L3 – Concavity and the Second Derivative Unit 2
MCV4U
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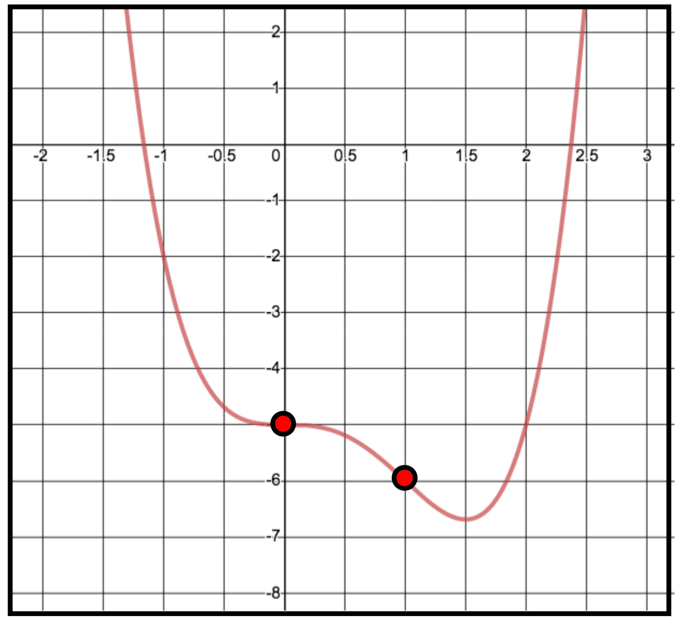
The **second derivative** is the derivative of the first derivative. It is the rate of change of the slope of the tangent.

Part 1: Discovery

Example 1:

a) Given the graph of $f(x) = x^4 - 2x^3 - 5$



$f'(x) = 4x^3 - 6x^2$

$f''(x) = 12x^2 - 12x$

When is $f''(x) = 0$?

$0 = 12x^2 - 12x$

$0 = 12x(x-1)$

$x = 0$ or $x = 1$

b) Use your pencil to simulate a tangent line to the function when $x = -1$. Drag the pencil slowly to the right, keeping it tangent to the curve, approaching $x = 0$. What is happening to the slope of the tangent? Is it above or below the curve? What is the value of $f''(-0.5)$?

- The curve is above the tangent line.
- The tangent line slopes are increasing.
- $f''(-0.5) = 9$; it's positive

c) Drag the pencil slowly to the right, keeping it tangent to the curve, moving through $x = 0$. What is happening to the slope of the tangent as it moves through $x = 0$? What is the value of $f''(0.5)$?

- The slope of the tangent stops increasing and starts decreasing.
- The curve goes from above the tangent line to below the tangent line.
- $f''(0.5) = -3$; it's negative

d) What happens to the slope of the tangent as it moves through $x = 1$?

- The slope of the tangent stops decreasing and starts increasing.
- The curve goes from below the tangent line to above the tangent line.

Summary of findings:

How $f''(x)$ effects $f(x)$:

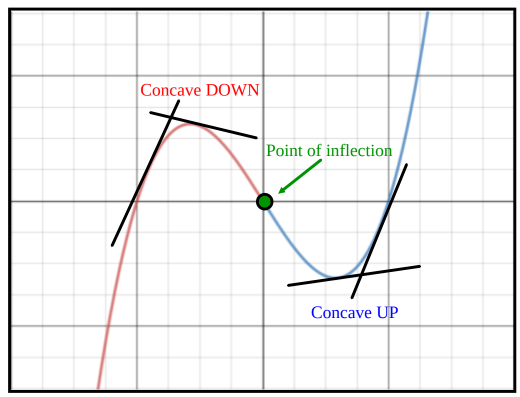
The graph of a function is concave up over an interval if the curve is above all of the tangents on the interval. The slopes of the tangent lines are increasing, therefore $f''(x) > 0$ over this interval.

The graph of a function is concave down over an interval if the curve is below all of the tangents on the interval. The slopes of the tangent lines are decreasing, therefore $f''(x) < 0$ over this interval.

$f(x)$ is concave **UP** on an interval if $f''(x) > 0$ over that interval (tangent line slopes are increasing)

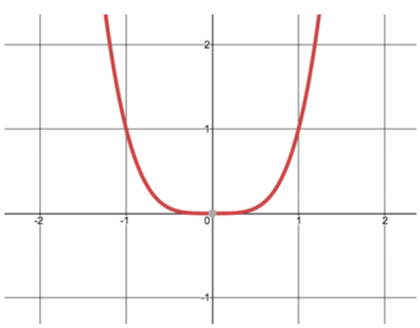
$f(x)$ is concave **DOWN** on an interval if $f''(x) < 0$ over that interval (tangent line slopes are decreasing)

A **POINT OF INFLECTION** is a point in the domain of the function at which the graph changes from being concave up to concave down or vice versa. The second derivative, $f''(x)$, is equal to zero at this point (or is undefined) and changes sign on either side. The tangent lines change from increasing to decreasing OR from decreasing to increasing.



However, just like that not every critical point is a local max / min, not every zero or restriction of the second derivative is an inflection point either. They are just the pool of points you need to check in order to find the inflection point(s) of a curve.

$f(x) = x^4$

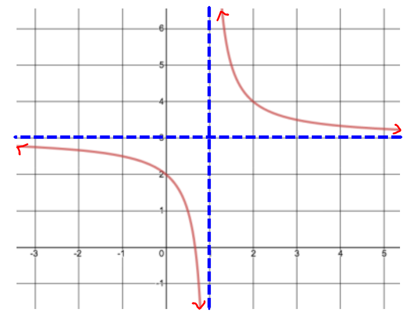


$f''(x) = 12x^2$

$f''(0) = 0$

But $x = 0$ is not a point of inflection; the function has no change in concavity. Tangent slopes are always increasing.

Note: It often happens that a graph has different concavity on the two sides of a vertical asymptote. However, because a curve is not continuous at a vertical asymptote, it can never have an inflection point there. We will look at these types of functions next lesson (rational functions).



The **second derivative test** can also be used to help check for local min/max points.

In the second derivative test we check the critical points themselves (those where $f'(x) = 0$), by evaluating $f''(x)$ AT each critical point.

If $f'(c) = 0$ and $f''(c) > 0$, then f has a local **minimum** at *c*.

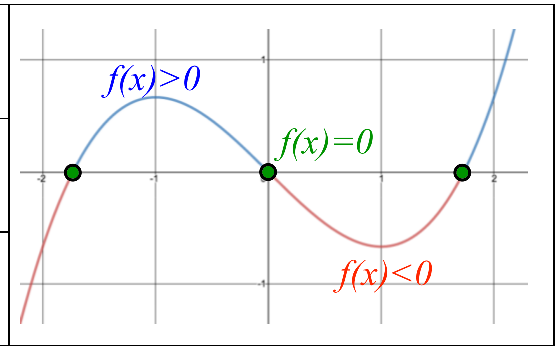
If $f'(c) = 0$ and $f''(c) < 0$, then f has a local **maximum** at *c*.

Note: *Even though it is often easier to use than the first derivative test, the second derivative test can fail at some points (eg.* $y = x^4$*). If the second derivative test fails, then the first derivative test must be used to classify the point in question.*

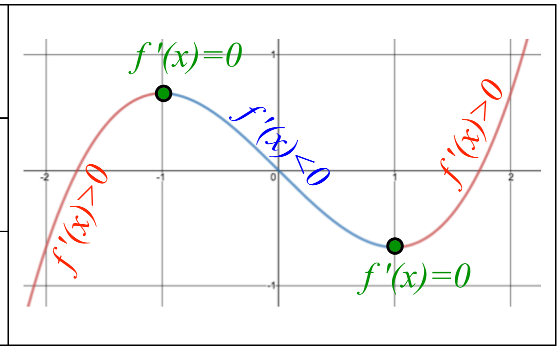
Summary Page of what we know so far

Relationship between $f(x)$, $f'(x)$, and $f''(x)$

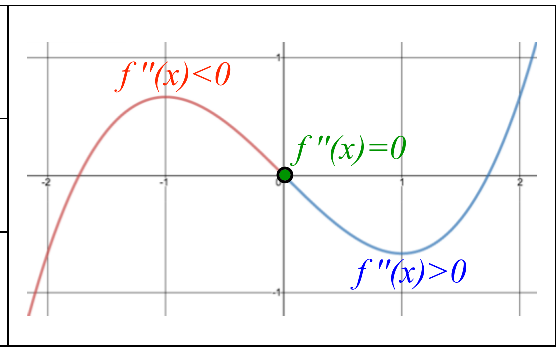
$f(x) = 0$	Zeros (x-intercepts) of the function
$f(x) > 0$	Function is positive (above x-axis)
$f(x) < 0$	Function is negative (below x-axis)



$f'(x) = 0$	Horizontal tangent; possible local extrema (turning point)
$f'(x) > 0$	$f(x)$ is increasing
$f'(x) < 0$	$f(x)$ is decreasing



$f''(x) = 0$	Possible point of inflection (change in concavity)
$f''(x) > 0$	$f(x)$ is concave up
$f''(x) < 0$	$f(x)$ is concave down



Tests of Critical Numbers:

Absolute Extrema on an Interval $[a, b]$	1. Find CN $x = c$, at $f'(x) = 0$ or undefined 2. Check endpoints and critical numbers; $f(a), f(c), f(b)$ 3. Choose the minimum and maximum values
Local Extrema – First Derivative Test of Critical Numbers	1. Find CN $x = c$, at $f'(x) = 0$ or undefined 2. Make a sign chart for $f'(x)$. Use test values. 3. Draw conclusions about $f(x)$ - If $f(x)$ changes from increasing to decreasing, $(c, f(c))$ is a local max - If $f(x)$ changes from decreasing to increasing, $(c, f(c))$ is a local min
Local Extrema – Second Derivative Test of Critical Numbers	1. Find CN $x = c$, at $f'(x) = 0$ or undefined 2. Calculate the second derivative $f''(x)$ 3. Test the critical numbers in $f''(x)$ - if $f''(c) > 0$, $f(x)$ is concave up and $(c, f(c))$ is a local min - if $f''(c) < 0$, $f(x)$ is concave down and $(c, f(c))$ is a local max - if $f''(c) = 0$, the test fails and you must use the First Derivative Test

Example 2: For the function $f(x) = x^4 - 6x^2 - 5$, find all points of inflection (POI) and the intervals of concavity.

$f'(x) = 4x^3 - 12x$

$f''(x) = 12x^2 - 12$

$0 = 12x^2 - 12$

$0 = 12(x^2 - 1)$

$0 = 12(x - 1)(x + 1)$

$x_1 = 1 \quad x_2 = -1$

Possible points of inflection:

$f(1) = (1)^4 - 6(1)^2 - 5 = -10$

$f(-1) = (-1)^4 - 6(-1)^2 - 5 = -10$

Use sign chart to check if there are changes of concavity on either side of these points.

Sign Chart:

Test value for x ($-\infty$ to -1)	-2	(-1 to 1) 0	(1 to ∞) 2
$f''(x)$	$+$	$-$	$+$
$f(x)$	Concave UP	Concave DOWN	Concave UP

POI at $(-1, -10)$

POI at $(1, -10)$

Concave up: $(-\infty, -1) \cup (1, \infty)$

Concave down: $(-1, 1)$

Example 3: For the function below, find the critical points. Then, classify them using the second derivative test.

$g(x) = x^3 - 3x^2 + 2$

$g'(x) = 3x^2 - 6x$

$0 = 3x^2 - 6x$

$0 = 3x(x-2)$

$x = 0$ and $x = 2$ are critical numbers

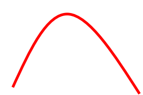
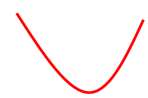
Critical points: $(0, 2)$ and $(2, -2)$

$g(0) = 2$

$g(2) = -2$

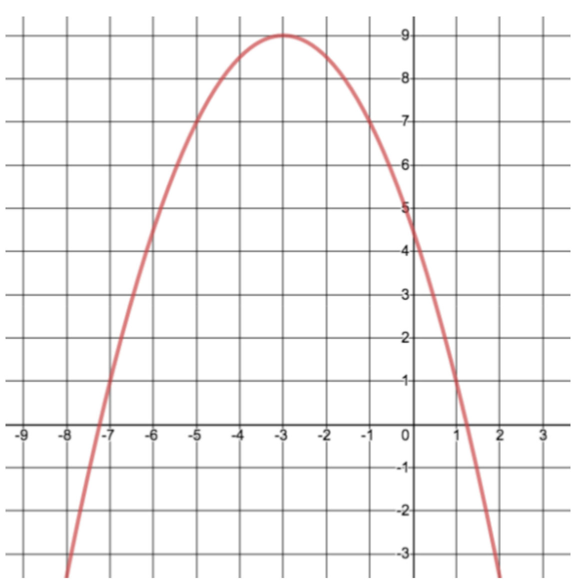
Second derivative test:

$g''(x) = 6x - 6$

	$x = 0$	$x = 2$
$g''(x)$	$-$	$+$
$g(x)$	Concave DOWN 	Concave UP 
	$(0, 2)$ is a local MAX	$(2, -2)$ is a local MIN

Example 4: Sketch a graph of a function that satisfies each set of conditions.

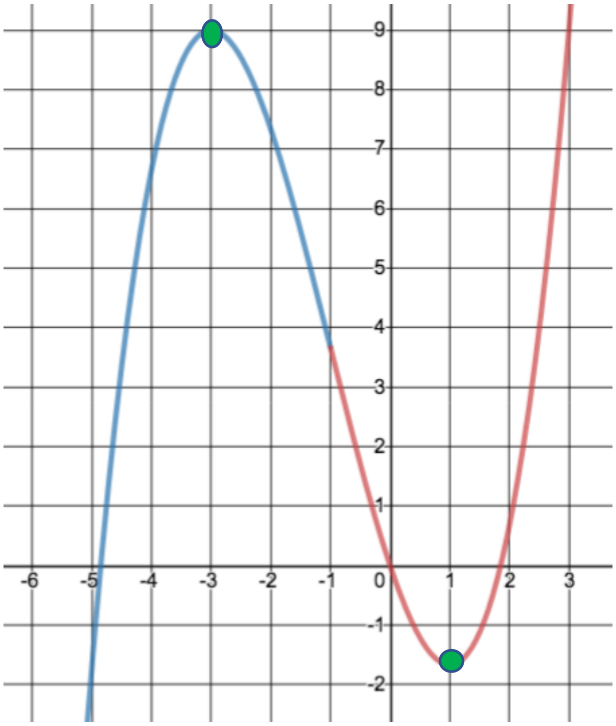
a) $f''(x) = -2$ for all x, $f'(-3) = 0$, $f(-3) = 9$



$f''(x) = -2$ for all x tells us that the function is always concave down.

$f'(-3) = 0$ means indicates there is a local max at $(-3, 9)$

b) $f''(x) < 0$ when $x < -1$, $f''(x) > 0$ when $x > -1$, $f'(-3) = 0$, $f'(1) = 0$



$f(x)$ is concave down when $x < -1$

$f(x)$ is concave up when $x > -1$

Local max at $x = -3$

Local min at $x = 1$