L4 – Vectors in 3-Space
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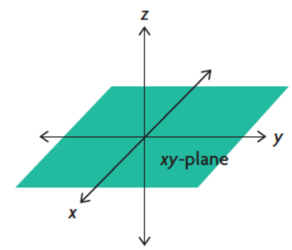
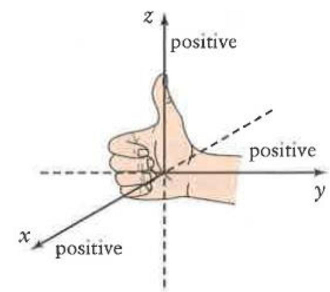
Unit 5

**Part 1: Plotting Points in 3-Dimensions**

https://www.geogebra.org/3d?lang=en

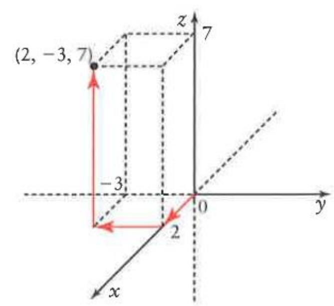



In placing points in 3-Dimensions ($R^3$), we choose three axes called $x$-, $y$-, and $z$-axis. Each axis is perpendicular. Each point is written using ordered triples $(x, y, z)$.




There are several ways to choose the orientation of the positive axes, but we will use what is called the right-handed system. If we imagine ourselves looking down the positive $z$-axis onto the $xy$ plane so that, when the positive $x$-axis is rotated $90°$ counterclockwise it becomes coincident with the positive $y$-axis, then this is called the right-handed system.

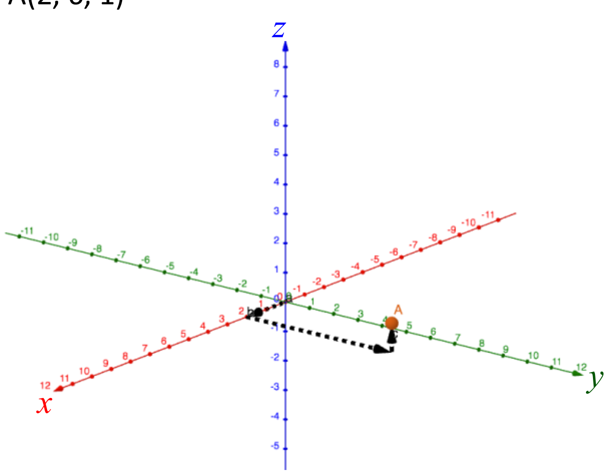
If you curl the fingers of your right hand from the positive $x$-axis to the positive $y$-axis your thumb will point along the positive $z$-axis.



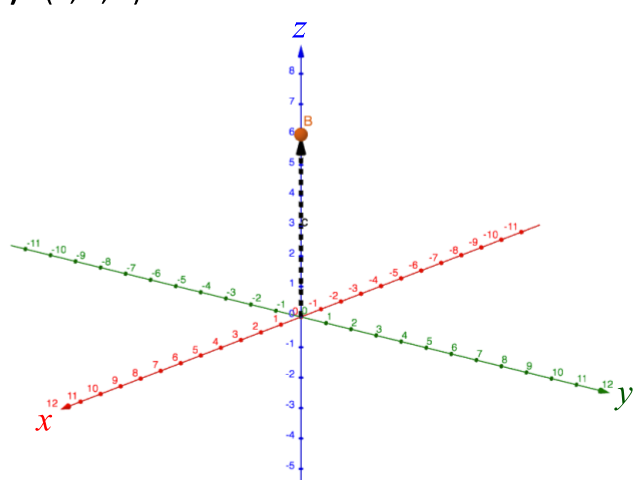

To plot the point of (2, -3, 7), start at the origin. Move two units along the positive $x$-axis, then 3 units parallel to the negative $y$-axis, and then 7 units parallel to the positive $z$-axis.

**Example 1:** Plot the following points in $R^3$

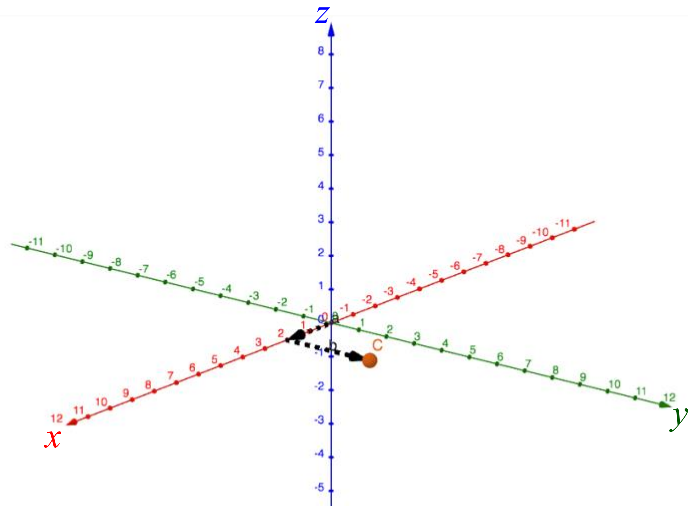
**a)** A(2, 6, 1)



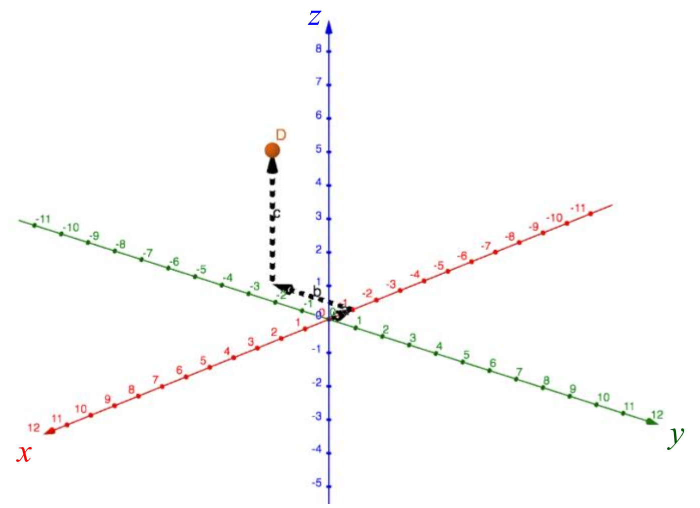

**b)** B(0, 0, 6)



**c)** C(2, 3, 0)

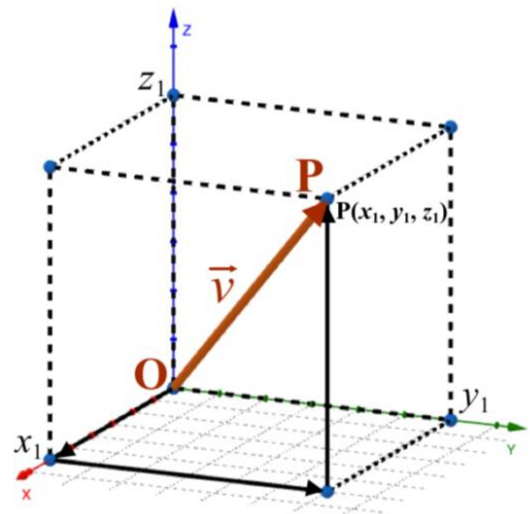



**d)** D(-1, -3, 4)




**Part 2: 3-D Cartesian Vectors**

Let $\vec{v}$ represent a vector in space. If $\vec{v}$ is translated so that its tail is at the origin, $O$, then its tip will be at some point $P(x_1, y_1, z_1)$. Then $\vec{v}$ is the position vector of the point $P$, and $\vec{v} = \overrightarrow{OP} = [x_1, y_1, z_1]$.



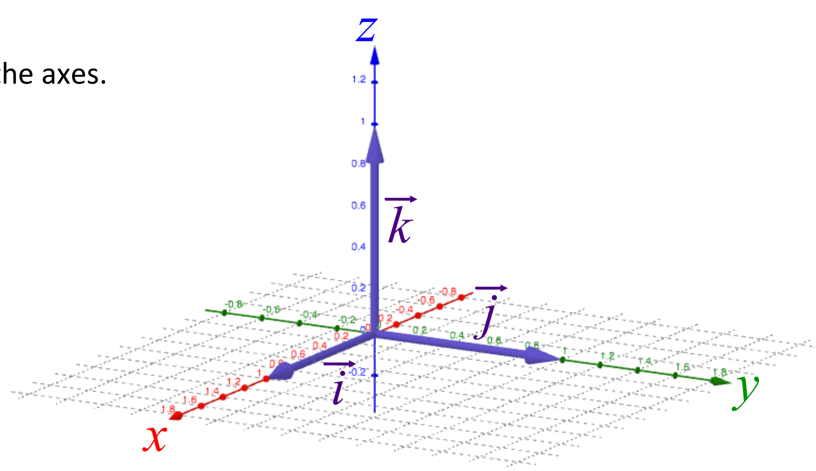

**Unit Vectors in $R^3$**

Unit vectors all have a magnitude of 1 and along the axes.
In 3-Dimensions, there are 3 unit vectors:

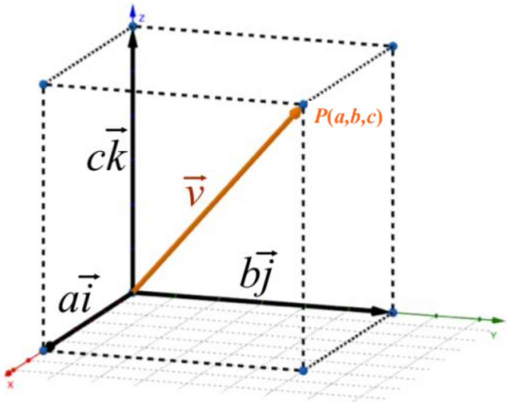
$x$-axis is $\vec{i} = [1,0,0]$

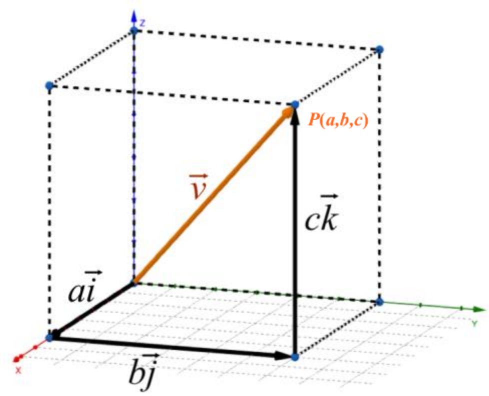
$y$-axis is $\vec{j} = [0,1,0]$

$z$-axis is $\vec{k} = [0,0,1]$



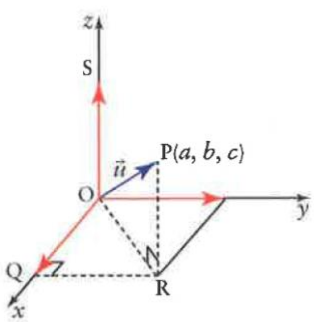
3-D vectors can be written as the sum of multiples of $\vec{\imath}, \vec{\jmath}$, and $\vec{k}$.




$$\vec{v} = [a, b, c] = [a, 0,0] + [0, b, 0] + [0,0, c] = a\vec{\imath} + b\vec{\jmath} + c\vec{k}$$

## Part 3: Magnitude of Vectors in $R^3$




$|\vec{u}|^2 = |\overrightarrow{OP}|^2$

$|\vec{u}|^2 = |\overrightarrow{OR}|^2 + |\overrightarrow{RP}|^2$

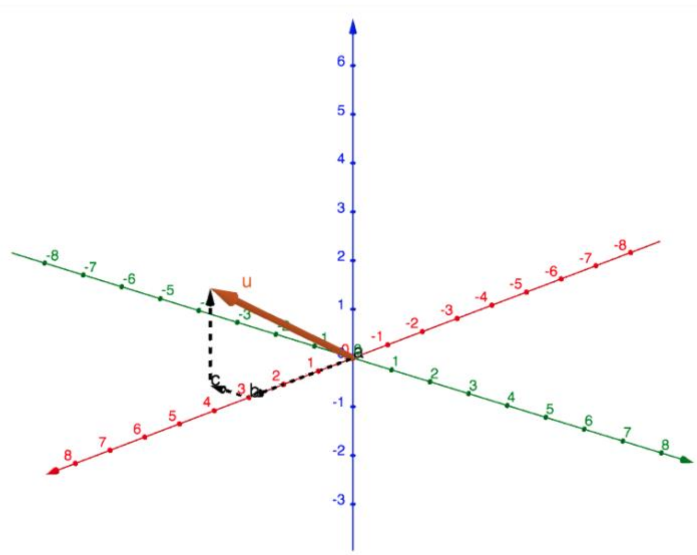
$|\vec{u}|^2 = (a^2 + b^2) + c^2$

$|\vec{u}|^2 = a^2 + b^2 + c^2$

$|\vec{\boldsymbol{u}}| = \sqrt{\boldsymbol{a^2 + b^2 + c^2}}$

**Example 2:** For $\vec{u} = [3, -1, 2]$…

**a)** sketch the position vector



**b)** write the vector in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$

$$\vec{u} = 3\vec{i} - 1\vec{j} + 2\vec{k}$$

**c)** find the magnitude

$$|\vec{u}| = \sqrt{(3)^2 + (-1)^2 + (2)^2}$$

$$|\vec{u}| = \sqrt{14}$$

**Example 3:** For the points $A(1,3,1)$ and $B$(5, 4, -2)

**a)** Find the magnitude of $\overrightarrow{AB}$

$$\overrightarrow{AB} = [5 - 1, 4 - 3, -2 - 1]$$

$$\overrightarrow{AB} = [4, 1, -3]$$

$$|\overrightarrow{AB}| = \sqrt{(4)^2 + (1)^2 + (-3)^2}$$

$$|\overrightarrow{AB}| = \sqrt{26}$$

Vector between 2 points:

$$\overrightarrow{P_1P_2} = [x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1]$$

**b)** Find a unit vector, $\vec{u}$, in the same direction as $\overrightarrow{AB}$

$$\vec{u} = \frac{1}{|\overrightarrow{AB}|}\overrightarrow{AB}$$

$$\vec{u} = \frac{1}{\sqrt{26}}[4, 1, -3]$$

$$\vec{u} = \left[\frac{4}{\sqrt{26}}, \frac{1}{\sqrt{26}}, \frac{-3}{\sqrt{26}}\right]$$

**Tools for 2-D vectors modified for 3-D vectors:**

**Vector Addition:** $\vec{u} + \vec{v} = [u_x + v_x,\ u_y + v_y,\ u_z + v_z]$

**Vector Subtraction:** $\vec{u} - \vec{v} = [u_x - v_x,\ u_y - v_y,\ u_z - v_z]$

**Vector between 2 points:** $\overrightarrow{P_1P_2} = [x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1]$

**Magnitude of a vector between 2 points:** $\left|\overrightarrow{P_1P_2}\right| = \sqrt{(x_2 - x_1)^2 + (\,y_2 - y_1)^2 + (\,z_2 - z_1)^2}$

**Dot Product:** for $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$, $\quad \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

**Example 4:** Given the vectors $\vec{u} = [2,3,-5]$, $\vec{v} = [8,-4,3]$, and $\vec{w} = [-6,-2,0]$, simplify each vector expression.

**a)** $-3\vec{v}$

$= -3[8,-4,3]$

$= [-24,12,-9]$

**b)** $\vec{u} + \vec{v} + \vec{w}$

$= [2,3,-5] + [8,-4,3] + [-6,-2,0]$

$= [2 + 8 + (-6), 3 + (-4) + (-2), -5 + 3 + 0]$

$= [4,-3,-2]$

**c)** $|\vec{u} - \vec{v}|$

$\vec{u} - \vec{v} = [2,3,-5] - [8,-4,3]$

$\vec{u} - \vec{v} = [2 - 8, 3 - (-4), -5 - 3]$

$\vec{u} - \vec{v} = [-6,7,-8]$

$|\vec{u} - \vec{v}| = \sqrt{(-6)^2 + (7)^2 + (-8)^2}$

$|\vec{u} - \vec{v}| = \sqrt{149}$

**d)** $\vec{u} \cdot \vec{v}$

$= [2,3,-5] \cdot [8,-4,3]$

$= 2(8) + 3(-4) + (-5)(3)$

$= 16 - 12 - 15$

$= -11$

**Example 5:** Determine if the vectors $\vec{a} = [6,2,4]$ and $\vec{b} = [9,3,6]$ are collinear.

Check if $\vec{b}$ is a scalar multiple of $\vec{a}$

$$[6,2,4] = k[9,3,6]$$

$x$:

$$6 = 9k$$
$$k = \frac{2}{3}$$

$y$:

$$2 = 3k$$
$$k = \frac{2}{3}$$

$z$:

$$4 = 6k$$
$$k = \frac{2}{3}$$

Therefore, $\vec{a}$ and $\vec{b}$ are collinear.

**Example 6:** Find $a$ such that $[1,2,3]$ and $[2, a, 6]$ are collinear.

$$[1,2,3] = k[2, a, 6]$$

$x$:

$$1 = 2k$$
$$k = \frac{1}{2}$$

$y$:

$$2 = ak$$
$$2 = a\left(\frac{1}{2}\right)$$
$$\boldsymbol{a = 4}$$

$z$:

$$3 = 6k$$
$$k = \frac{1}{2}$$

**Example 7:** Calculate the angle between $\vec{u} = [0, -1, -4]$ and $\vec{v} = [6, 1, -2]$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$$

$$\cos\theta = \frac{0(6) + (-1)(1) + (-4)(-2)}{\left[\sqrt{(0)^2 + (-1)^2 + (-4)^2}\right]\left[\sqrt{(6)^2 + (1)^2 + (-2)^2}\right]}$$

$$\cos\theta = \frac{7}{(\sqrt{17})(\sqrt{41})}$$

$$\theta \cong 74.6°$$

Angle between 2 vectors can be found using:

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

**Example 8:** Find a vector that is orthogonal to $[3,4,5]$

If $[x, y, z]$ is orthogonal to $[3,4,5]$,

$[x, y, z] \cdot [3,4,5] = 0$

$3x + 4y + 5z = 0$

An infinite number of vectors would satisfy this equation. Select values for any two variables and then solve for the third.

$3(2) + 4(1) + 5z = 0$

$5z = -10$

$z = -2$

$[2,1,-2]$ is orthogonal to $[3,4,5]$

**Shortcut:**

Make one variable equal 0, then swap the other two and change one sign.

Examples: $[0,5,-4]$, $[0-5,4]$, $[5,0,-3]$, $[-5,0,3]$, $[-4,3,0]$, and $[4,-3,0]$ would all be orthogonal to $[3,4,5]$

**Example 9:** Find the magnitude of the projection of $\vec{u} = [3,-3,2]$ onto $\vec{v} = [5,2,0]$

Formula reminder:

$$|proj_{\vec{b}}\, \vec{a}| = \left|\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\right|$$

$$|proj_{\vec{v}}\, \vec{u}| = \left|\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}\right|$$

$$|proj_{\vec{v}}\, \vec{u}| = \left|\frac{3(5) + (-3)(2) + 2(0)}{\sqrt{5^2 + 2^2 + 0^2}}\right|$$

$$|proj_{\vec{v}}\, \vec{u}| = \left|\frac{9}{\sqrt{29}}\right|$$

$$|proj_{\vec{v}}\, \vec{u}| = \frac{9}{\sqrt{29}} \cong 1.67$$