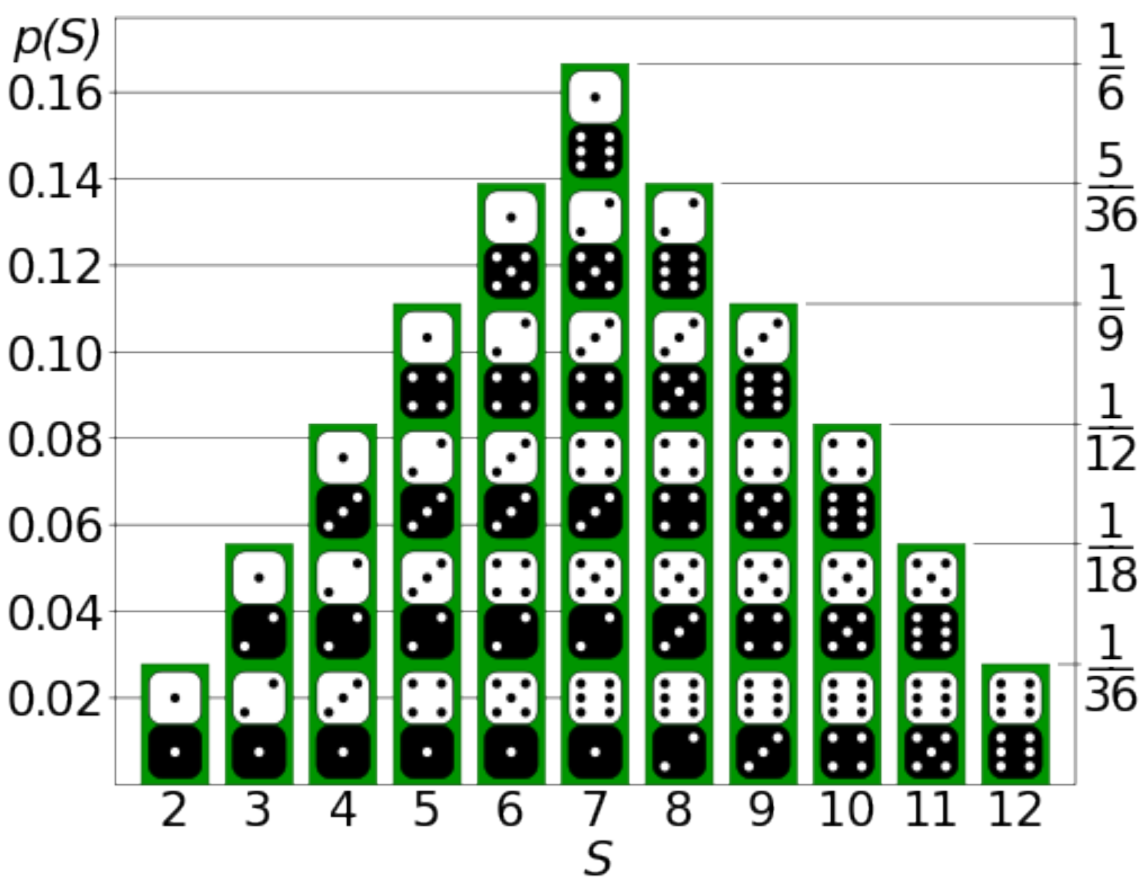
Chapter 5

Probability Distributions

MDM4U



Unit Outline

Section	Subject	Homework Notes	Lesson and Homework Complete (initial)
5.1	Probability Distributions		
5.2	Binomial Probability Distributions		
5.3	Geometric Probability Distributions		
5.4	Binomial Theorem		
5.5	Normal Approximation to the Binomial Distribution		

Unit Performance

Homework Completion: None Some Most All

Days absent:______

Test Review Complete? None Some All

Assignment Mark (%):______

Test Mark (%):______

Notes to yourself to help with exam preparation:

5.1 Probability Distributions

MDM4U
Jensen

Part 1: Introducing Probability Distribution

What is a *probability distribution*?

A table, formula or graph that provides the probabilities of a discrete random variable assuming any of all of its possible values.

Discrete Random Variable: A variable that has a unique value for each outcome.

A ***probability distribution*** must satisfy the following criteria:

1. The probability of each value of the discrete random variable is between 0 and 1, inclusive.

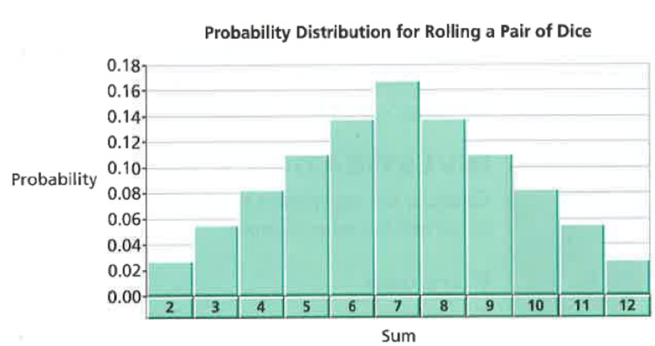
2. The sums of all the probabilities is 1.

Consider the experiment in which two six-sided dice are rolled. Suppose one die is red and the other is blue. Create a **table** to represent the probability distribution for the possible sums of the dice:

Sum	Theoretical Probability
2	$\frac{1}{36}$
3	$\frac{2}{36}$
4	$\frac{3}{36}$
5	$\frac{4}{36}$
6	$\frac{5}{36}$
7	$\frac{6}{36}$
8	$\frac{5}{36}$
9	$\frac{4}{36}$
10	$\frac{3}{36}$
11	$\frac{2}{36}$
12	$\frac{1}{36}$

Blue Die \ Red Die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

The **graph** below is another way to represent the probability distribution. This graph also provides the probability of each sum occurring when a pair of dice is rolled.



Part 2: Expected Value

The mean of a probability distribution is often called the expected value of the distribution. The mean is an average value and need not be a point of the sample space.

The formula used is:

$$E(x) = \mu = \sum x \cdot P(x)$$

The expected value of the distribution of dice sums is the result of adding each possible sum value multiplied by its probability.

$$E(sum) = 2 \cdot P(2) + 3 \cdot P(3) + \ \ldots + 12 \cdot P(12)$$

$$E(sum) = 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right) + 8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right)$$

$$= 7$$

Part 3: Making Probability Distributions

Example 1: Consider a simple game in which you roll a single die. If you roll an even number, you gain that number of points, and, if you roll an odd number, you lose that number of points. Show the probability distribution of points in this game and calculate the expected number of points per roll.

Number Rolled	Points, x	Probability, $P(x)$
1	-1	$\frac{1}{6}$
2	2	$\frac{1}{6}$
3	-3	$\frac{1}{6}$
4	4	$\frac{1}{6}$
5	-5	$\frac{1}{6}$
6	6	$\frac{1}{6}$

$$E(points) = -1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) - 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) - 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right)$$

$$= \frac{3}{6}$$

$$= 0.5$$

You would expect that the score in this game would average out to 0.5 points per roll.

Example 2: A summer camp has seven 4.6 meter canoes, ten 5.0 meter canoes, four 5,2 meter canoes, and four 6.1 meter canoes. Canoes are assigned randomly for campers going on a canoe trip. Show the probability distribution for the length of an assigned canoe. Then calculated the expected length of an assigned canoe.

Length of Canoe (m), x	**Probability, $P(x)$**
4.6	$\frac{7}{25}$
5.0	$\frac{10}{25}$
5.2	$\frac{4}{25}$
6.1	$\frac{4}{25}$

$$E(length) = 4.6\left(\frac{7}{25}\right) + 5.0\left(\frac{10}{25}\right) + 5.2\left(\frac{4}{25}\right) + 6.1\left(\frac{4}{25}\right)$$

$$= 5.1\ m$$

The expected length of the canoe is 5.1 meters.

Example 3: A school raffle sold 1500 tickets at $2 each. There are four prizes of $500, $250, $150, and $75.

a) Create a probability distribution for the amount of money you could win.

Winnings, x	**Probability, $P(x)$**
500	$\frac{1}{1500}$
250	$\frac{1}{1500}$
150	$\frac{1}{1500}$
75	$\frac{1}{1500}$
0	$\frac{1496}{1500}$

b) Calculate your expected gain if you buy a ticket

Expected gain = $E(winnings) - cost\ of\ ticket$

$$= \left[500\left(\frac{1}{1500}\right) + 250\left(\frac{1}{1500}\right) + 150\left(\frac{1}{1500}\right) + 76\left(\frac{1}{1500}\right) + 0\left(\frac{1496}{1500}\right)\right] - 2$$

$$= 0.65 - 2$$

$$= -1.35$$

You can expect to lost $1.35 if you buy a ticket

5.2 Hypergeometric Probability Distributions

MDM4U
Jensen

Part 1: What is a Hypergeometric Probability Distribution

When a coach chooses a starting line-up for a game, a coach obviously has to choose a different player for each position. Similarly, when you deal cards from a standard deck, there can be no repetitions. In such situations, each selection reduces the number of items that could be selected in the next trial. Thus, the probabilities in these trials are dependent.

A hypergeometric distribution has a specified number of dependent trials having two possible outcomes, success or failure. The random variable is the number of successful outcomes in the specified number of trials. The individual outcomes cannot be repeated within these trials.

Part 2: Completing a Hypergeometric Probability Distribution

Example 1: Determine the probability distribution for the number of women on a three person committee selected from a pool of 8 men and 10 women.

The selection process involves dependent events since each person who is already chosen for the jury cannot be selected again. The total number of ways the 3 people can be selected from the pool of 18 is:

$$n(S) = \binom{18}{3} = 816$$

There can be from 0 to 3 women on the committee. The men will fill the remaining positions. The probability distribution is as follows:

Number of Women, x	**Probability, $P(x)$**
0	$\frac{\binom{10}{0}\binom{8}{3}}{\binom{18}{3}} = 0.0686$
1	$\frac{\binom{10}{1}\binom{8}{2}}{\binom{18}{3}} = 0.3431$
2	$\frac{\binom{10}{2}\binom{8}{1}}{\binom{18}{3}} = 0.4412$
3	$\frac{\binom{10}{3}\binom{8}{0}}{\binom{18}{3}} = 0.1471$

The expected number of women is: $E(X) = \sum x \cdot P(x) = 0 \cdot P(0) + 1 \cdot P(1) + 2 \cdot P(2) + 3 \cdot P(3)$

$$= 0(0.0686) + 1(0.3431) + 2(0.4412) + 3(0.1471)$$
$$= 1.67$$

Part 3: Hypergeometric Probability Formula

To find the probability of x successes in r dependent trials, use the formula:

$$P(x) = \frac{\binom{a}{x}\binom{b}{r-x}}{\binom{n}{r}}$$

where a is the number of successful outcomes among a total of n possible outcomes, and b is the number of failures among all possible outcomes.

Looking back to example 1, we could have used the formula:

$P(x) = \frac{\binom{10}{x}\binom{8}{3-x}}{\binom{18}{3}}$ for each probability calculation.

$a = 10$ $b = 8$ $n = 18$ $r = 3$

Example 2: Suppose you are dealt a four-card hand from a standard deck of cards.

a) Create a table that shows the theoretical probability distribution of how many spades are in your hand.

There can be from 0 to 4 spades in your hand. The other suits can fill the remaining spots. We can use the following formula to calculate the probability of a three card hand with x spades in it:

$$P(x) = \frac{\binom{a}{x}\binom{n-a}{r-x}}{\binom{n}{r}} = \frac{\binom{13}{x}\binom{39}{4-x}}{\binom{52}{4}}$$

$a = 13$ $b = 39$ $n = 52$ $r = 4$

Number of Spades, x	Probability, $P(x)$
0	$\frac{\binom{13}{0}\binom{39}{4}}{\binom{52}{4}} = 0.3038$
1	$\frac{\binom{13}{1}\binom{39}{3}}{\binom{52}{4}} = 0.4388$
2	$\frac{\binom{13}{2}\binom{39}{2}}{\binom{52}{4}} = 0.2135$
3	$\frac{\binom{13}{3}\binom{39}{1}}{\binom{52}{4}} = 0.0412$
4	$\frac{\binom{13}{4}\binom{39}{0}}{\binom{52}{4}} = 0.0026$

b) What is the expected number of spades in a hand?

$$E(X) = \sum x \cdot P(x) = 0(0.3038) + 1(0.4388) + 2(0.2135) + 3(0.0412) + 4(0.0026) = 1$$

Example 3: A box contains eight yellow, four green, five purple, and three red candies jumbled together. You randomly pour five candies in to your hand. Create a table that shows the theoretical probability distribution of how many red candies are in your hand.

There can be from 0 to 3 red candies in your hand. The other candies will be one of the non-red colours. We can use the following formula to calculate the probability of pouring x red candies in to your hand:

$$P(x) = \frac{\binom{a}{x}\binom{b}{r-x}}{\binom{n}{r}} = \frac{\binom{3}{x}\binom{17}{5-x}}{\binom{20}{5}}$$

$a = 3$
$b = 17$
$n = 20$
$r = 5$

Number of Red Candies, x	**Probability, $P(x)$**
0	$\frac{\binom{3}{0}\binom{17}{5}}{\binom{20}{5}} = 0.0686$
1	$\frac{\binom{3}{1}\binom{17}{4}}{\binom{20}{5}} = 0.3431$
2	$\frac{\binom{3}{2}\binom{17}{3}}{\binom{20}{5}} = 0.4412$
3	$\frac{\binom{3}{3}\binom{17}{2}}{\binom{20}{5}} = 0.1471$

5.3 Binomial Probability Distributions

MDM4U
Jensen

Part 1: Properties of a Binomial Experiment

In a Binomial Experiment, the number of successes in n trials is a discrete random variable - X. X is termed a **binomial random variable** and its probability distribution is called a **binomial distribution.**

Properties of Binomial Experiments (Bernoulli Trials)

1. There are n identical trials

2. There are only two possible outcomes. Success or failure.

3. The probability of success is the same in every trial (trials are independent of one another)

Part 2: Investigating Binomial Experiments

Consider an experiment where you roll a single die 4 times.

a) What is the probability that your first two rolls are 6's and your next two rolls will be something other than a 6?

P[(roll 1 = 6) and (roll 2 = 6) and (roll 3 ≠ 6) and (roll 4 ≠ 6)]

$= \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}$

$= \frac{25}{1296}$

Note: *This is a Bernoulli Trial because there are only two possible outcomes - success is a roll of 6, failure is a roll that is not 6. The probability of success is the same for every roll. The trials are independent of each other.*

b) Find the probability of the event that the roll of 6 will appear exactly twice in any of the four available positions in the table.

Note: *The number of ways 6 can be placed in two of the four entries is the same as counting the number of ways two objects can be selected from four available objects. C(4, 2)*

$$P(two\ 6's) = \frac{25}{1296} \times C(4,2) = \frac{25}{1296} \times 6 = \frac{150}{1296} = \frac{25}{216}$$

c) Complete a theoretical probability distribution for the number of 6's showing in four rolls

Probability of zero 6's

$$= \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$$

Probability of one 6

$$= \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \binom{4}{1} = \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3 \binom{4}{1} = \frac{500}{1296} = \frac{125}{324}$$

Probability of two 6's

$$= \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \binom{4}{2} = \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 \binom{4}{2} = \frac{150}{1296} = \frac{25}{216}$$

Probability of three 6's

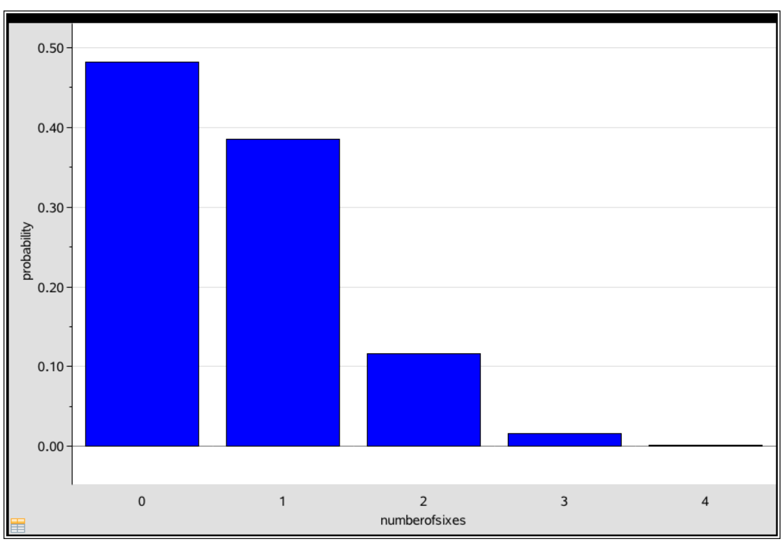
$$= \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \binom{4}{3} = \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^1 \binom{4}{3} = \frac{20}{1296} = \frac{5}{324}$$

Probability of four 6's

$$= \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \left(\frac{1}{6}\right)^4 = \frac{1}{1296}$$

Use a chart to display the probability distribution…

# of 6's, x	$P(x)$
0	$\binom{4}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^4 = \frac{625}{1296}$
1	$\binom{4}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^3 = \frac{500}{1296}$
2	$\binom{4}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 = \frac{150}{1296}$
3	$\binom{4}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^1 = \frac{20}{1296}$
4	$\binom{4}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^0 = \frac{1}{1296}$



Part 2: Binomial Probability Formula

In a binomial experiment with n Bernoulli trials, each with a probability of success p, the probability of k successes in the n trials is given by:

$$P(X = k) = \binom{n}{k}(p)^k(1-p)^{n-k}$$

Where X is the discrete random variable corresponding to the number of successes.

Example 1: Each child of a particular set of parents has probability 0.25 of having type O blood. Suppose the parents have 5 children.

a) Find the probability that exactly 3 of the children have type O blood

$n = 5$

$p = 0.25$

$k = 3$

$$P(X = k) = \binom{n}{k}(p)^k(1-p)^{n-k}$$

$$P(X = 3) = \binom{5}{3}(0.25)^3(0.75)^2$$

$$P(X = 3) = 0.08789$$

There is an 8.79% chance that 3 of their children will have type O blood.

b) Should the parents be surprised if fewer than 2 of the children have type O blood?

$$P(X < 2) = P(X = 0) + P(X = 1)$$

$$P(X < 2) = \binom{5}{0}(0.25)^0(0.75)^5 + \binom{5}{1}(0.25)^1(0.75)^4$$

$$P(X < 2) = 0.6328125$$

There is a 63.28% chance that fewer than 2 of their children have type O blood so they should not be surprised.

Example 2: During the 2010 NHL season when Crosby led the NHL in Goals and Points, he had a 17% shooting percentage. Determine the probability that in a game where he takes four shots, he gets three goals.

$n = 4$

$p = 0.17$

$k = 3$

$$P(X = 3) = \binom{4}{3}(0.17)^3(0.83)^1 = 0.0163$$

There is approximately a 1.63% chance of Crosby getting a hat-trick when he takes 4 shots.

Example 3: A basketball player has 3 shots in a free-throw competition. Historically, the player has a probability of 65% of scoring on a free throw. Assuming the probability is constant...

a) Determine the probability distribution for the number of free throws made

# of Free Throws Made, x	Probability, $P(x)$
0	$\binom{3}{0}(0.65)^0(0.35)^3 = 0.042875$
1	$\binom{3}{1}(0.65)^1(0.35)^2 = 0.238875$
2	$\binom{3}{2}(0.65)^2(0.35)^1 = 0.443625$
3	$\binom{3}{3}(0.65)^3(0.35)^0 = 0.274625$

b) Calculate the players expected number of free throws made in the competition.

Note: to calculate the expected value for a binomial probability distribution, you can use either of the following two formulas:

i) $E(X) = \sum x \cdot P(x)$ (this formula works for ALL types of probability distributions)

$= 0(0.042875) + 1(0.238875) + 2(0.443625) + 3(0.274625)$

$= 1.95$

ii) $E(X) = np$ (this formula only works for binomial probability distributions)

$= 3(0.65)$

$= 1.95$

Example 4: A candy company makes candy-coated chocolates, 40% of which are red. The production line mixes the candies randomly and packages ten per box.

a) What is the probability that 3 of the candies in a given box are red?

$P(X = 3) = \binom{10}{3}(0.4)^3(0.6)^7 = 0.21499$

There is about a 21.50% chance that exactly three of the candies are red.

b) What is the probability that a given box has at least 2 red candies?

$P(X \geq 2) = 1 - [P(0) + P(1)]$

$P(X \geq 2) = 1 - \left[\binom{10}{0}(0.4)^0(0.6)^{10} + \binom{10}{1}(0.4)^1(0.6)^9\right]$

$P(X \geq 2) = 1 - 0.0463574016$

$P(X \geq 2) = 0.9536$

There is about a 95.36% chance that a given box has at least 2 red candies.

Part 3: Using the Ti-84

binompdf(n, p, k) computes $P(X = k)$

binomcdf(n, p, k) computes $P(X \leq k)$

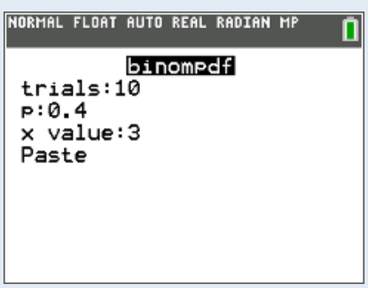
Note: the binomcdf command only computes the probability of getting k or FEWER successes

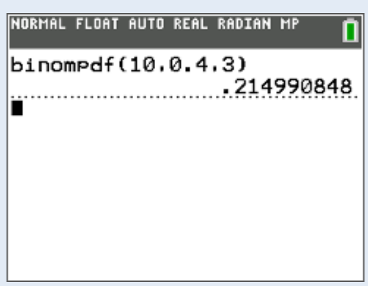
Example 4: A candy company makes candy-coated chocolates, 40% of which are red. The production line mixes the candies randomly and packages ten per box.

a) What is the probability that 3 of the candies in a given box are red?

How to use the calculator:

- 2nd → VARS (DISTR) → binompdf(→ trials: 10 → p: 0.4 → x value: 3 → PASTE





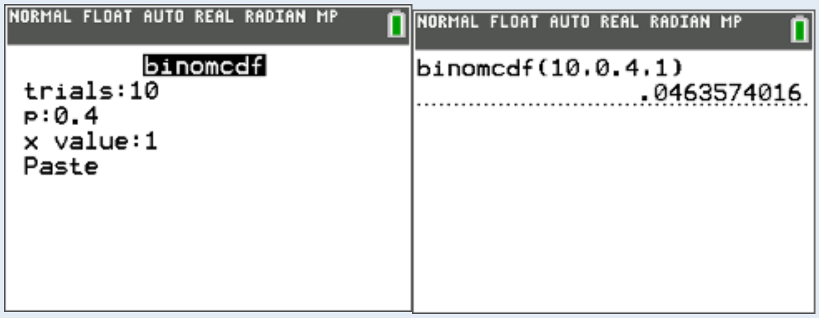
What you need to write:

$P(X = 3)$ = binompdf(n=10, p=0.4, k=3) = 0.21499

There is about a 21.50% chance that exactly three of the candies are red.

b) What is the probability that a given box has at least 2 red candies?

- 2^{nd} → VARS (DISTR) → binomcdf(→ trials: 10 → p: 0.4 → x value: 1 → PASTE



What you need to write:

$P(X \geq 2) = 1 - P(X \leq 1)$

$=$ 1 $-$ binomcdf(n=10, p=0.4, k=1) = 0.0463574016

$= 0.9536$

There is about a 95.36% chance that a given box has at least 2 red candies.

5.4 Geometric Probability Distributions
MDM4U
Jensen

Part 1: What is a Geometric Probability?

In some board games, you cannot move forward until you roll a specific number, which could take several tries. Manufacturers of products such as switches, relays, and hard drives need to know how many operations their products can perform before failing. In some sports competitions, the winner is the player who scores the most points before missing a shot. In each of these situations, the critical quantity is the waiting time, which is the number of trials it takes for a specific outcome to occur.

Like the binomial distribution, trials in a geometric distribution have only two possible outcomes, success or failure, whose probabilities do not change from one trial to the next. However, the random variable for a geometric distribution is the waiting time (not the number of successes) and this causes significant differences between the binomial and geometric distributions.

The waiting time is the number of trials you have to wait before the event of interest (success) happens. The number of trials isn't fixed – you simply count the number of trials until you get the first success.

Part 2: Getting out of Jail Activity

In the board game Monopoly, one way to get out of jail is to roll doubles. Suppose that this was the only a player could get out of jail. With a pair of dice, keep rolling them until you get out of jail (roll doubles).

a) Record the number of trials it takes you to get out of jail

Waiting time: ________

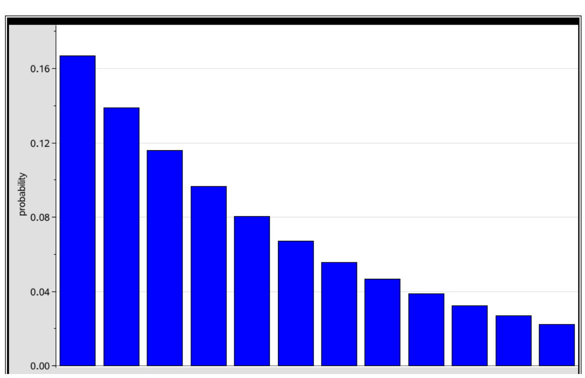
b) Record the following class statistics:

Mean waiting time: ______

Sketch of graph of waiting times:

Should be close to 6

Should have same shape as corresponding probability distribution



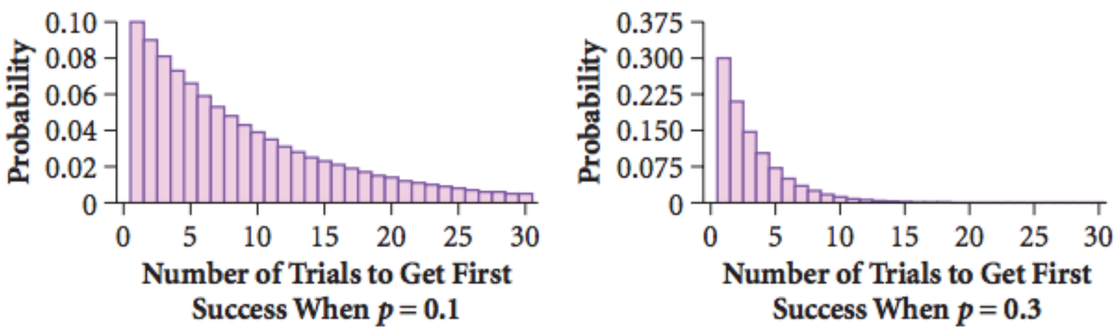
c) What do you think the probability of getting out of jail on your second attempt would be?

$$P(Y = 2) = P(fail\ on\ first\ try) \cdot P(succeed\ on\ second\ try) = \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) = \frac{5}{36} = 0.139$$

d) How many times would you expect to have to roll the dice before getting out of jail?

$$E(Y) = \frac{1}{p} = \frac{1}{\left(\frac{1}{6}\right)} = 6$$

The probability distribution we constructed in this activity is called a geometric distribution. Below are examples of geometric probability distributions for different p values.



<u>Part 3: Geometric Probability Formula</u>

A geometric setting arises when we perform independent trials of the same chance process and record the number of trials it takes to get one success Y. On each trial, the probability p of success must be the same. The possible values of Y are 1, 2, 3, ….. If k is any of these values,

$$P(Y = k) = (1 - p)^{k-1}p$$

k - number of trials until a success occurs (waiting time)

p - probability of success

<u>Part 4: Expected Value of a Geometric Random Variable Formula</u>

If Y is a geometric random variable with probability of success p on each trial, then its mean (expected value) is:

$$E(Y) = \frac{1}{p}$$

Part 5: Using the Geometric Probability Formula

Example 1: As a special promotion for its 20-ounce bottles of soda, a soft drink company printed a message on the inside of each cap. Some of the caps said, "Please try again," while others said, "You're a winner!" The company advertised the promotion with the slogan "1 in 6 wins a prize." Kramer decides to keep buying 20-ounce bottles of soda until he gets a winner.

a) Find the probability that he will have to buy exactly 5 bottles before getting a winner.

$P(Y = 5) = \left(\frac{5}{6}\right)^4 \left(\frac{1}{6}\right) = 0.0804$ There is about a 8.04% chance he will have to buy 5 bottles to get a winner.

b) What is the probability that he will have to buy 3 or fewer bottles to get a winner.

$P(Y \leq 3) = P(1) + P(2) + P(3)$

$$= \left(\frac{5}{6}\right)^0 \left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^1 \left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right)$$

$$= 0.4213$$

There is about a 42.13% chance that we will have to buy 3 or fewer bottles to get a winner.

c) Find the expected number of bottles he will have to buy before getting a winner.

$$E(Y) = \frac{1}{p} = \frac{1}{\left(\frac{1}{6}\right)} = 6$$

He should expect to have to buy 6 bottles before getting a winner.

Part 6: Using the Ti-84 For Geometric Distributions

geometpdf(p, k) computes $P(Y = k)$
geometcdf(p, k) computes $P(Y \leq k)$

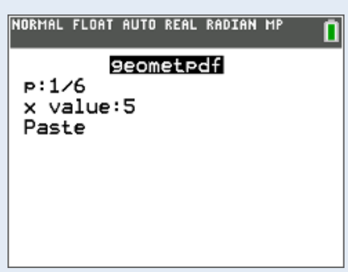
Example 1:

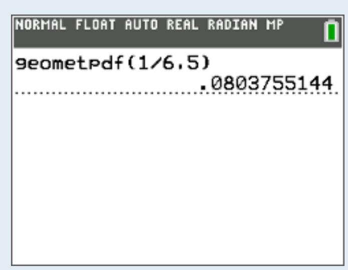
a) Find the probability that he will have to buy exactly 5 bottles before getting a winner.

- 2^{nd} → VARS (DISTR) → geometpdf → p: 1/6 → x value: 5 → Paste

$$P(Y = 5) = geometpdf\left(p = \frac{1}{6}, x = 5\right)$$

$$= 0.0804$$



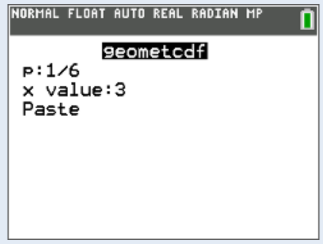


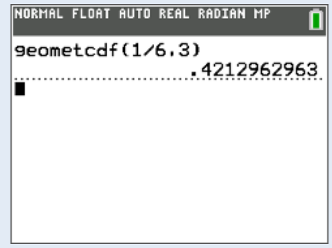
b) What is the probability that he will have to buy 3 or fewer bottles to get a winner.

- 2^{nd} → VARS (DISTR) → geometcdf → p: 1/6 → x value: 3 → Paste

$$P(Y \le 3) = geometcdf\left(p = \frac{1}{6}, x = 3\right)$$

$$= 0.4213$$





Example 2: Your teacher is planning to give you 10 problems for homework. As an alternative, you can agree to play the Lucky Day Game. Here's how it works. A student will be selected at random from your class and asked to pick a day of the week. Then your teacher will use technology to randomly choose a day of the week as the 'lucky day'. If the student picks the correct day, the class will have only one homework problem. If the student picks the wrong day, your teacher will select another student form the class at random and the process is repeated. If the second student gets it right, the class will have two homework problems. If he/she gets it wrong, the process is repeated until a student guesses the right day. Your teacher will assign a number of homework problems that is equal to the total number of guesses made by members of your class.

Do you think the class should play the lucky day game or just accept the 10 homework problems?

In this problem, the random variable of interest is Y = the number of picks it takes to correctly match the lucky day. On each trial, the probability of a correct pick p is 1/7.

a) Find the probability that the class receives exactly 10 homework problems as a result of playing the game.

$$P(Y = 10) = geometpdf\left(p = \frac{1}{7}, x = 10\right) = 0.0357$$

There is about a 3.57% chance the class will get exactly 10 homework problems.

b) Find the probability that the class will get fewer than 10 homework problems.

$$P(Y < 10) = P(Y \le 9) = geometcdf\left(p = \frac{1}{7}, x = 9\right) = 0.7503$$

There is about a 75.03% chance that the class will get less homework by playing the game.

c) How many homework problems should they expect if they play the game?

$E(Y) = \frac{1}{p} = \frac{1}{\left(\frac{1}{7}\right)} = 7$ They should expect to get 7 problems if they play the game.

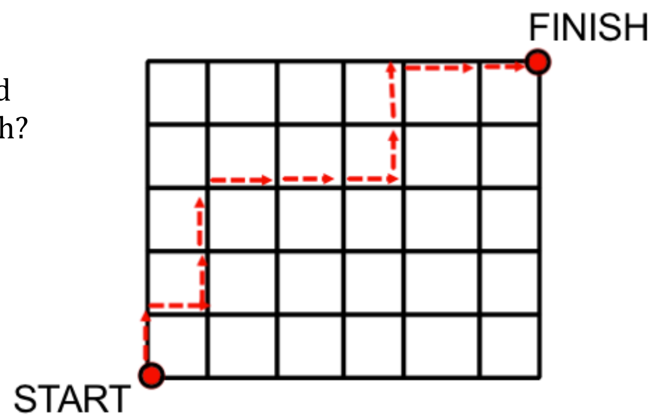
5.5 Binomial Theorem

MDM4U
Jensen

Part 1: Routes Through a Map Grid

Example 1:

City streets make up a grid, as shown. Travelling east and north, how many possible routes lead from start to finish?



One possible solutions is:

NENNEEENNEE

You must move 6 streets east and 5 streets north to get from start to finish. Therefore, every possible route requires 11 steps. Of these steps, 6 will move you east and 5 will move you north.

There are two possible solutions to determine the number of possible routes:

1) As a combination

1a) Determine the number of ways East can be inserted into 6 positions selected from 11 available positions.

$\binom{11}{6} = 462$

1b) Determine the number of ways North can inserted into 5 positions selected from 11 available positions.

$\binom{11}{5} = 462$

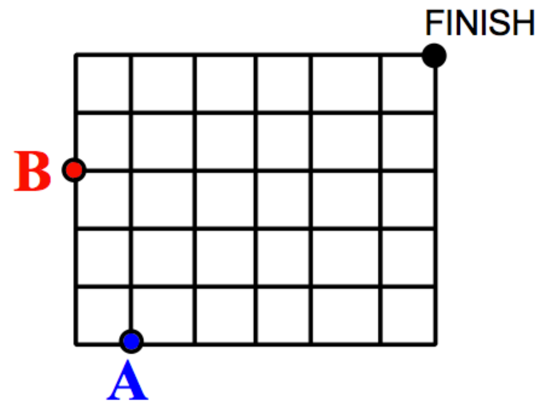
2) As a permutation

Think of NNNNNEEEEEE as a word with 11 letters. Of the 11 letters, 5 are identical N's and 6 are identical E's. The number of routes is equivalent to the number of different arrangements of the letters.

$\frac{11!}{5!6!} = 462$

Example 2:

City streets make up a grid, as shown.
Travelling east and north, how many possible routes lead from A **or** B to the finish?



From A: You must travel 5 North and 5 East; therefore the number of routes from A to finish =

$\binom{10}{5} = 252$ OR $\frac{10!}{5!5!} = 252$

From B: You must travel 2 North and 6 East; therefore the number of routes from B to finish=

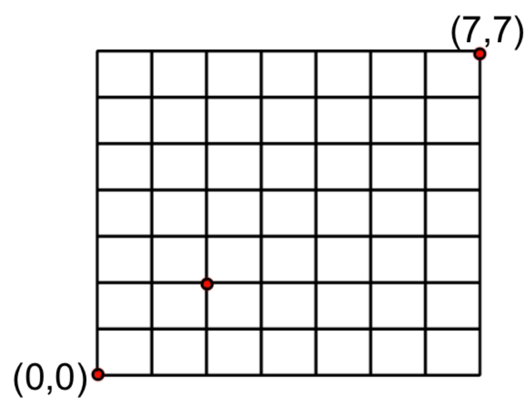
$\binom{8}{2} = 28$ OR $\binom{8}{6} = 28$ OR $\frac{8!}{2!6!} = 28$

Number of possible routes lead from A **or** B to the finish =

$n(A \cup B) = n(A) + n(B) = 252 + 28 = 280$

Example 3:

a) Starting at (0, 0) and moving only North and East, how many routes pass through (2, 2) and end at (7, 7) ?



Solution:

The routes that lead to (2, 2) from (0, 0) must each contain 2 north and 2 east movements.

$\binom{4}{2} = 6$ OR $\frac{4!}{2!2!} = 6$

The routes from (2, 2) to (7, 7) must contain 5 north and 5 east movements.

$\binom{10}{5} = 252$ OR $\frac{10!}{5!5!} = 252$

The total number of routes that pass through (2, 2) and end at (7, 7) is:

= number of routes from (0, 0) to (2, 2) × number of routes from (2, 2) to (7, 7)

=6×252

=1512

b) Starting at (0, 0) and moving only North and East, how many routes <u>avoid</u> (2, 2) and end at (7, 7) ?

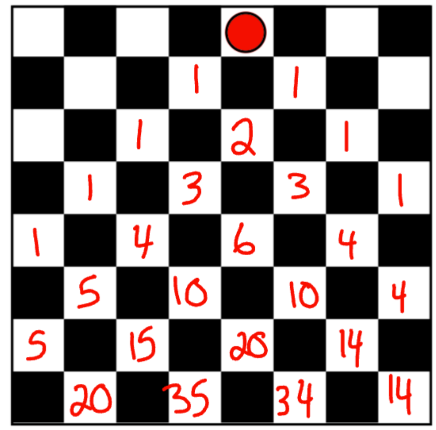
$= routes\ from\ (0,0)\ to\ (7,7) - routes\ that\ pass\ through\ (2,2)$

$= \binom{14}{7} - 1512$

$= 3432 - 1512$

$= 1920$

Example 4: A checker is placed on a game board. Determine the number of paths the checker may take to get to each allowable square on the board if it can move only diagonally forward one square at a time.



Notice the pattern resembles Pascal's Triangle

<u>Part 2: Binomial Theorem</u>

Blaise Pascal (French Mathematician) discovered a pattern in the expansion of $(a+b)^n$ ....

$(a+b)^0 = 1$

$(a+b)^1 = 1a + 1b$

$(a+b)^2 = 1a^2 + 2ab + 1b^2$

$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

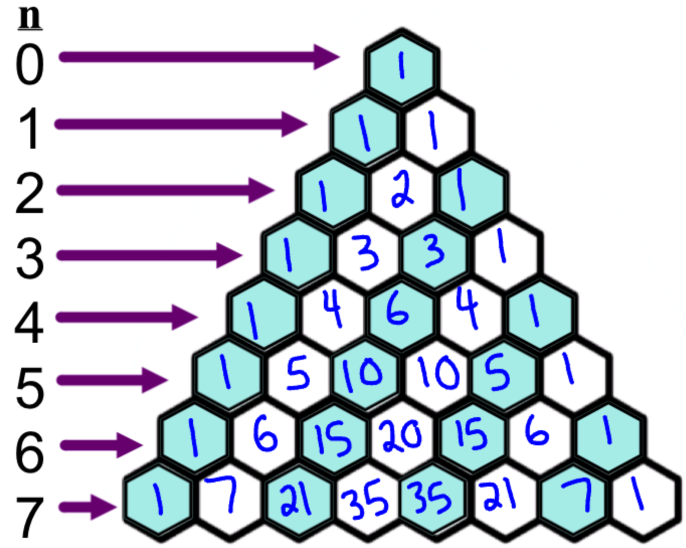
$(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$

Notice the exponents on the variables form a predictable pattern. The exponents of each term always sum to *n.* The exponents on a decrease by 1 and the exponents on b increase by 1.

Pascal's triangle can be used to determine the <u>coefficients</u> of each of the terms. There are many useful patterns in Pascal's Triangle but the main pattern used to complete the triangle is:

- the outside number is always 1
- a number inside a row is the sum of the two numbers above it

Looking at the coefficients...



The coefficients in the expansions can be determined using a strategy very similar to the one used previously to analyze paths through a map grid.

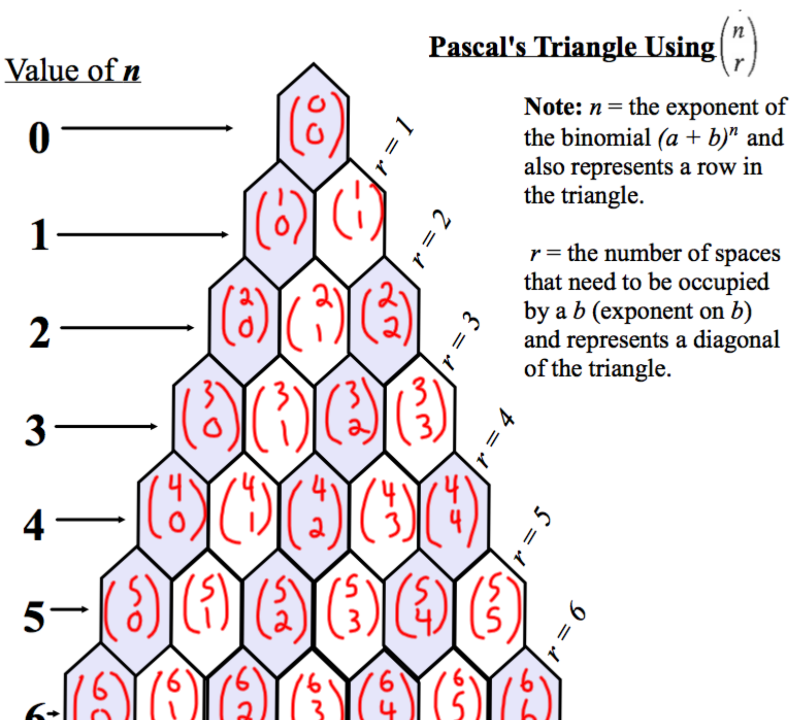
Consider the term that includes a^3b in the expansion of $(a+b)^4$. It is the result of multiplying the a-term from three of the factors with the b-term from the remaining factor.

There are four ways this can be done. This is why the coefficient of the term is 4.

a	a	a	b
a	a	b	a
a	b	a	a
b	a	a	a

There are four available spaces to record the a's and b's. For the term a^3b, one space has to be occupied by a b. Therefore, there are C(4, 1) ways of doing this. [C(4, 3) is equivalent]

Therefore, the coefficients of the general binomial expansion $(a+b)^n$ can be represented by Pascal's Triangle in terms of the combination formula.



We can generalize the results of the expansions of $(a+b)^n$ with the Binomial Theorem:

Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

The coefficients of the form $\binom{n}{r}$ are called binomial coefficients.

Notice that each term in the expansion of a binomial $= \binom{n}{r}a^{n-r}b^r$ where r is 0 for the first term and increases by 1 each term until r is equal to n.

Example 5:

Expand using the Binomial Theorem:

a) $(a+b)^6$

$$= \binom{6}{0}a^{6-0}b^0 + \binom{6}{1}a^{6-1}b^1 + \binom{6}{2}a^{6-2}b^2 + \binom{6}{3}a^{6-3}b^3 + \binom{6}{4}a^{6-4}b^4 + \binom{6}{5}a^{6-5}b^5 + \binom{6}{6}a^{6-6}b^6$$

$$= \binom{6}{0}a^6 + \binom{6}{1}a^5b^1 + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}a^1b^5 + \binom{6}{6}b^6$$

$$= 1a^6 + 6a^5b^1 + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6a^1b^5 + 1b^6$$

b) $(2x-3)^5$

$$= \binom{5}{0}(2x)^{5-0}(-3)^0 + \binom{5}{1}(2x)^{5-1}(-3)^1 + \binom{5}{2}(2x)^{5-2}(-3)^2 + \binom{5}{3}(2x)^{5-3}(-3)^3 + \binom{5}{4}(2x)^{5-4}(-3)^4 + \binom{5}{5}(2x)^{5-5}(-3)^5$$

$$= \binom{5}{0}(2x)^5(-3)^0 + \binom{5}{1}(2x)^4(-3)^1 + \binom{5}{2}(2x)^3(-3)^2 + \binom{5}{3}(2x)^2(-3)^3 + \binom{5}{4}(2x)^1(-3)^4 + \binom{5}{5}(2x)^0(-3)^5$$

$$= 1(32)(x^5)(1) + 5(16)(x^4)(-3) + 10(8)(x^3)(9) + 10(4)(x^2)(-27) + 5(2)(x)(81) + 1(1)(-243)$$

$$= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243$$

Part 3: General Term in Expansion of Binomial

The general term in the expansion of $(a+b)^n$ is:

$t_{r+1} = \binom{n}{r} a^{n-r} b^r$

This formula can be used to determine the $(r+1)^{st}$ term in a binomial expansion

Note: for the 1st term, $r = 0$
2nd term, $r = 1$
3rd term, $r = 2$

$$t_{r+1} = \binom{n}{r} a^{n-r} b^r$$

Example 6:

What is the 5th term in the following binomial?

$(x^2 - 2)^6$

$t_{4+1} = \binom{6}{4} (x^2)^{6-4} (-2)^4$

$t_5 = 15(x^2)^2(-2)^4$

$t_5 = 15x^4(16)$

$t_5 = 240x^4$

Example 7:

What is the 3rd term in the binomial $\left(x + \frac{1}{x}\right)^8$

$t_{2+1} = \binom{8}{2} (x)^{8-2} \left(\frac{1}{x}\right)^2$

$t_3 = 28(x)^6 \left(\frac{1^2}{x^2}\right)$

$t_3 = 28(x)^6 \left(\frac{1}{x^2}\right)$

$t_3 = 28(x)^6(1)(x)^{-2}$

$t_3 = 28x^4$

Example 8: Determine the constant term in the expansion of $\left(x+\frac{1}{x}\right)^{10}$

Note: *A constant term is a term with no variables. For this to happen, the exponent on the variables must be zero.*

$t_{r+1}=\binom{10}{r}(x)^{10-r}\left(\frac{1}{x}\right)^{r}$

$t_{r+1}=\binom{10}{r}(x)^{10-r}\left(\frac{1^{r}}{x^{r}}\right)$

$t_{r+1}=\binom{10}{r}(x)^{10-r}(1)(x)^{-r}$

$t_{r+1}=\binom{10}{r}(x)^{10-2r}$

The exponent on x must be zero, therefore:

$10-2r=0$
$r=5$

Therefore, the constant term is:

$t_{5+1}=\binom{10}{5}(x)^{10-2(5)}$

$t_{6}=\binom{10}{5}(x)^{0}$

$t_{6}=252$

Example 9: In the expansion of $\left(x^{3}+\frac{3}{x}\right)^{8}$, find…

a) the number of terms in the expansion

$n+1=8+1=9$

b) the 4th term

$t_{3+1}=\binom{8}{3}(x^{3})^{8-3}\left(\frac{3}{x}\right)^{3}$

$t_{4}=56(x^{3})^{5}\left(\frac{3^{3}}{x^{3}}\right)$

$t_{4}=56(x)^{15}(27)(x)^{-3}$

$t_{4}=1512x^{12}$

c) the constant term

$t_{r+1} = \binom{8}{r}(x^3)^{8-r}\left(\frac{3}{x}\right)^r$

$t_{r+1} = \binom{8}{r}(x)^{24-3r}\left(\frac{3^r}{x^r}\right)$

$t_{r+1} = \binom{8}{r}(x)^{24-3r}(3)^r(x)^{-r}$

$t_{r+1} = \binom{8}{r}(3)^r(x)^{24-4r}$

The exponent on x must be zero, therefore:

$24 - 4r = 0$
$r = 6$

Therefore, the constant term is:

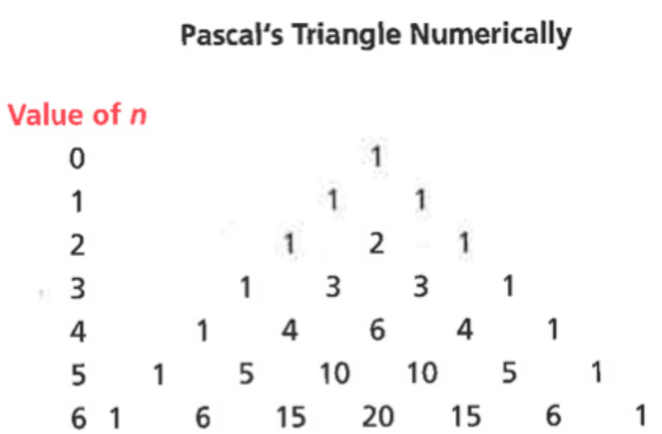
$t_{6+1} = \binom{8}{6}(3)^6(x)^{24-4(6)}$

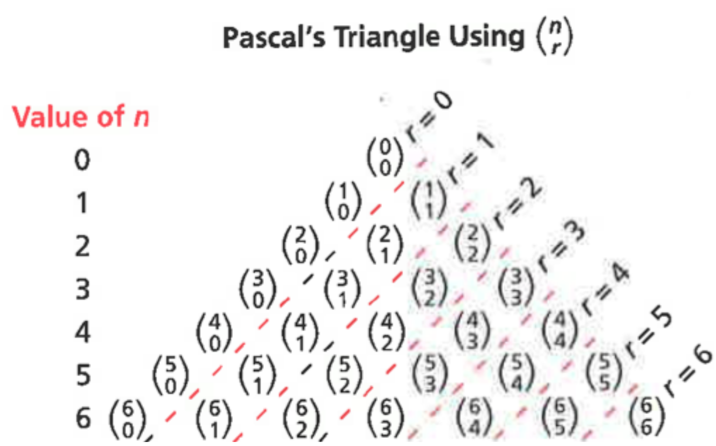
$t_7 = 28(729)(x)^0$

$t_7 = 20\,412$

Part 5: Pascal's Identity

Perhaps the most famous pattern in Pascal's Triangle stems from the relationship between the sum of consecutive values in one row and the value found in the next row immediately beneath.





This diagram suggests that $\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$

Pascal's Identity:

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

Verify Pascal's Identity using n = 6, and r = 4

$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$

$15 + 6 = 21$

$21 = 21$

Example 10: Rewrite each of the following using Pascal's Identity

a) $\binom{11}{7} + \binom{11}{8}$

$= \binom{12}{8}$

b) $\binom{19}{5} + \binom{19}{6}$

$= \binom{20}{6}$