Chapter 4

Probability

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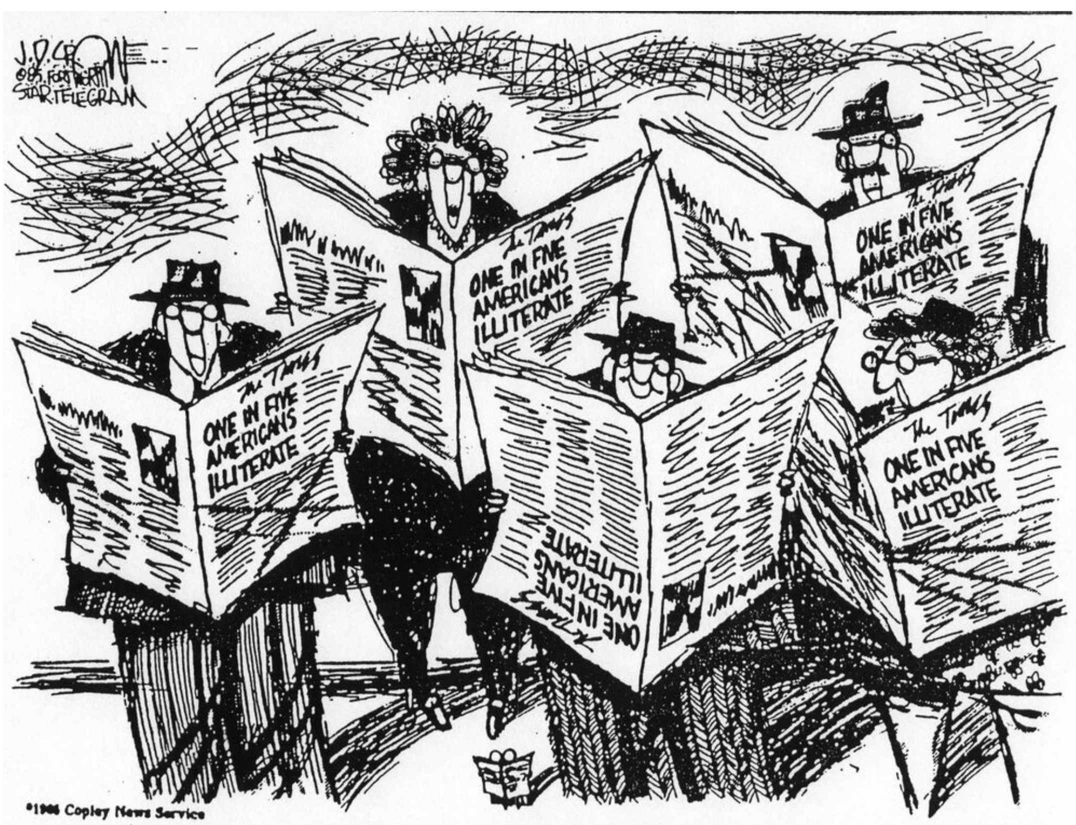

Section 4.1 - Intro to Probability

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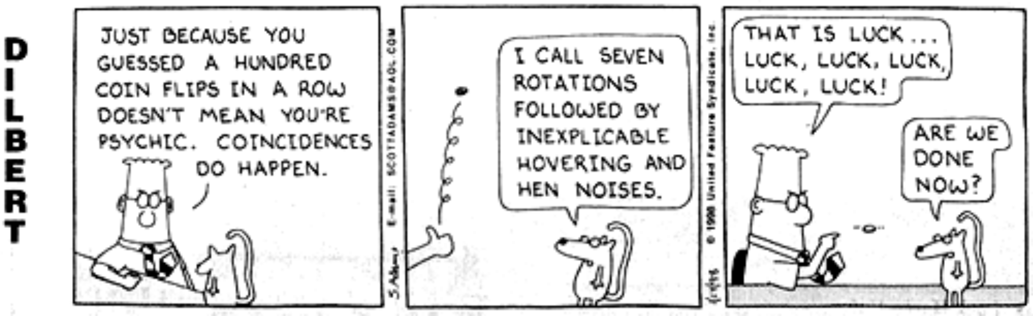


Activity #1: Monty Hall – Let's Make a Deal

Game description: This game is based on the old television show "Let's Make a Deal" hosted by Monty Hall. At the end of each show, the contestant who had won the most money was invited to choose from among 3 doors: Door #1, Door #2, and Door #3. Behind one of the three doors was a very nice prize, let's say a car. Behind the other 2 doors there was a goat. The contestant selected a door. Monty then revealed what was behind one of the OTHER doors (always a goat). The contestant was then offered a choice: stick with his current door, or switch to the remaining un-revealed door. He won what was behind his final choice of door.

Part I: SIMULATION

Instructions: Students pair up. Each pair of students should have 3 cards – a face card/ace (car) and 2 numbered cards (goats). Have one of the partners arrange the cards and act as Monty Hall and the other as the contestant. The contestant picks a door (card). Without showing the original pick, the show host shows one of the other cards (it must always be a goat). The contestant must now decide to switch or stick. The card is shown. Do this 10 times and record the results in the table below.

Trial	Stick/Switch?	Win/lose?
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

Finally, exchange roles. Repeat the simulation. Ultimately, each pair of students should have 20 observations between them. (For a "modern version" of the simulation, visit: http://istics.net/monty/#

Combine your data with your partner's data.

# of trials switched	_________	# of cars won (after switching)	_________
# of trials "stuck"	_________	# of cars won (after "sticking")	_________
total # of trials	_________	# of cars won (grand total)	_________

Now pool the class results. Don't double-count your data and your partner's!

# of trials switched	_________	# of cars won (after switching)	_________
# of trials "stuck"	_________	# of cars won (after "sticking")	_________
total # of trials	_________	# of cars won (grand total)	_________

Questions:

1. What proportion of all trials resulted in a win?
2. What proportion of all "switch" trials resulted in a win?
3. What proportion of all "stick" trials resulted in a win?
4. What proportion of all wins (i.e., all cars) were the result of the switching strategy?

What do these probabilities tell you about your intuitive answer? Does switching improve your chance of winning the car?

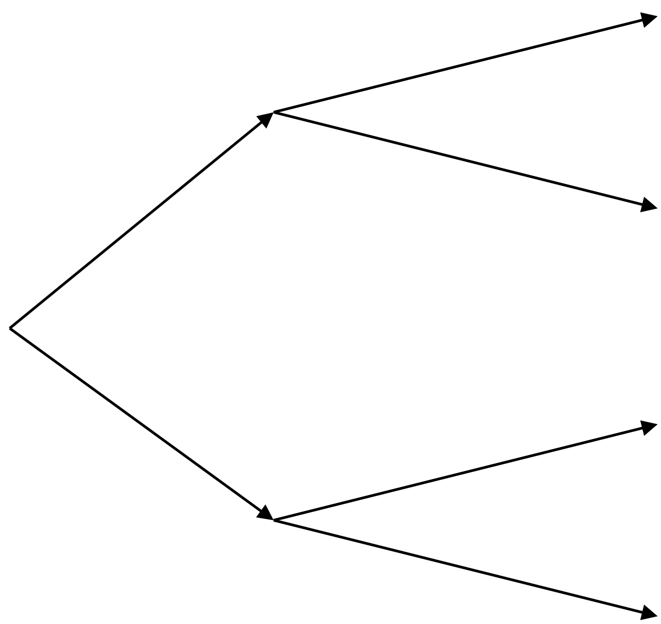
Part II: MATHEMATICAL PROBABILITY

We will now construct a tree diagram. First consider the "prize" behind the first door the contestant selects. This will form the first part of the tree diagram. The two options are "CAR" and "GOAT."

What is *P*(GOAT)? _________ What is the *P*(CAR)? _________

Second, consider the decision made by the contestant. This will form the second part of the tree diagram. The two options are "SWITCH" and "STICK." To be fair in our calculations, we assume that the probability of switching is 0.5. Fill in the remaining probabilities and complete your tree diagram below.

Finally, use the tree diagram:



What is *P*(CAR|SWITCH)? __________

Is there an advantage to switching? Does this agree with your original opinion?

Watch this video for an explanation of the probabilities: http://www.youtube.com/watch?v=mhlc7peGlGg

<u>Activity #2: Egg Roulette</u>

While watching the video, answer the following questions

1) What is the probability that the egg will be raw?

2) What is the probability that the egg will be raw?

3) What is the probability that the egg will be raw?

4) What is the probability that the egg will be raw?

5) What is the probability that the egg will be raw?

6) What is the probability that the egg will be raw?

7) What is the probability that the egg will be raw?

8) What is the probability that the egg will be raw?

9) What is the probability that the egg will be raw?

10) What is the probability that the egg will be raw?

11) What is the theoretical probability that the first five eggs chosen are not raw?

<u>Activity #3: Skunk</u>

Game Description: The object of SKUNK is to accumulate points by rolling dice. Points are accumulated by making several "good" rolls in a row but choosing to stop before a "bad" roll comes and wipes out all the points. **Each letter of SKUNK represents a different round of the game**; play begins with the “S” column and continues through the "K" column. The object of SKUNK is to accumulate the greatest possible point total over five rounds. The rules for play are the same for each of the five rounds. (letters)

- At the beginning of each round, every player stands. Then, the teacher rolls a pair of dice and records the total on an overhead or at the board.
- Players record the total of the dice in their column, unless a "one" comes up.
- If a "one" comes up, play is over for <u>that round only</u> and all the player's points in that column are wiped out.
- If "double ones" come up, <u>you get -10 for that round.</u>
- If a "one" doesn't occur, players may choose either to try for more points on the next roll (by continuing to stand) or to stop and keep what he or she has accumulated (by sitting down). Once a player sits during a round they may not stand again until the beginning of the next round.
- A round is over when all the students are seated or a one or double ones show.

Here is your game board:

S	K	U	N	K

S	K	U	N	K

Probability questions:

1) What is the probability of a one showing on any given roll?

2) What is the probability of rolling double ones on any given roll?

3) What was your strategy during the game? Why?

Section 4.2 - Theoretical Probability

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The probability of any outcome of a chance process is a number between 0 and 1 that describes the proportion of times the outcome would occur in a very long series of repetitions.

Part 1: Video on Probability

http://www.learner.org/courses/againstallodds/unitpages/unit19.html

Answer the following questions while watching the video:

1. What is a probability model (distribution)?

A probability model is the set of all possible outcomes together with the probabilities associated with those outcomes.

2. Describe the sample space for the sum of two dice.

S = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

3. What is the probability of rolling two dice and getting a sum of seven?

P(7) = 6/36 = 1/6

4. If you know the probability that event A occurs, how do you calculate the probability that event A does not occur?

P(not A) = 1 – P(A)

5. What probability can you find using the Addition Rule? 6. What probability can you find using the Multiplication Rule?

If events A and B are mutually exclusive, you can use the Addition Rule to calculate P(A or B), the probability that either A or B occurs. $P(A \cup B) = P(A) + P(B)$

6. What probability can you find using the Multiplication Rule?

If events A and B are independent, you can use the Multiplication Rule to calculate P(A and B), the probability that both A and B occur. $P(A \cap B) = P(A) \times P(B)$

Part 2: Theoretical Probability

Assuming that all outcomes are equally likely, the probability of an event in an experiment is the ratio of the number of outcomes that make up that event to the total number of possible outcomes.

The formula for the probability of an event A is:

$$P(A) = \frac{n(A)}{n(S)}$$

S is the collection of all possible outcomes of the experiment (the sample space)

A is the collection of outcomes that correspond to the event of interest (the event space)

n(S) and n(A) are the numbers of elements in the two sets.

Examples of Theoretical Probability

We know that there are 6 possible outcomes when a die is rolled. Only one of these outcomes is the event of rolling a 4. Since the outcomes are all equally likely to happen, it is reasonable to expect that the fraction of the time you roll a 4 is the ratio of the number of ways a 4 can occur to the number of possible outcomes.

$$P(4) = \frac{n(4)}{n(S)} = \frac{1}{6}$$

What is the probability of rolling an even number?

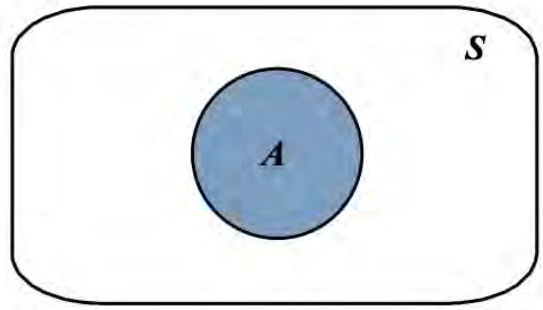
$$P(even) = \frac{n(even)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

Given a large enough number of trials, the experimental probability should trend towards the theoretical probability

Venn Diagrams

A Venn diagram can be used to show the relationship between the event space, $\underline{A}$, and the sample space, $\underline{S}$. Venn diagrams will be used more in section 4.3.

The Venn diagram shows A as the shaded region within S.



Example 1

If a single die is rolled, determine the probability of rolling:

a) an odd number

A = odd = {1, 3, 5}

S = {1, 2, 3, 4, 5, 6}

$$P(odd) = \frac{n(A)}{n(S)} = \frac{n(odd)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

b) a 5 or a 2

A = 5 or 2 = {2, 5}

S = {1, 2, 3, 4, 5, 6}

$$P(5\ or\ 2) = \frac{n(A)}{n(S)} = \frac{n(5\ or\ 2)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

c) A number greater than 2

A = >2 = {3, 4, 5, 6}

S = {1, 2, 3, 4, 5, 6}

$$P(>2) = \frac{n(A)}{n(S)} = \frac{n(>2)}{n(S)} = \frac{4}{6} = \frac{2}{3}$$

Example 2

A bag contains 20 marbles

4 Blue, 6 Orange, 7 Green, 3 Purple

What is the probability of pulling a green marble?

$$P(green) = \frac{n(green)}{n(S)} = \frac{7}{20}$$

Example 3

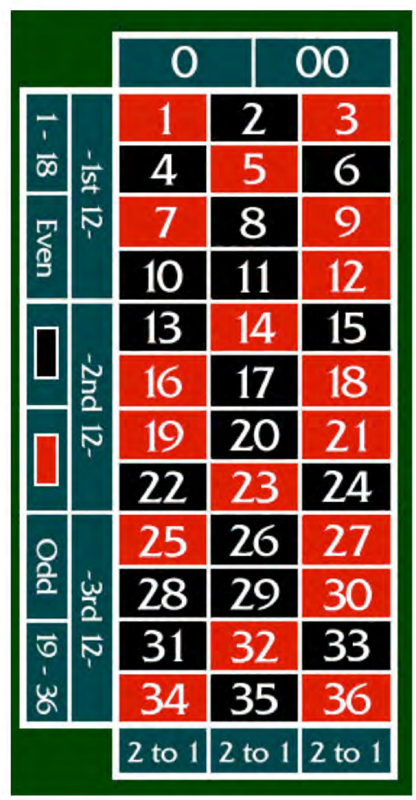
Roulette was invented by Blaise Pascal in his search for a perpetual motion machine...

$$P(red) = \frac{n(red)}{n(S)} = \frac{18}{38} = \frac{9}{19}$$

$$P(even) = \frac{n(even)}{n(S)} = \frac{18}{38} = \frac{9}{19}$$

$$P(14) = \frac{n(14)}{n(S)} = \frac{1}{38}$$

$$P(1st\ 12) = \frac{n(1st\ 12)}{n(S)} = \frac{12}{38} = \frac{6}{19}$$



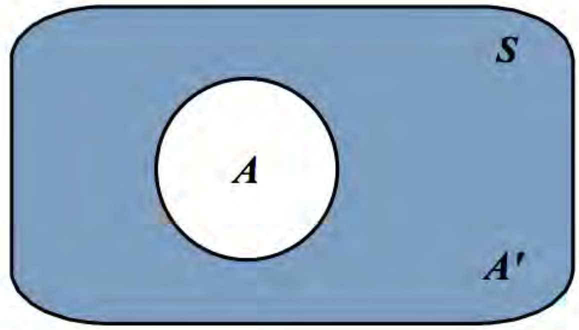
Part 3: Probability and Complimentary Events

The complement of a set (A), is written as $P(A')$ and consists of all the outcomes in the sample space that are NOT in A.

A' = {outcomes in S that are NOT in A}

P(A') = 1 – P(A)

The Venn diagram shows A' as the shaded region within S that is entirely outside of A.



Example 4

In example 1b, we calculated the probability of rolling a 2 or a 5 to be $\frac{1}{3}$. Use this to calculate the probability of NOT rolling a 2 or a 5.

$$P(2\ or\ 5') = 1 - P(2\ or\ 5)$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

Example 5

A standard deck of cards comprises 52 cards in four suits - clubs, hearts, diamonds, and spades. Each suit consists of 13 cards - ace through 10, jack, queen, and king.

a) What is the probability of drawing an ace from a well-shuffled deck?

$$P(ace) = \frac{n(ace)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

b) What is the probability of drawing anything but an ace?

$$P(ace') = 1 - P(ace)$$

$$= 1 - \frac{1}{13}$$

$$= \frac{12}{13}$$

c) What is the probability of selecting a face card (J, Q, K)?

$$P(face) = \frac{n(face)}{n(S)} = \frac{12}{52} = \frac{3}{13}$$

d) What is the probability of NOT selecting a face card (J, Q, K)?

$P(face') = 1 - P(face)$

$= 1 - \frac{3}{13}$

$= \frac{10}{13}$

Section 4.3 – Probability Using Sets

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Part 1: Warm-Up

When rolling a die with sides numbered from 1 to 20, what is the probability of rolling a number divisible by 5?

$$P(divisible\ by\ 5) = \frac{n(divisible\ by\ 5)}{n(S)} = \frac{4}{20} = \frac{1}{5}$$

Part 2: New Terms

1. Simple Event

- an event that consists of exactly one outcome

2. Compound event

- consists of two or more simple events

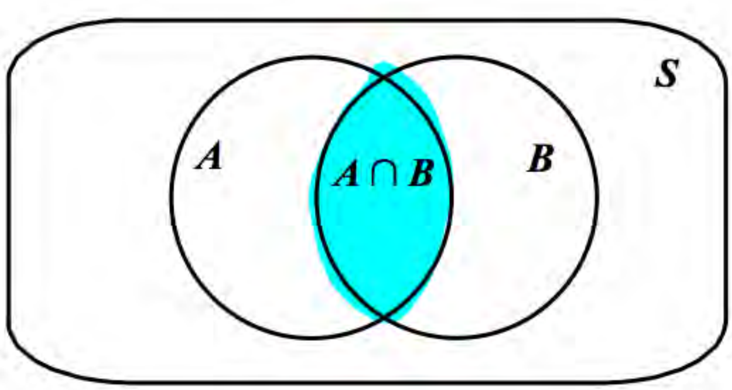
3. Subsets

- sets that exist as a set within a larger set

Part 3: Intersection of Sets

Given two sets, A and B, the set of common elements is called the intersection of A and B, and is written as $A \cap B$. These common elements are members of set A and are also elements of set B.

$A \cap B$ = {elements in both A AND B}

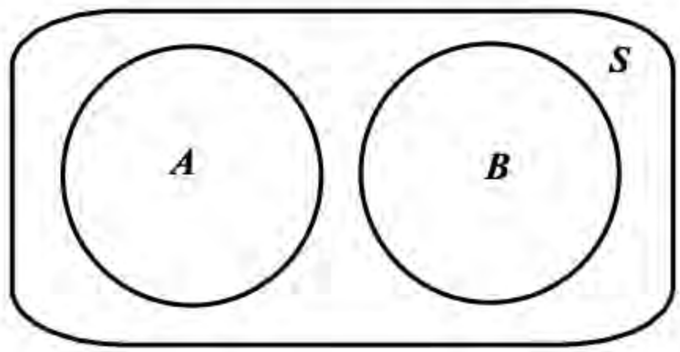


Note: A and B are both subsets of S

Disjoint Sets

If A and B have no elements in common they are said to be disjoint or mutually exclusive and their 'intersection' is the empty set (∅).

$$\mathbf{A \cap B = \varnothing}$$

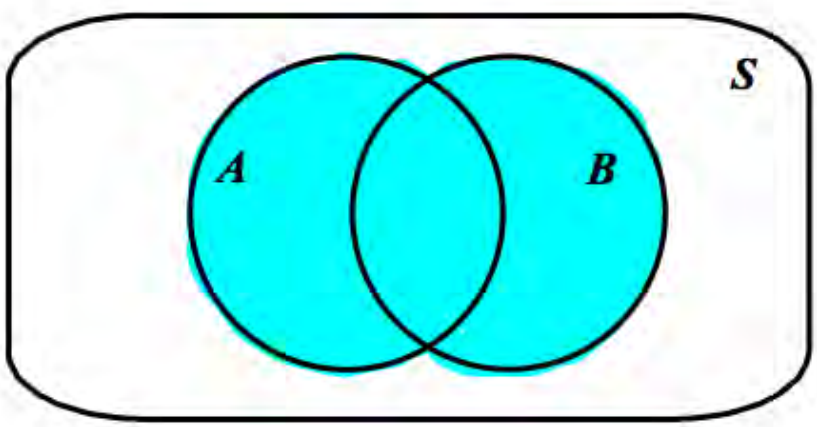


Note: n($A \cap B$) = 0

Part 4: Union of Sets

The set formed by combining the elements of A with those in B is called the union of A and B, and is written as $A \cup B$. The elements in $A \cup B$ are elements of A or they are elements of B.

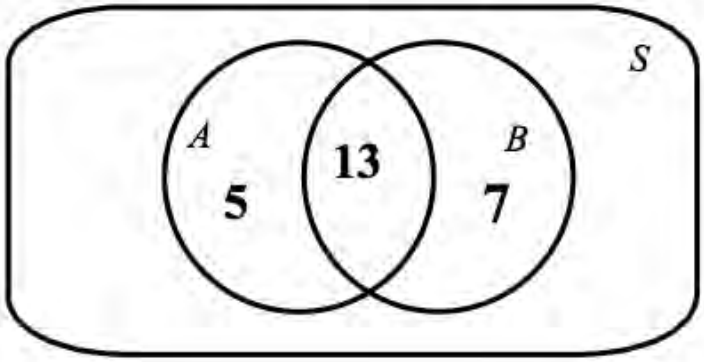
$A \cup B$ = {elements in A OR B}



The set $A \cup B$ is represented by the shaded region in the Venn diagram.

Example 1

Using the following Venn diagram, determine:



a) n(A) = 18

b) n(B) = 20

c) n($A \cap B$) = 13

d) n($A \cup B$) = 25

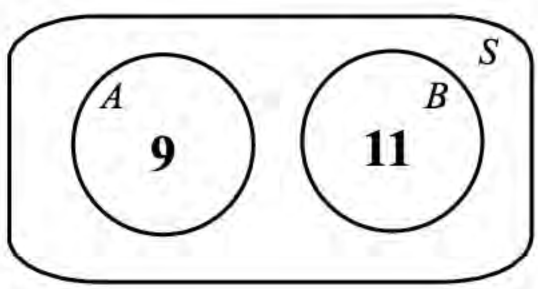
e) Show that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A \cup B) = 18 + 20 - 13$

$= 25$

Example 2:

If A and B are disjoint sets:



a) $n(A \cap B) = 0$

b) $n(A \cup B) = 20$

Part 5: Additive Principle for the Union of Two Sets

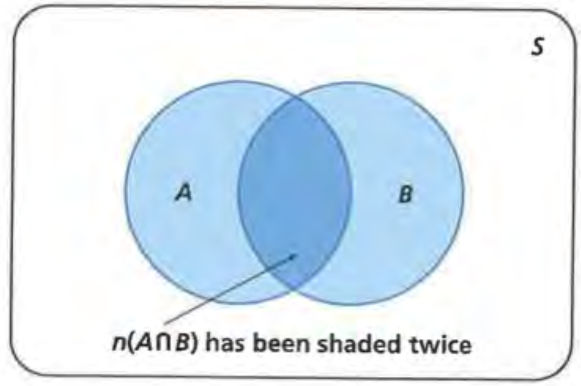
Given two sets, A and B, the number of elements in $A \cup B$ can be found by totaling the number of elements in both sets and then subtracting the number that have been counted twice. The double counted elements will be found in the intersection of the two sets ($A \cap B$).

Number of elements in set A or B:

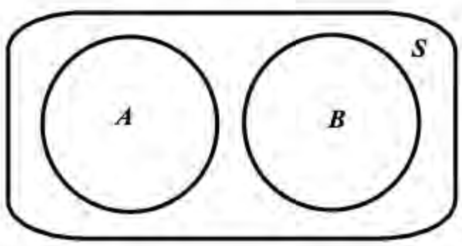
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Probability of the event that A or B occurs is:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Additive Principle for Mutually Exclusive Events



Note: n($A \cap B$) = 0

If events A and B are mutually exclusive (they can't occur at the same time); the rule can be simplified:

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A \cup B) = n(A) + n(B) - 0$

$n(A \cup B) = n(A) + n(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = P(A) + P(B) - 0$

$P(A \cup B) = P(A) + P(B)$

Example 3:

The Cav's are sending their basketball and hockey teams to OFSAA on the same bus. There are 10 students on the basketball team and 17 students on the hockey team. Three students play on both teams. How many students must the bus taking the teams be able to hold?

$n(basketball \cup hockey) = n(basketball) + n(hockey) - n(basketball \cap hockey)$

$n(basketball \cup hockey) = 10 + 17 - 3$

$= 24$

Example 4:

When rolling a 6 sided die, what is the probability you roll an even number **or** a number less than 3?

$P(even \cup less\ than\ 3) = P(even) + P(less\ than\ 3) - P(even \cap less\ than\ 3)$

$P(even \cup less\ than\ 3) = \frac{3}{6} + \frac{2}{6} - \frac{1}{6}$

$= \frac{4}{6}$

$= \frac{2}{3}$

Example 5:

A sporting goods store has 22 Bauer hockey sticks (14 right, 8 left) and 38 Easton hockey sticks (20 right, 18 left). If the sales representative randomly grabs a stick to make a sales pitch to you, what is the probability that it is a Bauer stick or a left handed stick?

$P(Bauer \cup left\ handed) = P(Bauer) + P(left\ handed) - P(Bauer \cap left\ handed)$

$P(Bauer \cup left\ handed) = \frac{22}{60} + \frac{26}{60} - \frac{8}{60}$

$= \frac{40}{60}$

$= \frac{2}{3}$

Example 6:

A blood bank catalogs the types of blood, including positive or negative Rh-factor, given by donors during the last five days. The umber of donors who gave each blood type is shown in the table. A donor is selected at random.

		Blood Type				
		O	A	B	AB	Total
Rh-factor	Positive	156	139	37	12	344
	Negative	28	25	8	4	65
	Total	184	164	45	16	409

a) Find the probability the donor has type O or type A blood.

$P(type\ O \cup type\ A) = P(type\ O) + P(type\ A)$

$= \frac{184}{409} + \frac{164}{409}$

$= \frac{348}{409}$

Note: *the events __are__ mutually exclusive because a donor can not have type O and type A blood.*

b) Find the probability the donor has type B blood or is Rh-negative.

$$P(type\ B \cup Rh\ negative) = P(type\ B) + P(Rh\ negative) - P(type\ B \cap Rh\ negative)$$

$$P(type\ B \cup Rh\ negative) = \frac{45}{409} + \frac{65}{409} - \frac{8}{409}$$

$$= \frac{102}{409}$$

> ***Note:*** *the events <u>are not</u> mutually exclusive because a donor can have type B blood and be Rh-negative.*

Example 7:

If two dice are rolled, one red and one green, find the probability that you roll:

a) a total of 2 or 12

$$P(2 \cup 12) = P(2) + P(12)$$

$$= \frac{1}{36} + \frac{1}{36}$$

$$= \frac{2}{36}$$

$$= \frac{1}{18}$$

D_2 \ D_1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Mutually Exclusive

b) a total of 4 or a pair will occur

Let A be the event of rolling a total of 4 and B be the event of rolling a pair.

$$P(4 \cup pair) = P(4) + P(pair) - P(4 \cap pair)$$

$$= \frac{3}{36} + \frac{6}{36} - \frac{1}{36}$$

$$= \frac{8}{36}$$

$$= \frac{2}{9}$$

D_2 \ D_1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

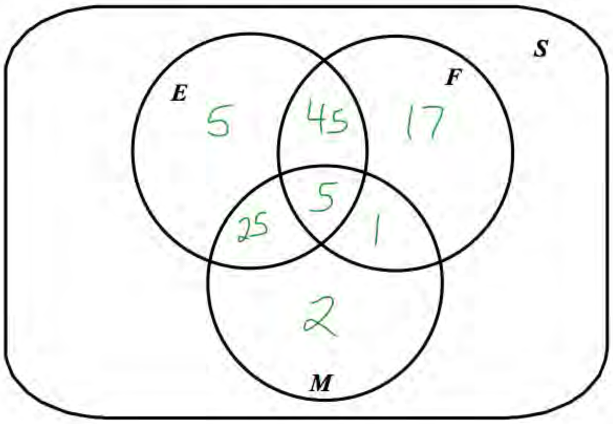
Not Mutually Exclusive

Example 8 (Using Venn Diagrams):

A survey of 100 grade 12 students in a local high school produced the following results:

a) Create a Venn Diagram of the information

Course Taken	No. of students
English	80
Mathematics	33
French	68
English and Mathematics	30
French and Mathematics	6
English and French	50
All three courses	5



b) How many students study English only? $n(english\ only) = 5$

c) How many students study French only? $n(french\ only) = 17$

d) How many students study Math only? $n(math\ only) = 2$

If you were asked to randomly select a student from the group of students described, what is the probability that:

e) the student selected is enrolled only in Math?

$$P(math\ only) = \frac{2}{100} = \frac{1}{50}$$

f) the student is enrolled in french or math?

$$P(french \cup math) = P(french) + P(math) - P(french \cap math)$$

$$= \frac{68}{100} + \frac{33}{100} - \frac{6}{100}$$
$$= \frac{95}{100}$$

$$= \frac{19}{20}$$

g) the student is enrolled in french and math?

$$P(french \cap math) = \frac{6}{100} = \frac{3}{50}$$

Section 4.4 – Conditional Probability

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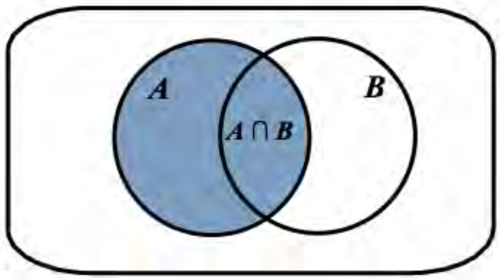
Part 1: Conditional Probability

A conditional probability is the probability of an event occurring, given that another event has already occurred.



formula(s):

$$P(B|A) = \frac{n(B \cap A)}{n(A)} \quad \text{OR} \quad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Note: P(B|A) is read as "probability of B, given A."

Example 1a: The table shows the results of a study in which researchers examined a child's IQ and the presence of a specific ten in the child. Find the probability that a child has a high IQ given that the child has the gene.

$$P(high\ IQ|gene\ present) = \frac{n(high\ IQ \cap gene\ present)}{n(gene\ present)}$$

$$= \frac{33}{72}$$

	Blood Type		
	Gene Present	Gene Not Present	Total
High IQ	33	19	52
Normal IQ	39	11	50
Total	72	30	102

Example 1b: Find the probability that a child does not have the gene, given that the child has a normal IQ.

$$P(gene\ not\ present|normal\ IQ) = \frac{n(gene\ not\ present \cap normal\ IQ)}{n(normal\ IQ)}$$

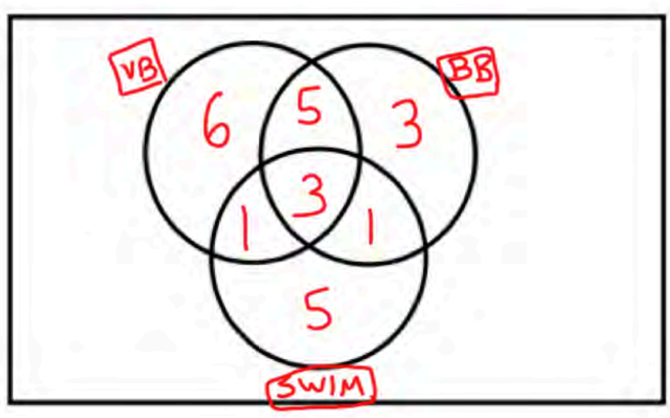
$$= \frac{11}{50}$$

	Blood Type		
	Gene Present	Gene Not Present	Total
High IQ	33	19	52
Normal IQ	39	11	50
Total	72	30	102

Example 2:

Draw a Venn Diagram
(24 students total)

Team	Number of Students
Volleyball	15
Basketball	12
Swimming	10
VB and BB	8
VB and Swim	4
BB and Swim	4
All three	3



There are often restrictions placed on probabilities. Consider the question from yesterdays homework. This time you are asked to determine the probability of selecting a swim team member given they are on the volleyball team.

$$P(swim|volleyball) = \frac{n(swim \cap volleyball)}{n(volleyball)}$$

$$= \frac{4}{15}$$

Example 3: The probability that Crosby gets a goal and an assist in a game is 20%. Crosby gets an assist in 75% of the games he plays. Determine the probability that Crosby gets a goal given that he gets an assist.

$$P(goal|assist) = \frac{P(goal \cap assist)}{P(assist)}$$

$$= \frac{0.2}{0.75}$$

$$= \frac{4}{15}$$

Part 2: Multiplication Law for Conditional Probability

The probability of events *A* and *B* occurring:

$$P(A \cap B) = P(A) \times P(B|A)$$

Note: this formula is used when events are *dependent*

Example 4: What is the probability of drawing two aces in a row from a well-shuffled deck of 52 playing cards? The first card drawn is not replaced.

$P(first\ ace \cap second\ ace) = P(first\ ace) \times P(second\ ace|first\ ace)$

$= \frac{4}{52} \times \frac{3}{51}$

$= \frac{12}{2652}$

$= \frac{1}{221}$

Example 5: Two cards are selected from a standard deck without replacement. Find the probability that they are both hearts.

$P(first\ heart \cap second\ heart) = P(first\ heart) \times P(second\ heart|first\ heart)$

$= \frac{13}{52} \times \frac{12}{51}$

$= \frac{156}{2652}$

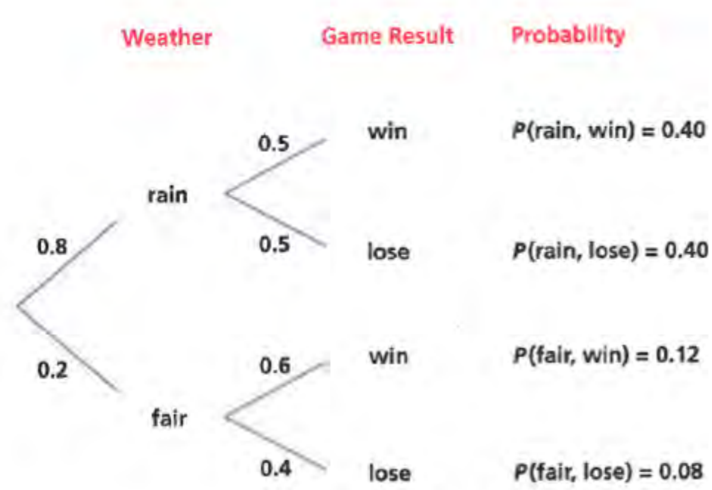
$= \frac{1}{17}$

Part 3: Tree Diagrams and Conditional Probability

Tree diagrams can be a helpful way of organizing outcomes in order to identify probabilities.

Example 6: Your soccer teams wins 50% of it's games when it rains and 60% of it's games on clear days. The weather for your next game calls for an 80% chance of rain. What is the probability that you team will win?

Start with a tree diagram:



There are two scenarios that result in a win: (rain, win) OR (fair, win)

P(win) = P(rain, win) + P(fair, win)

$P(win) = (0.8)(0.5) + (0.2)(0.6)$

$P(win) = 0.4 + 0.12$

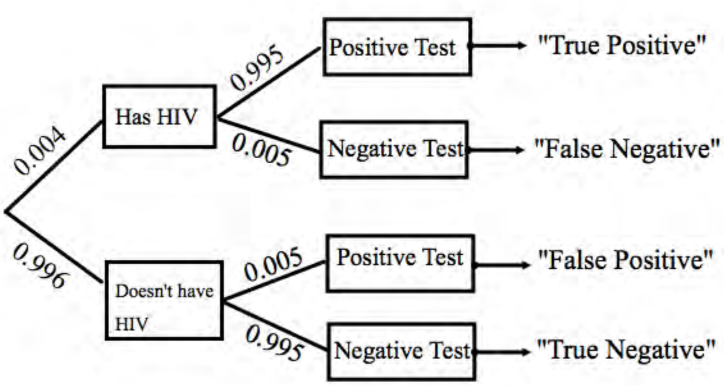
$P(win) = 0.52$

There is a 52% chance your team will win.

Example 7: In the population of the USA, estimates suggest that 1 in every 250 people (0.4%) has been infected with HIV. Tests that are given to detect HIV in people are 99.5% accurate. Chris is randomly selected from the population of the USA for an HIV test and his test is "positive". What is the chance that Chris is infected with HIV?

Start by making a guess: _________

Next make a tree diagram to help visualize the problem:



We want to calculate $P(has\ HIV \mid positive\ test)$. Using the conditional probability formula,

$$P(has\ HIV \mid positive\ test) = \frac{P(has\ HIV \cap positive\ test)}{P(positive\ test)}$$

Numerator:

$P(has\ HIV \ \cap \ positive\ test) = P(has\ HIV) \times P(positive\ test \mid \ has\ HIV)$

$= (0.004)(0.995)$

$= 0.00398$

Denominator:

to find the $P(positive\ test)$ we must consider both scenarios. (1) a would could have HIV and test positive OR (2) a would could not have HIV and test positive.

$P(positive\ test) = (0.004)(0.995) + (0.996)(0.005)$

$= 0.00896$

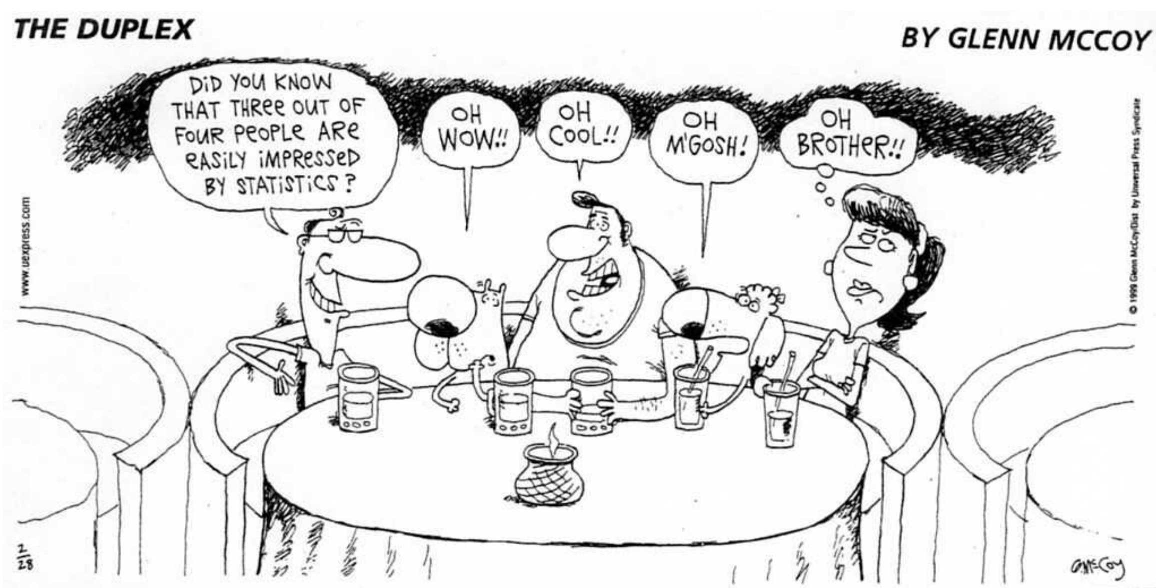
Final Calculation:

$P(has\ HIV \mid positive\ test) = \frac{P(has\ HIV \cap positive\ test)}{P(positive\ test)}$

$= \frac{0.00398}{0.00896}$

$= 0.4442$

Therefore there is only a 44.42% chance that Chris has HIV even though he tested positive.



Section 4.5 – Multiplication of Independent Events

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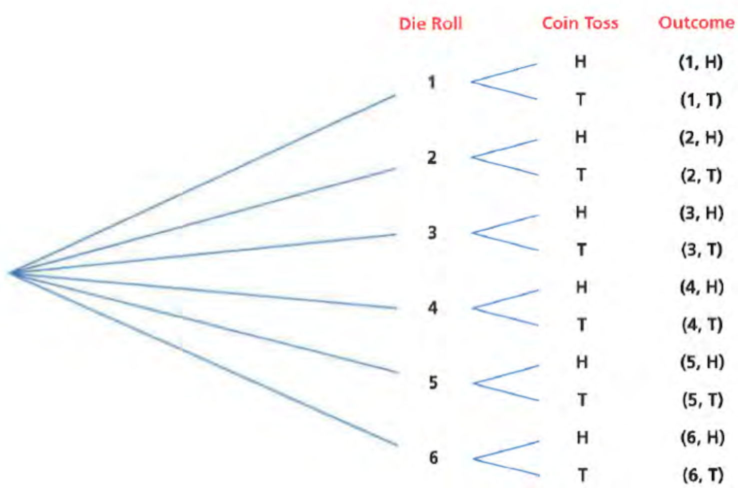
Example 1: Consider an experiment in which you first roll a six-sided die and then flip a coin.

a) Create a tree diagram to show the possible results



b) What is the probability of tossing tails and rolling an even number?

$$P(tails, even) = \frac{n(tails, even)}{n(S)}$$

$$= \frac{3}{12}$$

$$= \frac{1}{4}$$

Remember: $P(A) = \frac{n(A)}{n(S)}$

Part 1: Multiplicative Principle for Counting Ordered Pairs

If the outcome for each experiment (flipping coins and rolling dice) has no influence on the outcome of any other experiment we can use ordered pairs to write the outcomes. Ordered pairs are used for independent events.

The total number of outcomes is the product of the possible outcomes at each step in the sequence.

We can count these ordered pairs where: $n(a, b) = n(a) \times n(b)$

Another way to think about it….

- If one event can occur in *m* ways and a second event can occur in *n* ways, the number of ways the two events can occur in sequence is $m \times n$

Example 2: How many outcomes are possible in the die roll/coin toss experiment?

$n(die\ roll, coin\ toss) = n(die\ roll) \times n(coin\ toss)$

$= 6 \times 2$

$= 12$

Example 3: How many outcomes are possible in an experiment where you draw a random card from a standard deck and then roll a six-sided die.

$n(card, die) = n(card) \times n(die)$

$= 52 \times 6$

$= 312$

Example 4: You are going to the school dance tonight and can't decide what to wear. You've narrowed it down to 4 possible pairs of pants, 6 shirts, and 2 pairs of shoes. How many possible outfits can you make?

$n(outfits) = n(pants) \times n(shirts) \times n(shoes)$

$= 4 \times 6 \times 2$

$= 48$

Part 2: Independent Events

Consider the experiment where you roll a 6 sided die and then flip a coin.

1. What is the probability tossing the coin and getting tails if you know in advance that the die will show an even number?

$P(tails \mid even) = \dfrac{n(tails \cap even)}{n(even)}$

$= \dfrac{3}{6}$

$= \dfrac{1}{2}$

2. What is the probability of getting tails when you flip a coin?

$$P(A) = \frac{n(A)}{n(S)}$$

$= \frac{1}{2}$

3. Make a conclusion about your answers to 1&2

$P(tails \mid even) = P(tails)$ because they are independent events.

The general rule is:

If A and B are independent events, then $P(B| A) = P(B)$

<u>Part 3: Multiplicative Principle for Probabilities of Independent Events</u>

If A and B are independent events, then:

$P(A \cap B) = P(A) \times P(B)$

Example 5: In the die roll/coin toss experiment; use the multiplicative law for probabilities of independent events to determine the probability of rolling an even number and then flipping tails.

$P(even, tails) = P(even) \times P(tails)$

$= \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{4}$

> ***note:*** *sequences of independent events can be written as ordered pairs*
>
> *P(even roll, tails) = P(even roll ∩ tails)*

Example 6: A coin is tossed and a die is rolled. Find the probability of flipping heads and then rolling a 6.

$P(heads \cap 6) = P(heads) \times P(6)$

$= \frac{1}{2} \times \frac{1}{6}$

$= \frac{1}{12}$

Example 7: The probability that Crosby gets a point in a game is 85%

a) Find the probability that he gets a point in three consecutive games

$P(point, point, point) = P(point) \times P(point) \times P(point)$

$= (0.85)(0.85)(0.85)$

$= 61.4\%$

b) Find the probability that he gets no points in three consecutive games

$P(no\ point, no\ point, no\ point) = P(no\ point) \times P(no\ point) \times P(no\ point)$

$= (0.15)(0.15)(0.15)$

$= 0.3\%$

c) Find the probability that he gets a point in at least one of the three games.

The phrase "at least one" means one or more. The complement to the event "at least one" is the event "none."

$P(point\ in\ at\ least\ 1\ game) = 1 - P(point\ in\ no\ games)$

$= 1 - 0.003$

$= 0.997$

$= 99.7\%$

Section 4.6 – Permutations

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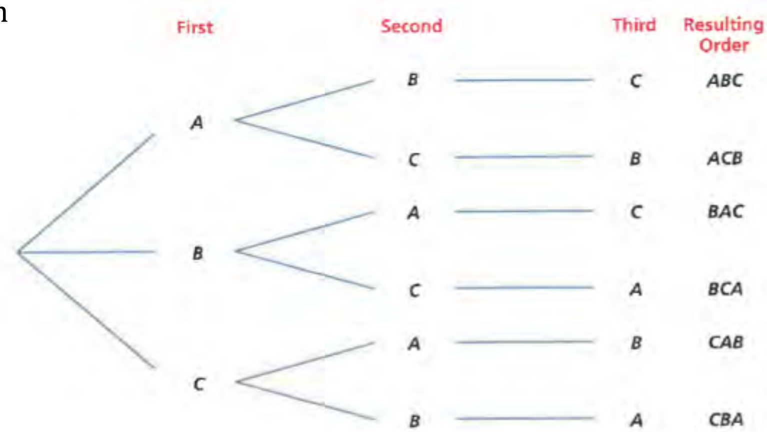
Part 1: Factorial Investigation

You are trying to put three children, represented by A, B, and C, in a line for a game. How many different orders are possible?

a) Use a tree diagram



b) Use the multiplication rule for counting *(find the product of the possible outcomes in each step of the sequence)*

$n(ordered\ arrangements) = n(choices\ for\ 1st) \times n(choices\ for\ 2nd) \times n(choices\ for\ 3rd)$

$= 3 \times 2 \times 1$

$= 6$

Permutations

The ordering problem in the investigation dealt with arranging three children to create sequences with different orders.

Sometimes when we consider *n* items, we need to know the number of different ordered arrangements of the *n* items that are possible.

A permutation is an ordered arrangement of objects. The number of different permutations of *n* distinct objects is $n!$

* Order Matters For Permutations *

Part 2: Factorials

factorial notation ($n!$) represents the number of ordered arrangements of n objects.

$$n! = n \times (n-1) \times (n-2) \times \ldots 1$$

examples:

i) $3! = 3 \times 2 \times 1 = 6$

ii) $\frac{6!}{4!} = \frac{6\times5\times\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}} = 6\times5 = 30$

Example 1: How many different ways can 7 people be seated at a dinner table?

$n(ordered\ arrangements) = 7! = 5040$

7 → MATH → PROB → ! → ENTER

Example 2: A horse race has 8 entries. Assuming that there are no ties, in how many different orders can the horses finish?

$n(ordered\ arrangments) = 8! = 40\ 320$

Example 3: In how many ways can the letters A, B, C, D, E, and F be arranged for a six-letter security code?

$n(codes) = 6! = 720$

Part 3: Distinguishable Permutations

You may want to order a group of n objects in which some of the objects are the same.

The formula for the number of permutations from a set of n objects in which a are alike, b are alike, c are alike, and so on is:

$$\frac{n!}{a!\,b!\,c!}$$

Example 4: Determine the number of arrangements possible using the letters of the word **MATHEMATICS.**

There are 11 letters and there are 2 M's, 2 A's, and 2 T's. Therefore, the number of arrangements is:

$$= \frac{11!}{2!\,2!\,2!}$$

$$= 4\,989\,600$$

Example 5: A building contractor is planning to develop a subdivision. The subdivision is to consist of 6 one story houses, 4 two story houses, and 2 split level houses. In how many distinguishable ways can the houses be arranged?

$$n(ordered\ arrangements) = \frac{12!}{6!\,4!\,2!} = 13\,860$$

Part 4: Permutations of part of a group

We have considered the number of ordered arrangements of *n* objects taken as an entire group; but what if we don't arrange the entire group?……

Counting rule for Permutations

The number of ways to arrange in order *n* distinct objects, taking them *r* at a time is:

$$P(n,r) = \frac{n!}{(n-r)!}$$

Example 6:

$$P(5, 3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\times4\times3\times\cancel{2}\times\cancel{1}}{\cancel{2}\times\cancel{1}} = 5\times4\times3 = 60$$

That means there are 60 ways of ordering objects taken three at a time from a set of five different objects.

Example 7:

Let's compute the number of possible ordered seating arrangements for eight people in five chairs.

i) by using the multiplication rule for counting

	Chair 1	Chair 2	Chair 3	Chair 4	Chair 5
# of choices for the chair	8	7	6	5	4

$n(ordered\ arrangments) = 8{\times}7{\times}6{\times}5{\times}4 = 6\,720$

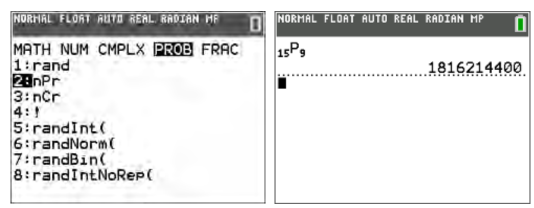
ii) by using the counting rule for permutations

$$P(8,5) = \frac{8!}{(8-5)!} = \frac{8!}{3!} = \frac{8{\times}7{\times}6{\times}5{\times}4{\times}\cancel{3}{\times}\cancel{2}{\times}\cancel{1}}{\cancel{3}{\times}\cancel{2}{\times}\cancel{1}} = 8{\times}7{\times}6{\times}5{\times}4 = 6\,720$$

Example 8:

There are 15 players on the school baseball team. How many ways can the coach complete the nine-person batting order?

$n(batting\ orders) = P(15,9) = 1\,816\,214\,400$



15 → MATH → PROB → nPr → 9 → ENTER

Example 9:

There are 8 teams in the Metropolitan Division in the NHL's Eastern Conference. How many ways can the teams finish first, second, and third?

$n(ordered\ arrangements\ for\ top\ 3) = P(8,3) = 336$

Part 5: Using Permutations to Determine Probability

Recall: theoretical probability is the ratio of the number of outcomes that make up the desired event to the total number of possible outcomes

$$P(A) = \frac{n(A)}{n(S)}$$

Example 10:

Four people are required to help out at a party: one to prepare the food, one to serve it, one to clear the tables, and one to wash up. Determine the probability that you and your three siblings will be chosen for these jobs if four people are randomly selected from a room of 12 people.

$$P(\text{you and siblings selected}) = \frac{n(\text{you and your siblings can be chosen for the four jobs}}{n(12 \text{ people can be chosen for the four jobs}}$$

$$P(\text{you and siblings selected}) = \frac{P(4,4) \text{ or } 4!}{P(12,4)} = \frac{24}{11\ 880} = \frac{1}{495}$$

Example 11:

A combination lock opens when the right combination of three numbers from 0 to 59 are entered in the correct order. The same number can't be used more than once.

a) What is the probability of getting the correct combination by chance?

$$P(\textbf{correct combination}) = \frac{n(\textbf{correct combinations})}{n(\textbf{possible combinations})}$$

$$P(\text{correct combination}) = \frac{1}{P(60,3)} = \frac{1}{205\ 320}$$

b) What is the probability of getting the right combination if you already know the first digit?

$$P(\text{correct combination}) = \frac{1}{P(59,2)} = \frac{1}{3\ 422}$$

In the situations examined so far, objects were selected from a set and then, once selected, were removed from the collection so that they could not be chosen again. If the object is replaced, lets examine how this affects the possible number of arrangements...

Example 12:

a) How many ways are there to draw two cards from a standard deck of 52 cards if the card **<u>is not</u>** replaced after drawing it. *(the order you draw them in matters)*

$$n(draw\ 2\ cards\ without\ replacement) = P(52, 2) = \frac{52!}{(52-2)!} = \frac{52!}{50!} = 52\times 51 = 2652$$

b) How many ways are there to draw two cards from a standard deck of 52 cards if the card **<u>is</u>** replaced after drawing it. *(the order you draw them in matters)*

$$n(draw\ 2\ cards\ with\ replacement) = n(choices\ for\ 1st)\times n(choices\ for\ 2nd) = 52\times 52 = 2\ 704$$

Note: you can't use the counting rule for permutations, you must use the multiplication rule for counting.

Example 13:

The access code for a car's security system consists of four digits. Each digit can be 0 through 9. How many access codes are possible if:

a) each digit can be used only once and not repeated?

$$n(codes\ no\ repeats) = P(10, 4) = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 10\times 9\times 8\times 7 = 5\ 040$$

b) each digit can be repeated?

$$n(codes\ with\ repeats) = 10\times 10\times 10\times 10 = 10\ 000$$

***Note:** because each digit can be repeated, there are 10 choices for each of the four digits.*

Section 4.7 – Combinations

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Jensen

Part 1: Handshake Problem

Problem: There are 26 students in this class. If every student shook every other students hand once, how many different handshakes would take place?

Guess: _____

Ideas on how to solve the problem:

Each of the 26 students will have 25 handshakes (one with each other student in the class). Therefore 26 x 25 seems to make sense....

However, this would count the handshake between each pair twice, and in this instance the order does NOT matter. Therefore we must divide our answer by 2 (because there are 2! arrangements within each pair that include the same people)

Solution: $n(handshakes) = C(26, 2) = \frac{26!}{(26-2)!2!} = \frac{26!}{24!2!} = \frac{26 \times 25}{2} = 325$

Part 2: Combinations

PERMUTATIONS are ordered sets where: ABC/BCA/CAB etc. are all different possibilities, because with permutations ORDER MATTERS.

HOWEVER, for many counting problems, order is not important!

COMBINATIONS are unordered sets where: ABC/BCA/CAB etc. are considered the same, because with combinations order DOES NOT matter.

Permutations	Combinations
When we consider permutations, we are considering groupings *and* order. $P(n, r) = \frac{n!}{(n-r)!}$	When we consider combinations, we are considering only the number of different groupings. Order within the groupings is not considered. $C(n, r) = \frac{n!}{(n-r)!\,r!}$

Example 1:

You won four free tickets for the 2016 Stanley Cup Final (Pittsburgh vs. Edmonton….hopefully). Determine how many ways you can select 3 of your 10 best friends to come to the game with you.

before you start you must ask yourself, does the order you pick your friends really matter? **If yes --> permutation** **If no --> combination**

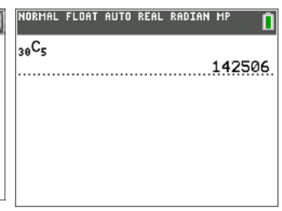
$$n(groups\ of\ 3\ friends) = C(10,3) = \frac{10!}{(10-3)!\,3!} = \frac{10!}{7!\,3!} = 120$$

Example 3:

From a class of 30 students, determine how many ways a five-person committee can be selected to organize a class party:

a) With no restrictions

$$n(committees) = C(30,5) = 142\ 506$$



30 → MATH → PROB → nCr → 5 → ENTER

b) With Marnie on the committee

$$n(committees) = C(29,4) = 23\ 751$$

Example 4:

How many 5-card hands can be dealt out of a deck of 52 cards?

$$n(hands) = \binom{52}{5} = 2\ 598\ 960$$

Note: $\binom{52}{5}$ *means the same as* $C(52,5)$

Example 5:

A coach of a co-ed basketball team must select five players to start the game from a team that consists of six females and five males. How many ways can this be achieved if Tanya must choose three females and two males to start the game. (assume order does not matter)

> Remember the **Multiplication Rule for Counting**:
>
> To find the number of outcomes for a series of events, find the product of the possible outcomes at each step in the sequence.
>
> n(a,b) = n(A) x n(B)

$n(3\ females\ \cap 2\ males) = n(3\ females) \times n(2\ males)$

$= \binom{6}{3} \times \binom{5}{2}$

$= 20 \times 10$

$= 200$

Example 6:

In how many ways can 6 people be selected from a group that consists of four adults and eight children if the group must contain at least two adults?

Solution 1: Direct Reasoning

In this situation, the condition that the group must have at least two adults must be satisfied. This can happen three ways:

2 adults and 4 children **or** 3 adults and 3 children **or** 4 adults and 2 children

> Remember the **Additive Principle for the Union of Sets**:
>
> $n(a \cup b) = n(A) + n(B)$
>
> (for disjoint sets)

$n(atleast\ 2\ adults) = n(2A, 4C) + n(3A, 3C) + n(4A, 2C)$

$= \binom{4}{2}\binom{8}{4} + \binom{4}{3}\binom{8}{3} + \binom{4}{4}\binom{8}{2}$

$= 6(70) + 4(56) + 1(28)$

$= 420 + 224 + 28$

$= 672$

Solution 2: Indirect Reasoning

In this situation, the solution can also be found by subtracting the number of ways the condition is not satisfied (0 or 1 adult in the group) from the total number of combinations.

$n(atleast\ 2\ adults) = n(groups) - [n(0A, 6C) + n(1A, 5C)]$

$= \binom{12}{6} - \left[\binom{4}{0}\binom{8}{6} + \binom{4}{1}\binom{8}{5}\right]$

$= 924 - [1(28) + 4(56)]$

$= 924 - 252$

$= 672$

Part 3: Using Combinations to Find Probabilities

Example 7:

Five cards are dealt at random from a deck of 52 playing cards. Determine the probability that you will have:

a) the 10-J-Q-K-A of the same suit (a royal flush)?

$$\boldsymbol{P(royal\ flush) = \frac{n(royal\ flush)}{n(hands)} = \frac{4}{C(52,5)} = \frac{4}{2\ 598\ 960} = \frac{1}{649\ 740}}$$

b) four of a kind?

$$P(4\ of\ a\ kind) = \frac{n(4\ of\ a\ kind\ hands)}{n(hands)} = \frac{13\binom{48}{1}}{\binom{52}{5}} = \frac{624}{2\ 598\ 960} = \frac{1}{4165}$$

Note: There are 13 cards in each suit and the hand must contain four of the same card. The remaining card in the hand can be any card of the remaining 48.

Example 8:

A company that has 40 employees chooses a committee of 7 to represent employee retirement issues. When the committee was formed, none of the 18 minority employees were selected. Do you think the committee selection was biased? Give mathematical evidence for your decision.

Possible solution:

The number of ways 7 employees can be chosen from 40:

$$n(committees) = \binom{40}{7} = 18\ 643\ 560$$

The number of ways 7 employees can be chosen from 22 non-minorities:

$$n(committees\ with\ no\ minorities) = \binom{22}{7} = 170\ 544$$

The probability that it contained no minorities if it was chosen randomly is:

$$P(no\ minorities) = \frac{n(committees\ with\ no\ minorities)}{n(committees)} = \frac{170\ 544}{18\ 643\ 560} = 0.00915$$

Since there is only about a 0.9% chance the committee would contain no minorities if it was chosen at random, it seems reasonable to conclude that the committee selection was biased.