L1 – Derivative of a Polynomial Functions Unit 1
MCV4U
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In advanced functions, you should have been introduced to the idea that the instantaneous rate of change is represented by the slope of the **tangent** at a point on the curve. You also learned that you can determine this value by taking the derivative of the function using the Newton Quotient.

Newton's Quotient:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Newton Quotient Example:

a) Find the equation of the derivative of $f(x) = 3x^2 + 2x + 4$

$$f'(x) = \lim_{h \to 0} \frac{3(x+h)^2 + 2(x+h) + 4 - (3x^2 + 2x + 4)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) + 2x + 2h + 4 - 3x^2 - 2x - 4}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 + 2x + 2h + 4 - 3x^2 - 2x - 4}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{6xh + 3h^2 + 2h}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{h(6x + 3h + 2)}{h}$$

$$f'(x) = 6x + 3(0) + 2$$

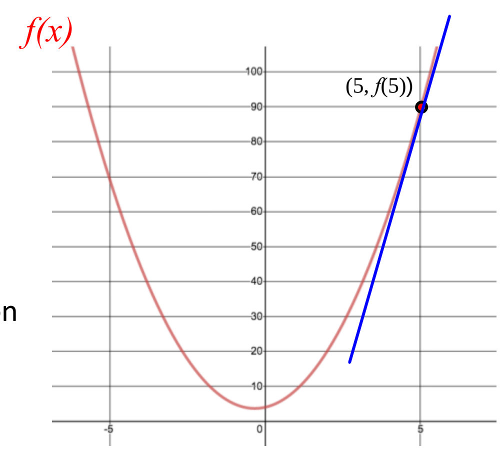
$$f'(x) = 6x + 2$$

b) Calculate $f'(5)$. What does it represent?

$$f'(5) = 6(5) + 2$$

$$f'(5) = 32$$

This tells us that the instantaneous rate of change of the original function when $x = 5$ is 32. Graphically speaking, this means the slope of the tangent line drawn on the original function at $(5, f(5))$ is 32.



Mathematicians have derived a set of rules for calculating derivatives that make this process more efficient.

Rule	Derivative	Example
Constant Rule If $f(x) = c$ where c is a constant	$f'(x) = 0$	$f(x) = 87$ $f'(x) = 0$
Power Rule If $f(x) = x^n$	$f'(x) = nx^{n-1}$	$f(x) = x^5$ $f'(x) = 5x^4$
Constant Multiple Rule If $f(x) = c \cdot g(x)$ where c is a constant	$f'(x) = c \cdot g'(x)$	$f(x) = 3x^5$ $f'(x) = 3 \cdot 5x^4$ $f'(x) = 15x^4$
Sum Rule If $h(x) = f(x) + g(x)$	$h'(x) = f'(x) + g'(x)$	$h(x) = x^5 + x^4$ $h'(x) = 5x^4 + 4x^3$
Difference Rule If $h(x) = f(x) - g(x)$	$h'(x) = f'(x) - g'(x)$	$h(x) = x^5 - x^4$ $h'(x) = 5x^4 - 4x^3$

Proof of Power Rule:

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{\binom{n}{0}x^n h^0 + \binom{n}{1}x^{n-1}h^1 + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n}x^0 h^n - x^n}{h}$$

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{\binom{n}{1}x^{n-1}h^1 + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n}x^0 h^n}{h}$$

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{h\left[\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h^1 + \cdots + \binom{n}{n}x^0 h^{n-1}\right]}{h}$$

$$\frac{d}{dx}x^n = \lim_{h\to 0}\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h^1 + \cdots + \binom{n}{n}x^0 h^{n-1}$$

$$\frac{d}{dx}x^n = \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}0^1 + \cdots + \binom{n}{n}x^0 0^{n-1}$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

Use Binomial Theorem:

$$t_{r+1} = \binom{n}{r}a^{n-r}b^r$$

Example 1: Determine the equation of the derivative of each of the following functions:

a) $f(x) = 3x^5$

$f'(x) = 15x^4$

b) $f(x) = 71$

$f'(x) = 0$

c) $f(x) = \sqrt{x}$

$f(x) = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}x^{\frac{1}{2}-1}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$f'(x) = \frac{1}{2\sqrt{x}}$

d) $y = \sqrt[3]{x}$

$y = x^{\frac{1}{3}}$

$y' = \frac{1}{3}x^{\frac{1}{3}-1}$

$y' = \frac{1}{3}x^{-\frac{2}{3}}$

$y' = \frac{1}{3x^{\frac{2}{3}}}$

e) $y = \frac{1}{x}$

$y = x^{-1}$

$y' = -1x^{-2}$

$y' = \frac{-1}{x^2}$

f) $y = -\frac{1}{x^5}$

$y = -1x^{-5}$

$\frac{dy}{dx} = 5x^{-6}$

$\frac{dy}{dx} = \frac{5}{x^6}$

Example 2: Differentiate each function

a) $y = 5x^6 - 4x^3 + 6$

$y' = 30x^5 - 12x^2 + 0$

$y' = 30x^5 - 12x^2$

b) $f(x) = -3x^5 + 8\sqrt{x} - 9.3$

$f(x) = -3x^5 + 8x^{\frac{1}{2}} - 9.3$

$f'(x) = -15x^4 + 4x^{-\frac{1}{2}} - 0$

$f'(x) = -15x^4 + \frac{4}{\sqrt{x}}$

c) $g(x) = (2x - 3)(x + 1)$

$g(x) = 2x^2 - x - 3$

$g'(x) = 4x - 1$

d) $h(x) = \frac{-8x^6+8x^2}{4x^5}$

$h(x) = \frac{-8x^6}{4x^5} + \frac{8x^2}{4x^5}$

$h(x) = -2x + 2x^{-3}$

$h'(x) = -2 - 6x^{-4}$

$h'(x) = -2 - \frac{6}{x^4}$

Example 3: Determine an equation for the tangent to the curve $f(x) = 4x^3 + 3x^2 - 5$ at $x = -1$.

Point on the tangent line:

$f(-1) = 4(-1)^3 + 3(-1)^2 - 5$

$f(-1) = -6$

$(-1, -6)$

Slope of tangent line:

$f'(x) = 12x^2 + 6x$

$f'(-1) = 12(-1)^2 + 6(-1)$

$f'(-1) = 6$

Slope of the tangent is 6

Remember: *The equation of the derivative tells you the slope of the tangent to the original function.*

Equation of tangent line:

$y = mx + b$

$-6 = 6(-1) + b$

$b = 0$

$y = 6x$

Example 4: Determine the point(s) on the graph of $y = x^2(x + 3)$ where the slope of the tangent is 24.

$y = x^3 + 3x^2$

$\frac{dy}{dx} = 3x^2 + 6x$

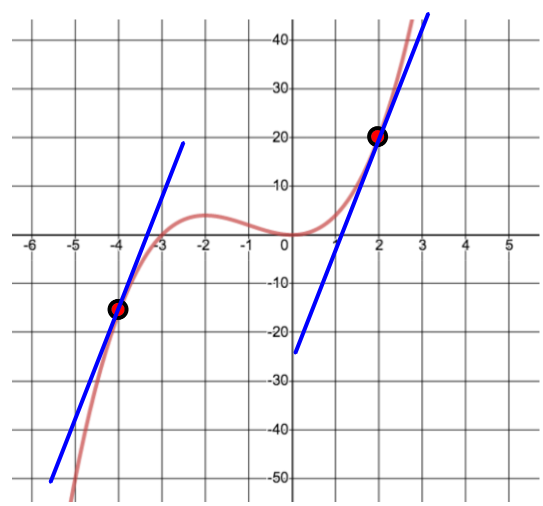
$24 = 3x^2 + 6x$

$0 = 3x^2 + 6x - 24$

$0 = 3(x^2 + 2x - 8)$

$0 = (x + 4)(x - 2)$

$x_1 = -4 \qquad x_2 = 2$



Now find y-coordinates of points:

Point 1:

$y = (-4)^3 + 3(-4)^2$

$y = -16$

$(-4, 16)$

Point 2:

$y = (2)^3 + 3(2)^2$

$y = 20$

$(2, 20)$