f(x) = (x-1)(x+2)(x+3)$

Degree	Leading Coefficient	End Behaviour	x-intercepts	y-intercept
The product of all factors of x is: $(x)(x)(x) = x^3$ The function is cubic. **DEGREE 3**	The product of all the x coefficients is: $(1)(1)(1) = 1$ **Leading Coefficient is 1**	Cubic with a positive leading coefficient extends from: **Q3 to Q1**	**The x-intercepts are 1, -2, and -3** (1, 0) (-2, 0) (-3, 0)	Set x equal to 0 and solve: $y = (0-1)(0+2)(0+3)$ $y = (-1)(2)(3)$ $y = -6$ **The y-intercept is at (0, -6)**



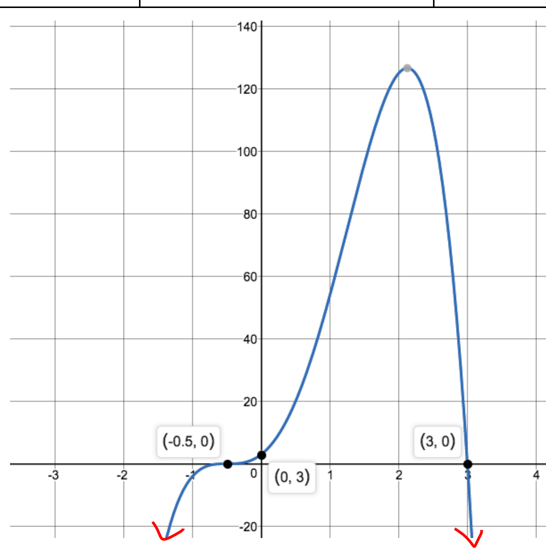
b) $g(x) = -2(x-1)^2(x+2)$

Degree	Leading Coefficient	End Behaviour	x-intercepts	y-intercept
The product of all factors of x is: $(x^2)(x) = x^3$ The function is cubic. **DEGREE 3**	The product of all the x coefficients is: $(-2)(1)^2(1) = -2$ **Leading Coefficient is -2**	Cubic with a negative leading coefficient extends from: **Q2 to Q4**	**The x-intercepts are 1 (order 2), and -2.** (1, 0) (-2, 0)	Set x equal to 0 and solve: $y = -2(0-1)^2(0+2)$ $y = (-2)(1)(2)$ $y = -4$ **The y-intercept is at (0, -4)**



c) $h(x) = -(2x+1)^3(x-3)$

Degree	Leading Coefficient	End Behaviour	x-intercepts	y-intercept
The product of all factors of x is: $(x^3)(x) = x^4$ The function is quartic. **DEGREE 4**	The product of all the x coefficients is: $(-1)(2)^3(1) = -8$ **Leading Coefficient is -8**	A quartic with a negative leading coefficient extends from: **Q3 to Q4**	**The x-intercepts are $-\frac{1}{2}$ (order 3), and 3.** $\left(-\frac{1}{2}, 0\right)$ $(3, 0)$	Set x equal to 0 and solve: $y = -[2(0)+1]^3[0-3]$ $y = (-1)(1)(-3)$ $y = 3$ **The y-intercept is at (0, 3)**

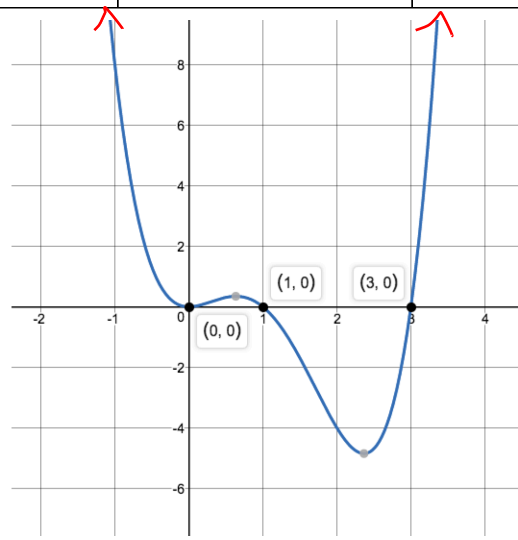


d) $j(x) = x^4 - 4x^3 + 3x^2$

> ***Note:*** *must put in to factored form to find x-intercepts*

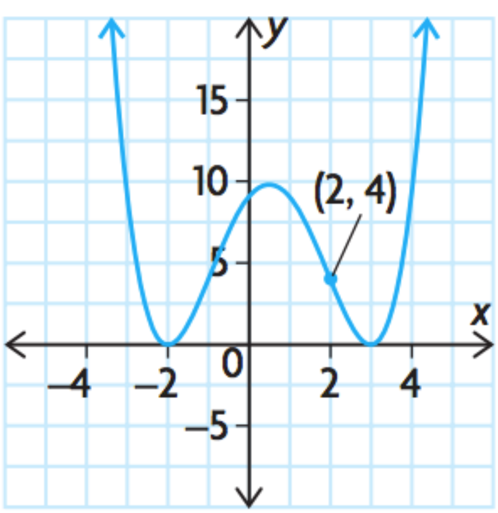
$$j(x) = x^2(x^2 - 4x + 3)$$
$$j(x) = x^2(x - 3)(x - 1)$$

Degree	Leading Coefficient	End Behaviour	x-intercepts	y-intercept
The product of all factors of x is: $(x^2)(x)(x) = x^4$ The function is quartic. **DEGREE 4**	The product of all the x coefficients is: $(1)^2(1)(1) = 1$ **Leading Coefficient is 1**	A quartic with a positive leading coefficient extends from: **Q2 to Q1**	**The x-intercepts are 0 (order 2), 3, and 1.** $(0, 0)$ $(3, 0)$ $(1, 0)$	Set x equal to 0 and solve: $y = (0)^2(0-3)(0-1)$ $y = (0)(-3)(-1)$ $y = 0$ **The y-intercept is at $(0, 0)$**



Example 3: Representing the Graph of a Polynomial Function with its Equation

a) Write the equation of the function shown below:



The function has x-intercepts at -2 and 3. Both are of order 2.

$f(x) = k(x+2)^2(x-3)^2$

$4 = k(2+2)^2(2-3)^2$

$4 = k(4)^2(-1)^2$

$4 = 16k$

$k = \frac{1}{4}$

$$\boldsymbol{f(x) = \frac{1}{4}(x+2)^2(x-3)^2}$$

> **Steps:**
>
> **1)** Write the equation of the family of polynomials using factors created from x-intercepts
>
> **2)** Substitute the coordinates of another point (x, y) into the equation.
>
> **3)** Solve for a
>
> **4)** Write the equation in factored form

b) Find the equation of a polynomial function that is degree 4 with zeros -1 (order 3) and 1, and with a y-intercept of -2.

$f(x) = k(x+1)^3(x-1)$

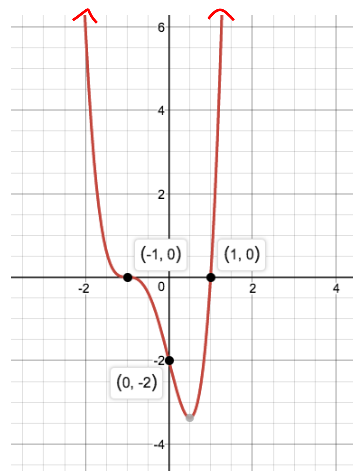
$-2 = k(0+1)^3(0-1)$

$-2 = k(1)^3(-1)$

$-2 = -1k$

$k = 2$

$\boldsymbol{f(x) = 2(x+1)^3(x-1)}$



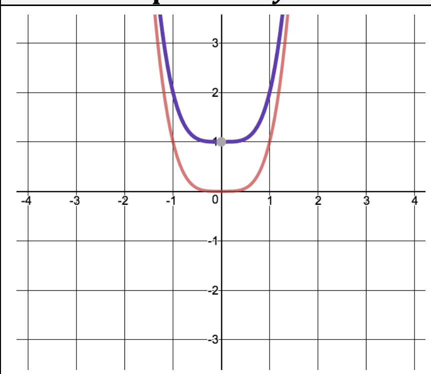
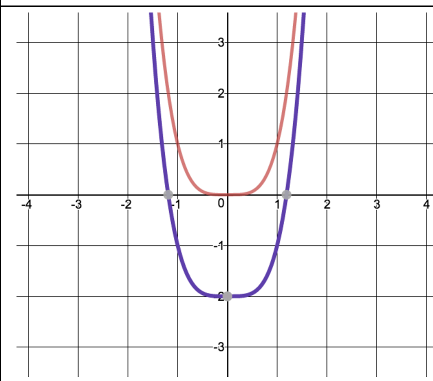
L4 – 1.4 – Transformations Lesson
MHF4U
Jensen

In this section, you will investigate the roles of the parameters a, k, d, and c in polynomial functions of the form $\boldsymbol{f(x) = a[k(x-d)]^n + c}$. You will apply transformations to the graphs of basic power functions to sketch the graph of its transformed function.

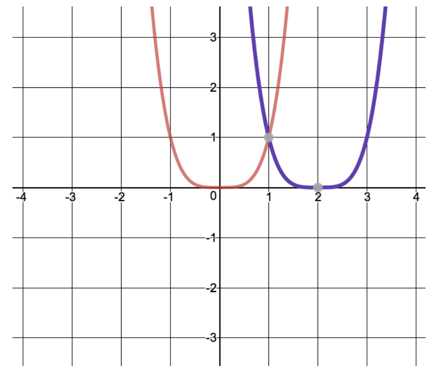
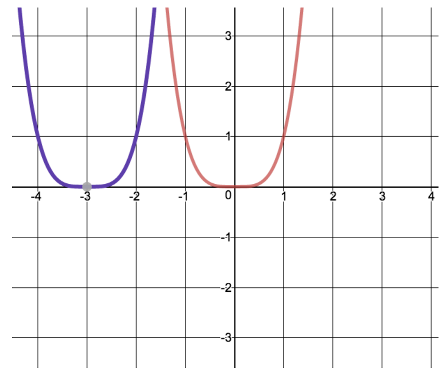
Part 1: Transformations Investigation

In this investigation, you will be looking at transformations of the power function $y = x^4$. Complete the following table using graphing technology to help. The graph of $y = x^4$ is given on each set of axes; sketch the graph of the transformed function on the same set of axes. Then comment on how the value of the parameter a, k, d, or c transforms the parent function.

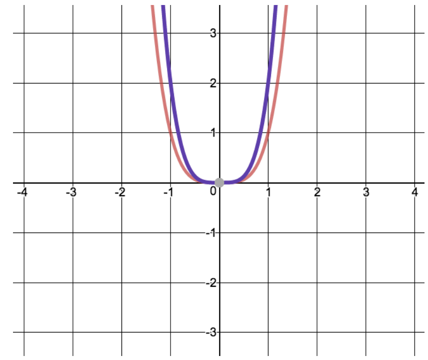
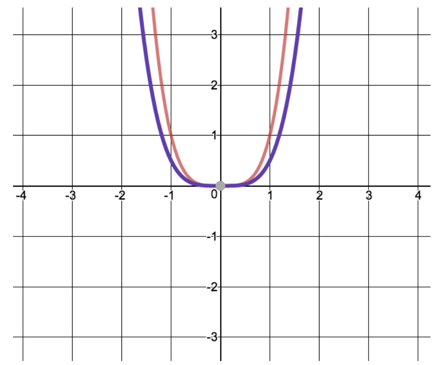
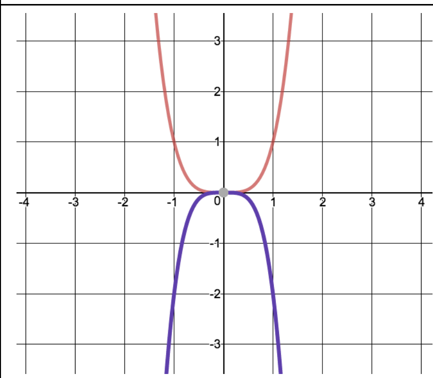
Effects of c on $y = x^4 + c$

Transformed Function	Value of c	Transformations to $y = x^4$	Graph of transformed function compared to $y = x^4$
$y = x^4 + 1$	$c = 1$	Shift up 1 unit	
$y = x^4 - 2$	$c = -2$	Shift down 2 units	

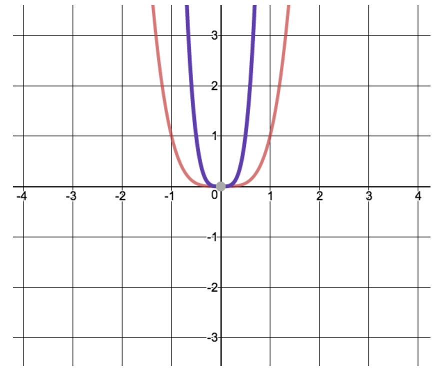
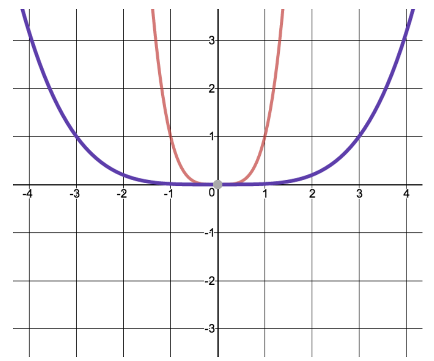
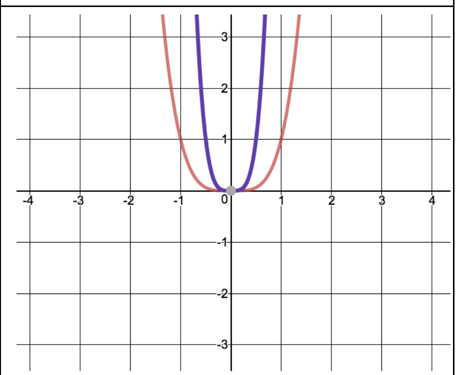
Effects of d on $y = (x - d)^4$

Transformed Function	**Value of d**	**Transformations to $y = x^4$**	**Graph of transformed function compared to $y = x^4$**
$y = (x - 2)^4$	$d = 2$	Shift right 2 units	
$y = (x + 3)^4$	$d = -3$	Shift left 3 units	

Effects of a on $y = ax^4$

Transformed Function	**Value of a**	**Transformations to $y = x^4$**	**Graph of transformed function compared to $y = x^4$**
$y = 2x^4$	$a = 2$	Vertical stretch by a factor of 2	
$y = \frac{1}{2}x^4$	$a = \frac{1}{2}$	Vertical compression by a factor of $\frac{1}{2}$	
$y = -2x^4$	$a = -2$	Vertical stretch by a factor of 2 and a vertical reflection.	

Effects of k on $y = (kx)^4$

Transformed Function	Value of k	Transformations to $y = x^4$	Graph of transformed function compared to $y = x^4$
$y = (2x)^4$	$k = 2$	Horizontal compression by a factor of $\frac{1}{2}$	
$y = \left(\frac{1}{3}x\right)^4$	$k = \frac{1}{3}$	Horizontal stretch by a factor of 3	
$y = (-2x)^4$	$k = -2$	Horizontal compression by a factor of $\frac{1}{2}$ and a horizontal reflection	

Summary of effects of a, k, d, and c in polynomial functions of the form $f(x) = a[k(x-d)]^n + c$

Value of c in $f(x) = a[k(x-d)]^n + c$	
$c > 0$	Shift c units up
$c < 0$	Shift c units down

Value of d in $f(x) = a[k(x-d)]^n + c$	
$d > 0$	Shift d units right
$d < 0$	Shift $\|d\|$ units left

Value of a in $f(x) = a[k(x-d)]^n + c$	
$a > 1$ or $a < -1$	Vertical stretch by a factor of $\|a\|$
$-1 < a < 1$	Vertical compression by a factor of $\|a\|$
$a < 0$	Vertical reflection (reflection in the x-axis)

Value of k in $f(x) = a[k(x-d)]^n + c$	
$k > 1$ or $k < -1$	Horizontal compression by a factor of $\frac{1}{\|k\|}$
$-1 < k < 1$	Horizontal stretch by a factor of $\frac{1}{\|k\|}$
$k < 0$	Horizontal reflection (reflection in the y-axis)

Note:

a and c cause **VERTICAL** transformations and therefore effect the y-coordinates of the function.

k and d cause **HORIZONTAL** transformations and therefore effect the x-coordinates of the function.

When applying transformations to a parent function, make sure to apply the transformations represented by a and k BEFORE the transformations represented by d and c.

Part 2: Describing Transformations from an Equation

Example 1: Describe the transformations that must be applied to the graph of each power function, $f(x)$, to obtain the transformed function, $g(x)$. Then, write the corresponding equation of the transformed function. Then, state the domain and range of the transformed function.

a) $f(x) = x^4, g(x) = 2f\left[\frac{1}{3}(x-5)\right]$

$a = 2$; vertical stretch by a factor of 2 $(2y)$

$k = \frac{1}{3}$; horizontal stretch by a factor of 3 $(3x)$

$d = 5$; shift 5 units right $(x+5)$

$$g(x) = 2\left[\frac{1}{3}(x-5)\right]^4$$

b) $f(x) = x^5, g(x) = \frac{1}{4}f[-2(x-3)] + 4$

$a = \frac{1}{4}$; vertical compression by a factor of $\frac{1}{4}$ $\left(\frac{y}{4}\right)$

$k = -2$; horizontal compression by a factor of $\frac{1}{2}$ and a horizontal reflection $\left(\frac{x}{-2}\right)$

$d = 3$; shift right 3 units $(x+3)$

$c = 4$; shift 4 units up $(y+4)$

$$g(x) = \frac{1}{4}[-2(x-3)]^5 + 4$$

Part 3: Applying Transformations to Sketch a Graph

Example 2: The graph of $f(x) = x^3$ is transformed to obtain the graph of $g(x) = 3[-2(x+1)]^3 + 5$.

a) State the parameters and describe the corresponding transformations

$a = 3$; vertical stretch by a factor of 3 $(3y)$

$k = -2$; horizontal compression by a factor of $\frac{1}{2}$ and a horizontal reflection $\left(\frac{x}{-2}\right)$

$d = -1$; shift left 1 unit $(x - 1)$

$c = 5$; shift up 5 units $(y + 5)$

b) Make a table of values for the parent function and then use the transformations described in part a) to make a table of values for the transformed function.

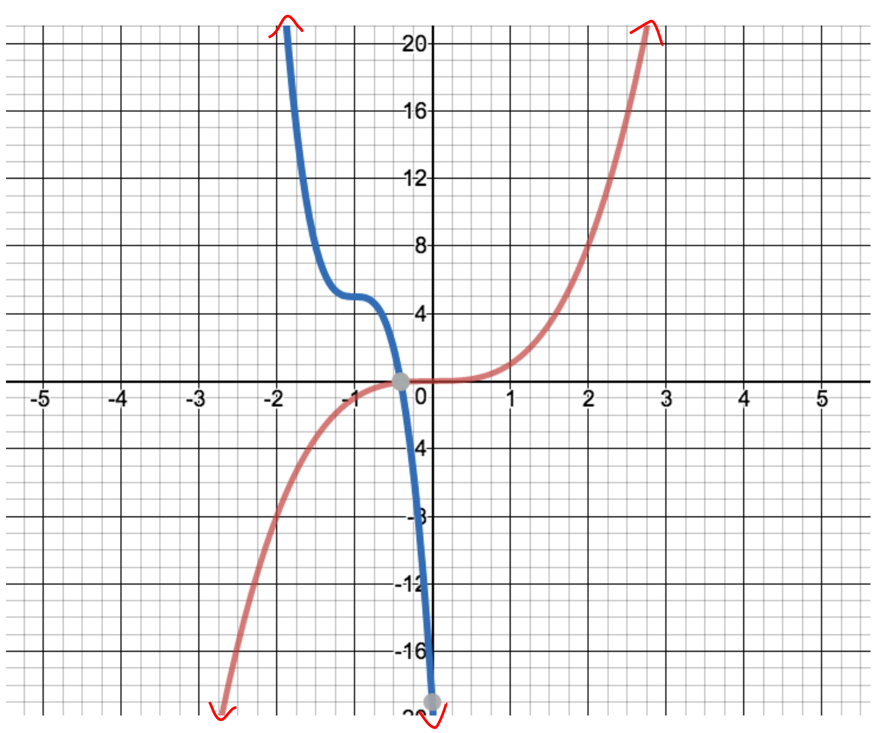
$f(x) = x^3$	
x	y
-2	-8
-1	-1
0	0
1	1
2	8

→

$g(x) = 3[-2(x+1)]^3 + 5$	
$\frac{x}{-2} - 1$	$3y + 5$
0	-19
-0.5	2
-1	5
-1.5	8
-2	29

Note: When choosing key points for the parent function, always choose x-values between -2 and 2 and calculate the corresponding values of y.

c) Graph the parent function and the transformed function on the same grid.



Example 3: The graph of $f(x) = x^4$ is transformed to obtain the graph of $g(x) = -\left(\frac{1}{3}x + 2\right)^4 - 1$.

a) State the parameters and describe the corresponding transformations

Note: k value must be factored out in to the form $[k(x+d)]$

$g(x) = -\left[\frac{1}{3}(x+6)\right]^4 - 1$

$a = -1$; vertical reflection $(-1y)$

$k = \frac{1}{3}$; horizontal stretch by a factor of 3 $(3x)$

$d = -6$; shift left 6 units $(x - 6)$

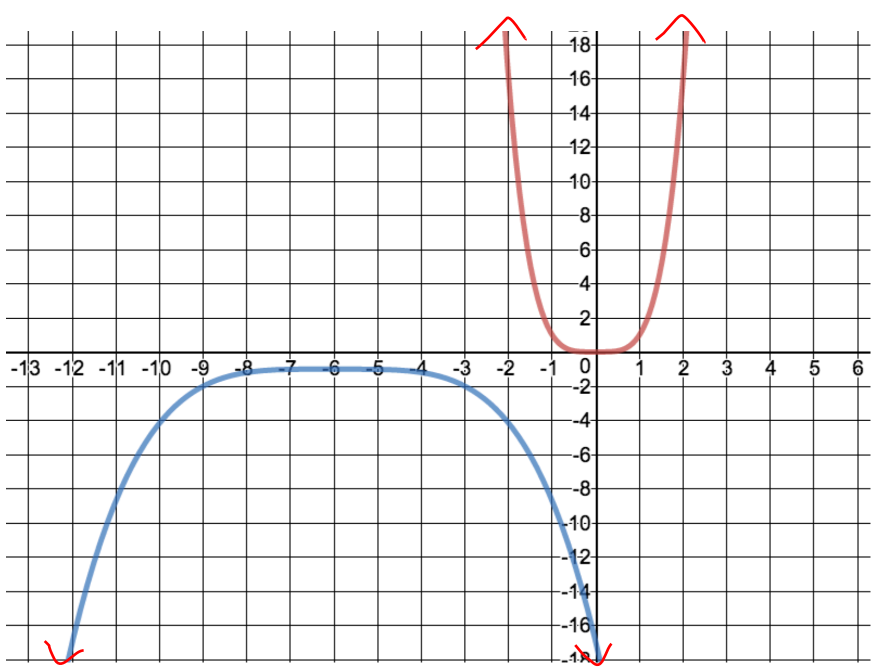
$c = -1$; shift down 1 unit $(y - 1)$

b) Make a table of values for the parent function and then use the transformations described in part a) to make a table of values for the transformed function.

$f(x) = x^4$	
x	y
-2	16
-1	1
0	0
1	1
2	16

$g(x) = -\left[\frac{1}{3}(x+6)\right]^4 - 1$	
$3x - 6$	$-y - 1$
-12	-17
-9	-2
-6	-1
-3	-2
0	-17

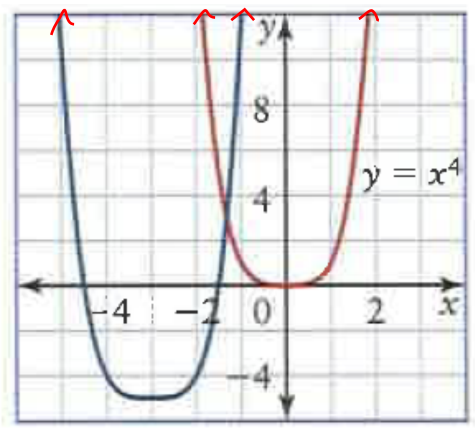
c) Graph the parent function and the transformed function on the same grid.



Part 4: Determining an Equation Given the Graph of a Transformed Function

Example 4: Transformations are applied to each power function to obtain the resulting graph. Determine an equation for the transformed function. Then state the domain and range of the transformed function.

a)



Notice the transformed function is the same shape as the parent function. Therefore, it has not been stretched or compressed.

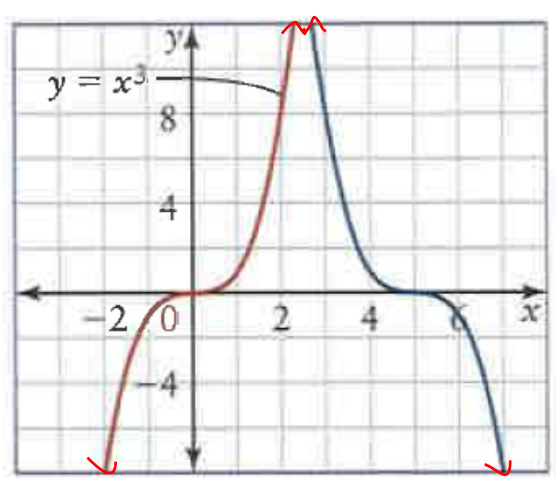
$d = -3$; it has been shifted left 3 units

$c = -5$; it has been shifted down 5 units

$\boldsymbol{g(x) = (x+3)^4 - 5}$

Domain: $(-\infty, \infty)$ **Range:** $[\mathbf{-5}, \infty)$

b)



Notice the transformed function is the same shape as the parent function. Therefore, it has not been stretched or compressed.

$a = -1$; it has been reflected vertically

$d = 5$; it has been shifted right 5 units

$\boldsymbol{g(x) = -(x-5)^3}$

Domain: $(-\infty, \infty)$ **Range:** $(-\infty, \infty)$

L5 – 1.3 – Symmetry in Polynomial Functions
MHF4U
Jensen

In this section, you will learn about the properties of even and odd polynomial functions.

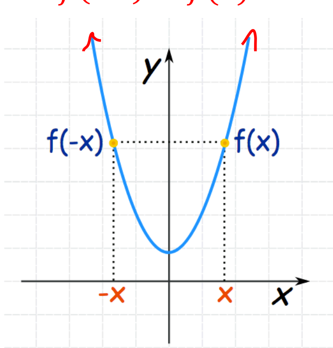
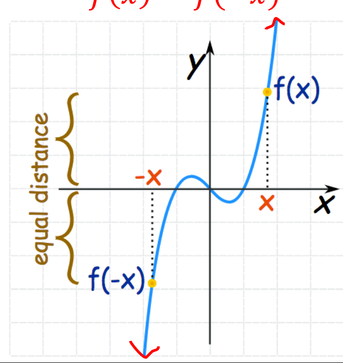
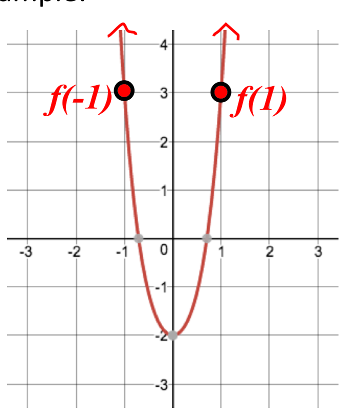
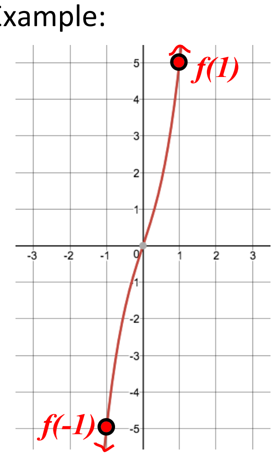
Symmetry in Polynomial Functions

Line Symmetry – there is a vertical line over which the polynomial remains unchanged when reflected.

Point symmetry / Rotational Symmetry – there is a point about which the polynomial remains unchanged when rotated $180°$

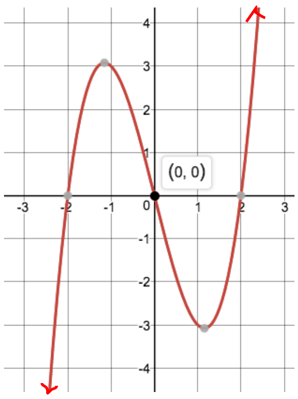
Section 1: Properties of Even and Odd Functions

A polynomial function of even or odd degree is NOT necessarily and even or odd function. The following are properties of all even and odd functions:

Even Functions	Odd Functions
An even degree polynomial function is an EVEN FUNCTION if: • Line symmetry over the y-axis • The exponent of each term is even • May have a constant term	An odd degree polynomial function is an ODD FUNCTION if: • Point symmetry about the origin (0, 0) • The exponent of each term is odd • No constant term
Rule: $f(-x) = f(x)$  	Rule: $-f(x) = f(-x)$ 
Example:  $f(x) = 2x^4 + 3x^2 - 2$ Notice: $f(1) = 3$ $f(-1) = 3$ $\therefore f(1) = f(-1)$	Example:  $f(x) = 2x^3 + 3x$ Notice: $f(1) = 5$ $f(-1) = -5$ $\therefore -f(1) = f(-1)$

Example 1: Identify each function as an even function, odd function, or neither. Explain how you can tell.

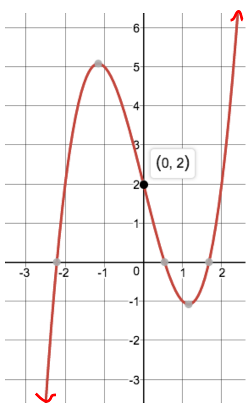
a) $y = x^3 - 4x$



This is an odd function because:

- It has point symmetry about the origin
- All terms in the equation have an odd exponent and there is no constant term

b) $y = x^3 - 4x + 2$

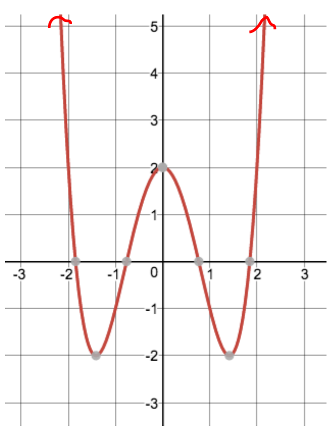


Neither

This function has point symmetry. However, the origin is not the point about which the function is symmetrical. Therefore, it is not an odd or even function.

From the equation we can tell it is NOT an odd function because there is a constant term.

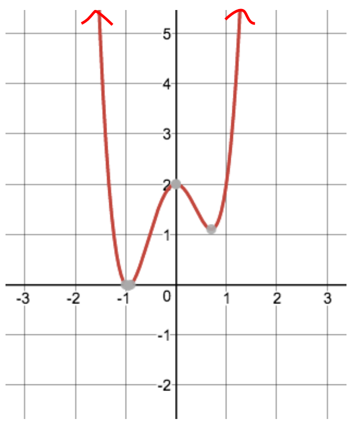
c) $y = x^4 - 4x^2 + 2$



This is an even function because:

- It has line symmetry about the y-axis
- All terms in the equation have an even exponent. Even functions are allowed to have a constant term.

d) $y = 3x^4 + x^3 - 4x^2 + 2$

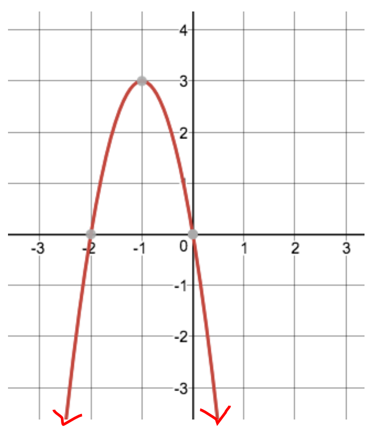


Neither

This function does not have line or point symmetry.

From the equation we can tell it is NOT an even or odd function because there is a mix of even and odd exponents.

e) $y = -3x^2 - 6x$



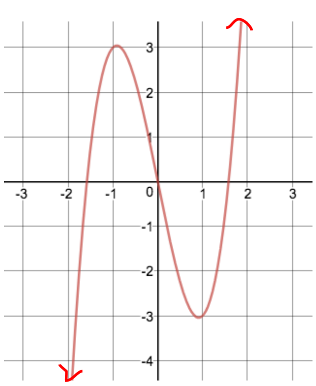
Neither

This function has line symmetry. However, the y-axis is not the line about which the function is symmetrical. Therefore, it is not an odd or even function.

From the equation we can tell it is NOT an even or odd function because there is a mixture of even and odd exponents.

<u>**Example 2:**</u> Choose all that apply for each function

a)



i) no symmetry

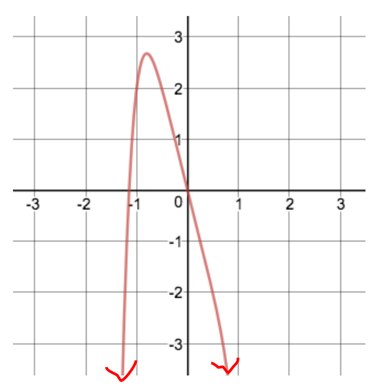
ii) point symmetry

iii) line symmetry

iv) odd function

v) even function

b)



i) no symmetry

ii) point symmetry

iii) line symmetry

iv) odd function

v) even function

c) $P(x) = 5x^3 + 3x^2 + 2$

__Note:__ all cubic functions have point symmetry

i) no symmetry

ii) point symmetry

iii) line symmetry

iv) odd function

v) even function

d) $P(x) = x^6 + x^2 - 11$

i) no symmetry

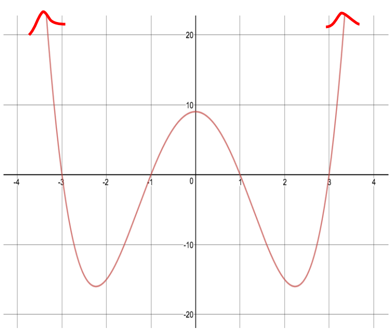
ii) point symmetry

iii) line symmetry

iv) odd function

v) even function

e)



i) no symmetry

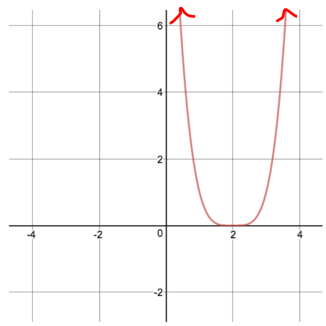
ii) point symmetry

iii) line symmetry

iv) odd function

v) even function

f)



i) no symmetry

ii) point symmetry

iii) line symmetry

iv) odd function

v) even function

g) $P(x) = 5x^5 - 4x^3 + 8x$

i) no symmetry

ii) point symmetry

iii) line symmetry

iv) odd function

v) even function

__Example 3:__ Without graphing, determine if each polynomial function has line symmetry about the y-axis, point symmetry about the origin, or neither. Verify your response algebraically.

a) $f(x) = 2x^4 - 5x^2 + 4$

The function is even since the exponent of each term is even. The function has line symmetry about the y-axis.

Verify $f(x) = f(-x)$

$f(-x) = 2(-x)^4 - 5(-x)^2 + 4$
$f(-x) = 2x^4 - 5x^2 + 4$
$f(-x) = f(x)$

b) $f(x) = -3x^5 + 9x^3 + 2x$

The function is odd since the exponent of each term is odd. The function has point symmetry about the origin.

Verify $-f(x) = f(-x)$

$-f(x) = -(-3x^5 + 9x^3 + 2x)$
$-f(x) = 3x^5 - 9x^3 - 2x$

$f(-x) = -3(-x)^5 + 9(-x)^3 + 2(-x)$
$f(-x) = 3x^5 - 9x^3 - 2x$

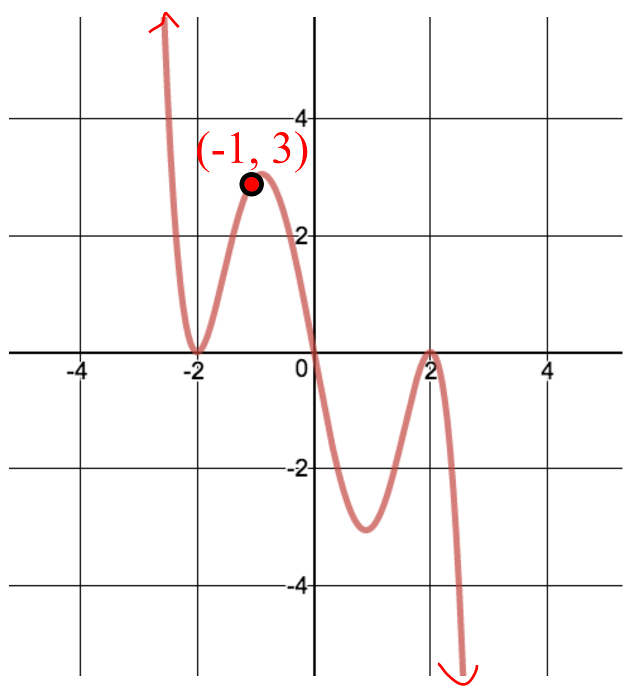
$\therefore\ -f(x) = f(-x)$

c) $x^6 - 4x^3 + 6x^2 - 4$

Some exponents are even and some are odd, so the function is neither even nor odd. It does not have line symmetry about the y-axis or point symmetry about the origin.

Section 2: Connecting from throughout the unit

Example 4: Use the given graph to state:



a) x-intercepts

-2 (order 2), 0 (order 1), and 2(order 2)

b) number of turning points

2 local min and 2 local max
4 turning points

c) least possible degree

Least possible degree is 5

b) any symmetry present

Point symmetry about the origin. Therefore, this is an odd function.

c) the intervals where $f(x) < 0$

$(0, 2) \cup (2, \infty)$

d) Find the equation in factored form

$P(x) = k(x)(x+2)^2(x-2)^2$

$3 = k(-1)(-1+2)^2(-1-2)^2$

$3 = k(-1)(1)^2(-3)^2$

$3 = -9k$

$k = -\frac{1}{3}$

$P(x) = -\frac{1}{3}x(x+2)^2(x-2)^2$