L2 – Dot Product | Unit 5
MCV4U
Jensen

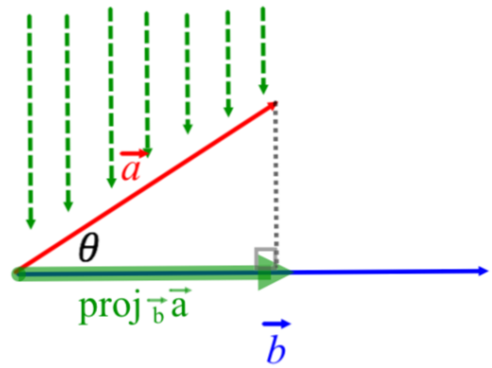
Part 1: Dot Product of Geometric Vectors

The dot product of two vectors is the product of the magnitude of one vector, vector $\vec{b}$, with the magnitude of the other vector, $\vec{a}$ that is applied in the same direction as $\vec{b}$. To determine the magnitude of $\vec{a}$ that is applied in the same direction as $\vec{b}$, we can consider the projection of vector $\vec{a}$ on to vector $\vec{b}$ ($proj_{\vec{b}}\, \vec{a}$).

In other words, the dot product of $\vec{a} \cdot \vec{b} = |\vec{b}|\, |proj_{\vec{b}}\, \vec{a}|$

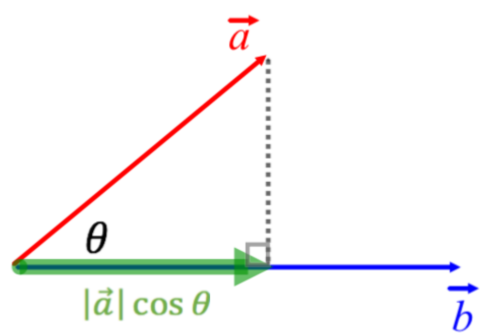
Notice: $\cos\theta = \frac{|proj_{\vec{b}}\, \vec{a}|}{|\vec{a}|}$, therefore $|proj_{\vec{b}}\, \vec{a}| = |\vec{a}|\cos\theta$

From this, the dot product of $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$



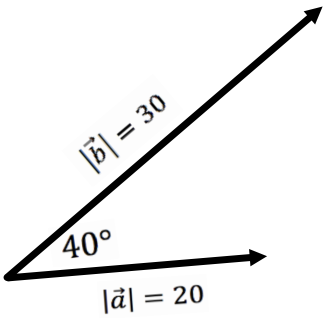
The Dot Product:

For two vectors $\vec{a}$ and $\vec{b}$, the dot product is defined as
$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$ when the vectors are arranged tail to tail, and $0 \le \theta \le 180°$. The dot product is a scalar, not a vector, and the units depend on the application.



Example 1: Determine the dot product of each pair of vectors.

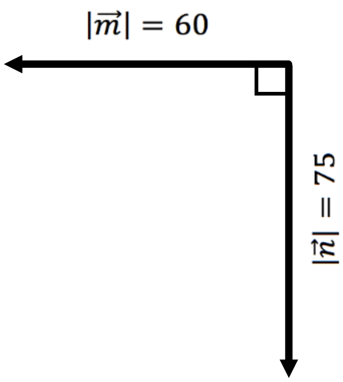
a)



$\vec{a} \cdot \vec{b} = (30)(20)\cos 40$

$\vec{a} \cdot \vec{b} \cong 459.6$

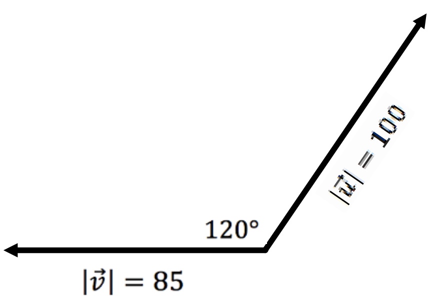
b)



$\vec{m} \cdot \vec{n} = (60)(75)\cos 90$

$\vec{m} \cdot \vec{n} = 0$

c)



$\vec{u} \cdot \vec{v} = (85)(100)\cos 120$

$\vec{u} \cdot \vec{v} = -4250$

Properties of the Dot Product

- For non-zero vectors $\vec{u}$ and $\vec{v}$, $\vec{u}$ and $\vec{v}$ are perpendicular if and only if $\vec{u} \cdot \vec{v} = 0$
- For any vectors $\vec{u}$ and $\vec{v}$, $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$. This is the commutative property.
- For any vector $\vec{u}$, $\vec{u} \cdot \vec{u} = |\vec{u}|^2$
- For any vectors $\vec{u}$ and $\vec{v}$ and scalar $k \in \mathbb{R}$, $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$. This is the associative property of the dot product.
- For any vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$. This is the distributive property of the dot product.

Based on the angle θ, we can predict whether our answer will be $+$, $-$, or 0:

If $\theta < 90°$ then $\vec{u} \cdot \vec{v} > 0$

If $\theta > 90°$ then $\vec{u} \cdot \vec{v} < 0$

If $\theta = 90°$ then $\vec{u} \cdot \vec{v} = 0$

Part 2: Dot Product of Cartesian Vectors

The dot product of two Cartesian vectors $\vec{a} = [a_x, a_y]$ and $\vec{b} = [b_x, b_y]$ is $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y$

Example 2: Calculate $\vec{u} \cdot \vec{v}$

a) $\vec{u} = [5, -3], \vec{v} = [4, 7]$

$\vec{u} \cdot \vec{v} = 5(4) + (-3)(7)$

$\vec{u} \cdot \vec{v} = -1$

b) $\vec{u} = [-2, 9], \vec{v} = [-1, 0]$

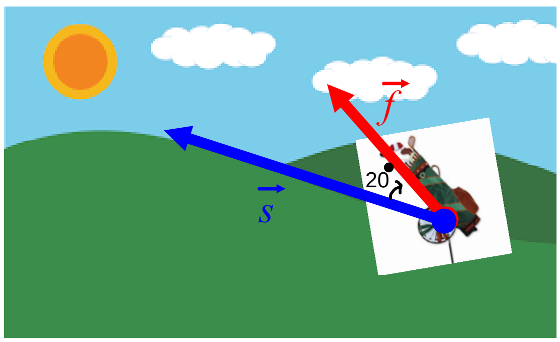
$\vec{u} \cdot \vec{v} = (-2)(-1) + (9)(0)$

$\vec{u} \cdot \vec{v} = 2$

Part 3: Mechanical Work

One of the applications of the dot product is to calculate the mechanical work (or simply the work) performed. Mechanical work is the product of the magnitude of the displacement travelled by an object and the magnitude of the force applied in the direction of the motion. The units are newton-meters $(N \cdot m)$, also known as joules (J).

Example 3: Max is pulling his golf cart up a hill with a force of 120 N at an angle of 20° to the surface of the hill. This hill is 100 meters long. Find the work that Max performs.



Let $\vec{f}$ represent the force

Let $\vec{s}$ represent the displacement along the hill

The work done equals $\vec{f} \cdot \vec{s}$

$W = (120)(100)\cos 20$

$W \cong 11276.3$ Joules