L3 – Applications of the Dot Product Unit 5
MCV4U
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Warm-Up

Example 1: A desk is pushed with a force of 50 N at an angle of 45 degrees below the horizontal. If the desk is pushed 5 meters, how much work is done?

$W = \vec{f} \cdot \vec{s}$

$W = |\vec{f}||\vec{s}| \cos\theta$

$W = 50(5)\cos 45$

$W = 250\left(\frac{1}{\sqrt{2}}\right)$

$W = \frac{250}{\sqrt{2}}$ or $125\sqrt{2}$ joules

> ***Remember:*** *Mechanical work is the product of the magnitude of the displacement travelled by an object and the magnitude of the force applied in the direction of the motion.*

Part 1: Angle Between 2 Vectors

To determine the angle between two vectors, you can rearrange the dot product formula, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$, to isolate $\cos\theta$:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Example 2: Determine the angle between each pair of vectors.

a) $\vec{g} = [5, 1]$ and $\vec{h} = [-3, 8]$

$\cos\theta = \frac{\vec{g} \cdot \vec{h}}{|\vec{g}||\vec{h}|}$

$\cos\theta = \frac{5(-3)+1(8)}{\left(\sqrt{(5)^2+(1)^2}\right)\left(\sqrt{(-3)^2+(8)^2}\right)}$

$\cos\theta = \frac{-7}{(\sqrt{26})(\sqrt{73})}$

$\theta \cong 99.2°$

b) $\vec{a} = [-3, 6]$ and $\vec{b} = [4, 2]$

$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$\cos\theta = \frac{(-3)(4)+6(2)}{\left(\sqrt{(-3)^2+(6)^2}\right)\left(\sqrt{(4)^2+(2)^2}\right)}$

$\cos\theta = \frac{0}{(\sqrt{45})(\sqrt{20})}$

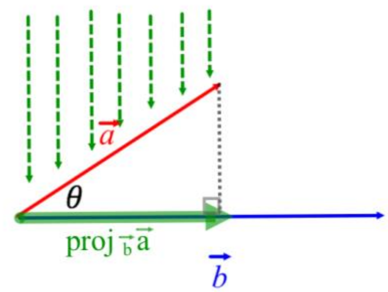
$\theta = 90°$

Part 2: Vector Projections

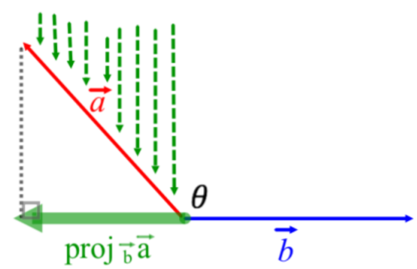
You can think of a vector projection like a shadow. The vertical arrows in the diagrams represent light from above.

Think of the projection of $\vec{a}$ on $\vec{b}$ as the shadow that $\vec{a}$ casts on $\vec{b}$.

If the angle between $\vec{a}$ and $\vec{b}$ is less than $90°$, then the projection of $\vec{a}$ on $\vec{b}$, or $proj_{\vec{b}}\,\vec{a}$, is the vector component of $\vec{a}$ in the direction of $\vec{b}$.

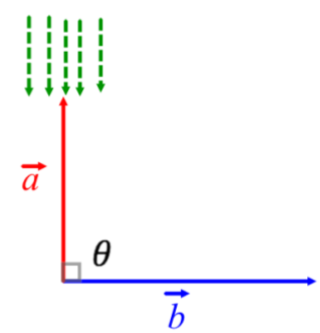


If the angle between $\vec{a}$ and $\vec{b}$ is between $90°$ and $180°$, the direction of $proj_{\vec{b}}\,\vec{a}$ is in the opposite direction of $\vec{b}$.



If $\vec{a}$ is perpendicular to $\vec{b}$, then $\vec{a}$ casts 'no shadow' on to $\vec{b}$. So if $\theta = 90°$, $proj_{\vec{b}}\,\vec{a} = 0$.

Note: *This is why the dot product $\vec{a} \cdot \vec{b}$ would be zero for perpendicular vectors.*



Formulas for Vector Projection:

Geometric Formulas:

$$proj_{\vec{b}}\,\vec{a} = |\vec{a}|\cos\theta\,(\hat{b})$$

$$proj_{\vec{b}}\,\vec{a} = |\vec{a}|\cos\theta\left(\frac{\vec{b}}{|\vec{b}|}\right)$$

OR

Cartesian Formulas:

$$proj_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}(\hat{b})$$

$$proj_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\left(\frac{\vec{b}}{|\vec{b}|}\right)$$

$$proj_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}(\vec{b})$$

Formulas for Magnitude of Vector Projection:

If $0° < \theta < 90°$ $\quad \left|proj_{\vec{b}}\,\vec{a}\right| = |\vec{a}|\cos\theta$

If $90° < \theta < 180°$ $\quad \left|proj_{\vec{b}}\,\vec{a}\right| = -|\vec{a}|\cos\theta$

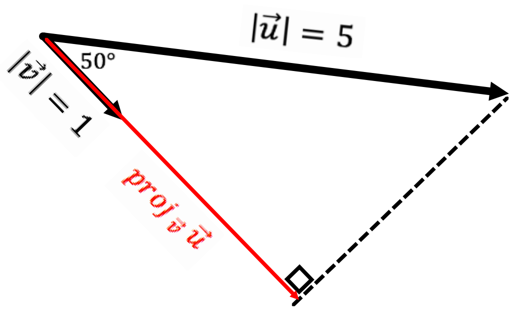
OR

$$\left|proj_{\vec{b}}\,\vec{a}\right| = \left|\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right|$$

Note: $\frac{\vec{b}}{|\vec{b}|}$ is a unit vector in the direction of $\vec{b}$. Sometimes the symbol $\hat{b}$ is used to denote a unit vector in the direction of $\vec{b}$.

Example 3: Determine the following projections of one vector on another.

a) Determine the projection of $\vec{u}$ on $\vec{v}$



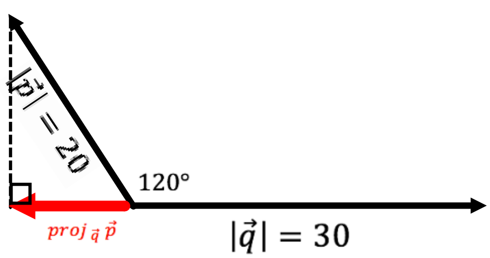
$proj_{\vec{v}}\,\vec{u} = |\vec{u}|\cos\theta(\hat{v})$

$proj_{\vec{v}}\,\vec{u} = 5\cos 50(\hat{v})$

$proj_{\vec{v}}\,\vec{u} \cong 3.2\hat{v}$

3.2 units in the same direction as $\vec{v}$

b) Determine $proj_{\vec{q}}\,\vec{p}$



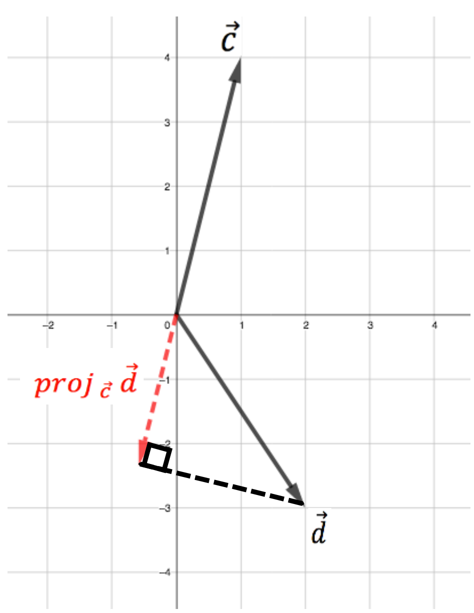
$proj_{\vec{q}}\,\vec{p} = |\vec{p}|\cos\theta(\hat{q})$

$proj_{\vec{v}}\,\vec{u} = 20\cos 120(\hat{q})$

$proj_{\vec{v}}\,\vec{u} \cong -10\hat{q}$

10 units in the opposite direction as $\vec{q}$

c) Determine the projection of $\vec{d} = [2,-3]$ on $\vec{c} = [1,4]$



$proj_{\vec{c}}\,\vec{d} = \frac{\vec{d}\cdot\vec{c}}{\vec{c}\cdot\vec{c}}(\vec{c}) = \frac{\vec{d}\cdot\vec{c}}{|\vec{c}|^2}(\vec{c})$

$proj_{\vec{c}}\,\vec{d} = \left(\frac{2(1)+(-3)(4)}{1^2+4^2}\right)[1,4]$

$proj_{\vec{c}}\,\vec{d} = \left(\frac{-10}{17}\right)[1,4]$

$proj_{\vec{c}}\,\vec{d} = \left[\frac{-10}{17}, \frac{-40}{17}\right]$

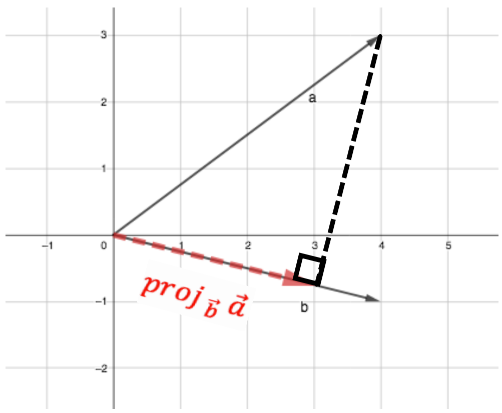
d) Find the magnitude of the projection of $\vec{a} = [4,3]$ on $\vec{b} = [4,-1]$

$$\left|proj_{\vec{b}}\, \vec{a}\right| = \left|\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\right|$$

$$\left|proj_{\vec{b}}\, \vec{a}\right| = \left|\frac{4(4)+(3)(-1)}{\sqrt{(4)^2+(-1)^2}}\right|$$

$$\left|proj_{\vec{b}}\, \vec{a}\right| = \frac{13}{\sqrt{17}}$$

e) Find the projection of $\vec{a} = [4,3]$ on $\vec{b} = [4,-1]$



$$proj_{\vec{b}}\, \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}(\vec{b})$$

$$proj_{\vec{b}}\, \vec{a} = \frac{4(4)+(3)(-1)}{(4)^2+(-1)^2}[4,-1]$$

$$proj_{\vec{b}}\, \vec{a} = \frac{13}{17}[4,-1]$$

$$proj_{\vec{b}}\, \vec{a} = \left[\frac{52}{17}, \frac{-13}{17}\right]$$

Part 3: Dot Product with Sales

Example 4: A shoe store sold 350 pairs of Nike shoes and 275 pairs of Adidas shoes in a year. Nike shoes sell for $175 and Adidas shoes sell for $250.

a) Write a Cartesian vector, $\vec{s}$, to represent the numbers of pairs of shoes sold.

$\vec{s} = [350,275]$

b) Write a Cartesian vector, $\vec{p}$, to represent the prices of the shoes.

$\vec{p} = [175,250]$

c) Find the dot product $\vec{s} \cdot \vec{p}$. What does this dot product represent?

$\vec{s} \cdot \vec{p} = 350(175) + 275(250)$

$\vec{s} \cdot \vec{p} = 130\ 000$

The dot product represents the revenue, $130 000, from sales of the shoes.