**L3 – Concavity and the Second Derivative** Unit 2

MCV4U

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The ___________________________ is the derivative of the first derivative. It is the rate of change of the slope of the tangent.

**Part 1: Discovery**

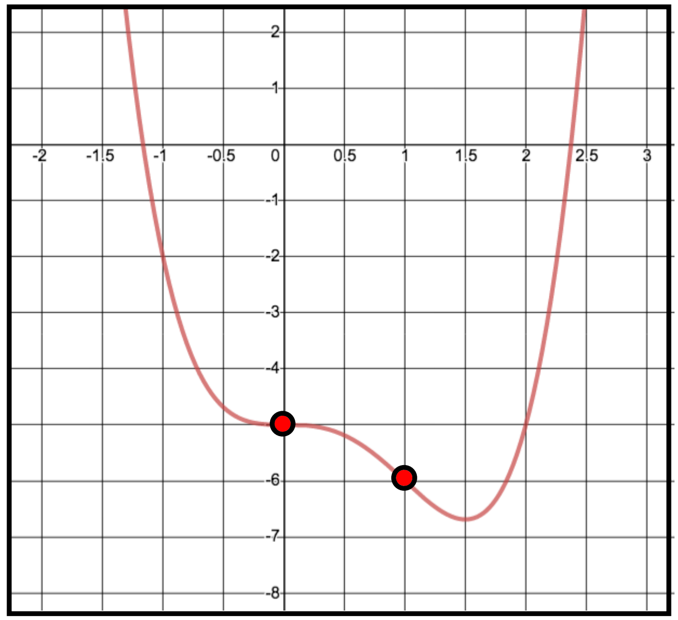
**Example 1:**

**a)** Given the graph of $f(x) = x^4 - 2x^3 - 5$

$f'(x) =$

$f''(x) =$

When is $f''(x) = 0$ ?




**b)** Use your pencil to simulate a tangent line to the function when $x = -1$. Drag the pencil slowly to the right, keeping it tangent to the curve, approaching $x = 0$. What is happening to the slope of the tangent? Is it above or below the curve? What is the value of $f''(-0.5)$?

**c)** Drag the pencil slowly to the right, keeping it tangent to the curve, moving through $x = 0$. What is happening to the slope of the tangent as it moves through $x = 0$? What is the value of $f''(0.5)$?

**d)** What happens to the slope of the tangent as it moves through $x = 1$?

**Summary of findings:**

How $f''(x)$ effects $f(x)$:

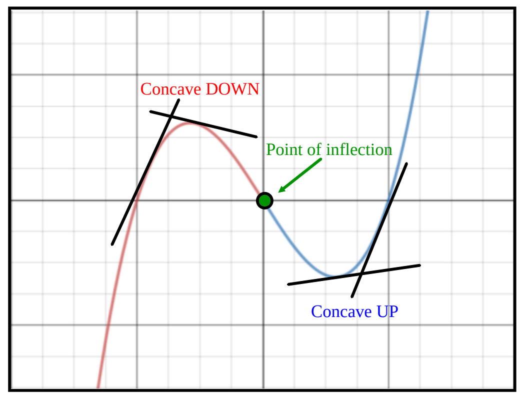
The graph of a function is concave up over an interval if the curve is above all of the tangents on the interval. The slopes of the tangent lines are increasing, therefore $f''(x) > 0$ over this interval.

The graph of a function is concave down over an interval if the curve is below all of the tangents on the interval. The slopes of the tangent lines are decreasing, therefore $f''(x) < 0$ over this interval.

$f(x)$ is concave _______ on an interval if $f''(x) > 0$ over that interval (tangent line slopes are increasing)

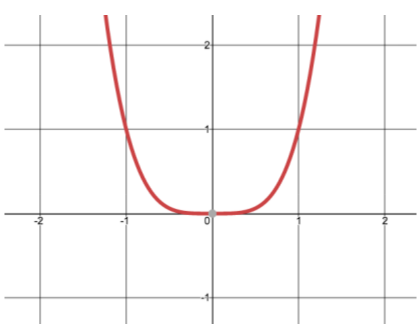
$f(x)$ is concave _______ on an interval if $f''(x) < 0$ over that interval (tangent line slopes are decreasing)

A ________________________________ is a point in the domain of the function at which the graph changes from being concave up to concave down or vice versa. The second derivative, $f''(x)$, is equal to zero at this point (or is undefined) and changes sign on either side. The tangent lines change from increasing to decreasing OR from decreasing to increasing.




However, just like that not every critical point is a local max / min, not every zero or restriction of the second derivative is an inflection point either. They are just the pool of points you need to check in order to find the inflection point(s) of a curve.

$f(x) = x^4$

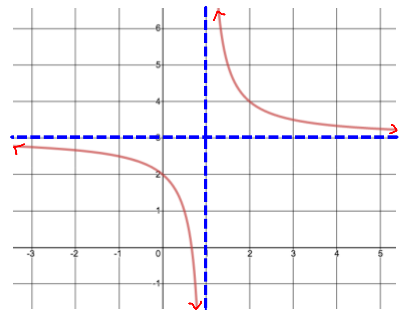


$f''(x) = 12x^2$

$f''(0) = 0$

But $x = 0$ is not a point of inflection; the function has no change in concavity. Tangent slopes are always increasing.

**Note:** It often happens that a graph has different concavity on the two sides of a vertical asymptote. However, because a curve is not continuous at a vertical asymptote, it can never have an inflection point there. We will look at these types of functions next lesson (rational functions).



The ___________________________________ can also be used to help check for local min/max points.

In the second derivative test we check the critical points themselves (those where $f'(x) = 0$), by evaluating $f''(x)$ AT each critical point.

If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local ___________________ at *c*.

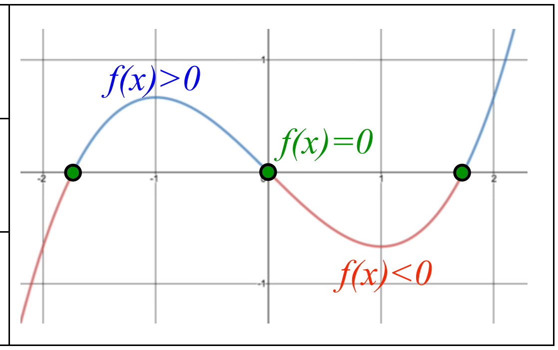
If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local ___________________ at *c*.

***Note:*** *Even though it is often easier to use than the first derivative test, the second derivative test can fail at some points (eg. $y = x^4$). If the second derivative test fails, then the first derivative test must be used to classify the point in question.*

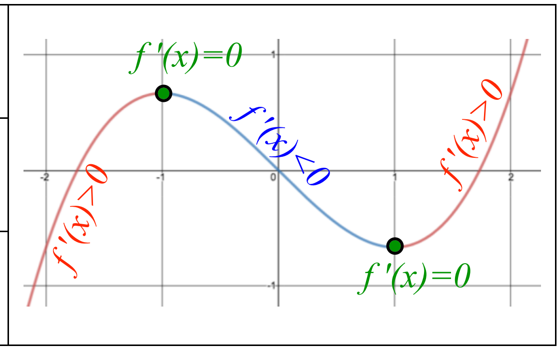
Summary Page of what we know so far

### Relationship between $f(x)$, $f'(x)$, and $f''(x)$

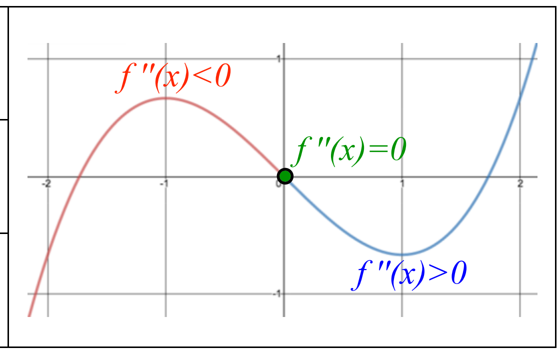
| | |
|---|---|
| $f(x) = 0$ | Zeros ($x$-intercepts) of the function |
| $f(x) > 0$ | Function is positive (above $x$-axis) |
| $f(x) < 0$ | Function is negative (below $x$-axis) |




| | |
|---|---|
| $f'(x) = 0$ | Horizontal tangent; possible local extrema (turning point) |
| $f'(x) > 0$ | $f(x)$ is increasing |
| $f'(x) < 0$ | $f(x)$ is decreasing |




| | |
|---|---|
| $f''(x) = 0$ | Possible point of inflection (change in concavity) |
| $f''(x) > 0$ | $f(x)$ is concave up |
| $f''(x) < 0$ | $f(x)$ is concave down |




### Tests of Critical Numbers:

| | |
|---|---|
| Absolute Extrema on an Interval $[a, b]$ | 1. Find CN $x = c$, at $f'(x) = 0$ or undefined<br>2. Check endpoints and critical numbers; $f(a), f(c), f(b)$<br>3. Choose the minimum and maximum values |
| Local Extrema – First Derivative Test of Critical Numbers | 1. Find CN $x = c$, at $f'(x) = 0$ or undefined<br>2. Make a sign chart for $f'(x)$. Use test values.<br>3. Draw conclusions about $f(x)$<br>- If $f(x)$ changes from increasing to decreasing, $(c, f(c))$ is a local max<br>- If $f(x)$ changes from decreasing to increasing, $(c, f(c))$ is a local min |
| Local Extrema – Second Derivative Test of Critical Numbers | 1. Find CN $x = c$, at $f'(x) = 0$ or undefined<br>2. Calculate the second derivative $f''(x)$<br>3. Test the critical numbers in $f''(x)$<br>- if $f''(c) > 0$, $f(x)$ is concave up and $(c, f(c))$ is a local min<br>- if $f''(c) < 0$, $f(x)$ is concave down and $(c, f(c))$ is a local max<br>- if $f''(c) = 0$, the test fails and you must use the First Derivative Test |

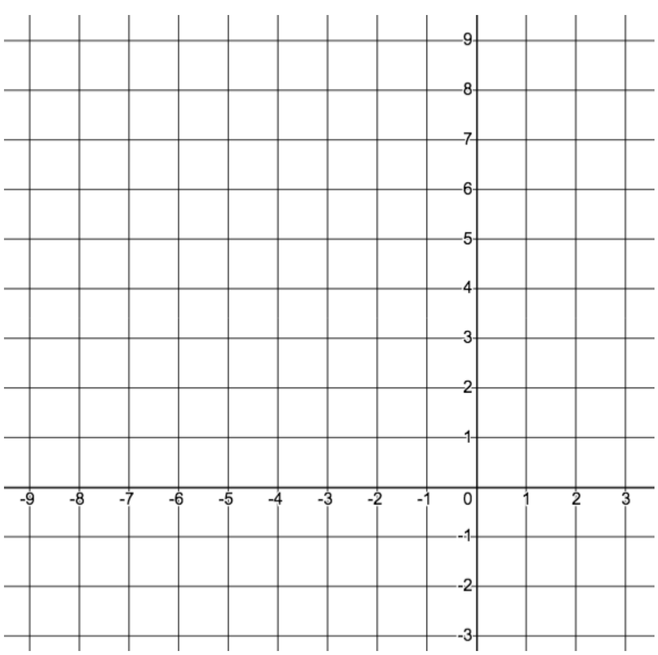
**Example 2:** For the function $f(x) = x^4 - 6x^2 - 5$, find all points of inflection (POI) and the intervals of concavity.

**Example 3:** For the function below, find the critical points. Then, classify them using the second derivative test.

$g(x) = x^3 - 3x^2 + 2$

**Example 4:** Sketch a graph of a function that satisfies each set of conditions.

**a)** $f''(x) = -2$ for all $x$, $f'(-3) = 0$, $f(-3) = 9$



**b)** $f''(x) < 0$ when $x < -1$, $f''(x) > 0$ when $x > -1$, $f'(-3) = 0$, $f'(1) = 0$