Part 1: Warm-Up

Find the intervals of concavity and the coordinates of any points of inflection for $y = \frac{1}{3}x^3 - 12x^2 + 5$

$y' = x^2 - 24x$

$y'' = 2x - 24$

$0 = 2(x - 12)$

$x = 12$

Possible point of inflection:

$$y = \frac{1}{3}(12)^3 - 12(12)^2 + 5 = -1147$$

$(12, -1147)$ *is a possible point of inflection*

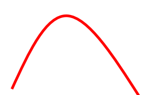
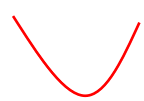
Remember:

$f''(x) = 0$ or undefined is a possible POI

If $f''(x) < 0$, $f(x)$ is concave DOWN

If $f''(x) > 0$, $f(x)$ is concave UP

$-\infty$ 12 ∞

Test value for x	11	13
y''	$-$	$+$
y	Concave DOWN 	Concave UP 

POI at
$(12, -1147)$

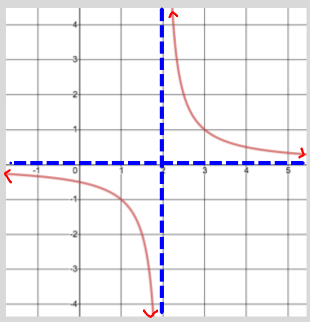
Concave up: $(12, \infty)$

Concave down: $(-\infty, 12)$

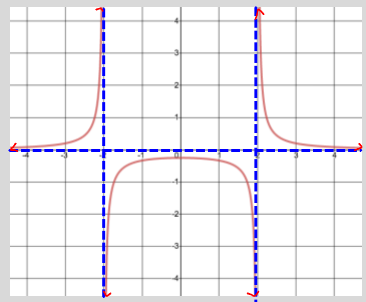
Part 2: Reminder of some simple rational functions

Degree of denominator > degree of numerator:

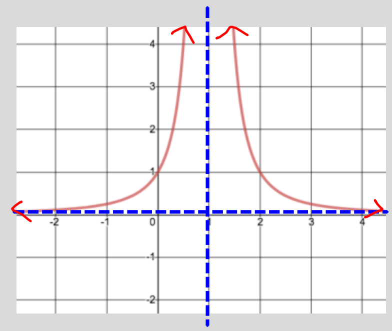
$y = \frac{1}{x-2}$



$y = \frac{1}{x^2-4}$



$y = \frac{1}{(x-1)^2}$

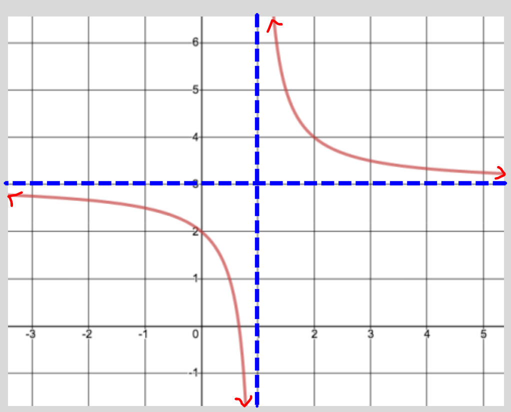


Notice: Horizontal asymptotes all are at $y = 0$

Vertical asymptotes are at zeros of the denominator

Degree of denominator = degree of numerator:

$y = \frac{3x-2}{x-1}$

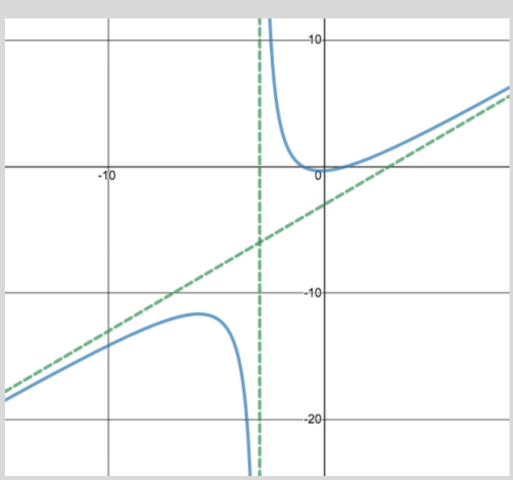


Notice: HA at quotient of leading coefficients

VA at zero of the denominator

Degree of denominator < degree of numerator:

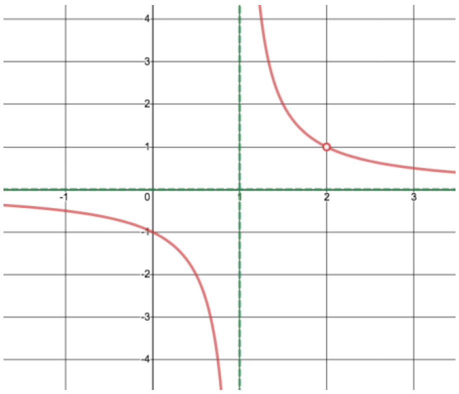
$y = \frac{x^2-1}{x+3}$



Notice: Oblique asymptote at quotient of numerator and denominator; VA at zero of the denominator

Vertical Asymptote vs. Hole in Graph

$f(x) = \frac{(x-2)}{(x-1)(x-2)}$



Notice: VA at $x = 1$; $f(1) = \frac{-1}{0}$

Hole at $(2, 1)$; $f(2) = \frac{0}{0}$

(remove discontinuity to find y-value of hole)

Conclusion: If $f(a) = \frac{\#}{0}$, $x = a$ is a VA

If $f(a) = \frac{0}{0}$, there is a hole in the graph when $x = a$

Limit Definition of Asymptotes:

For the rational function $y = \frac{f(x)}{g(x)}$

There is a Vertical Asymptote at $x = a$ when $g(a) = 0$ and $\lim_{x \to a} \frac{f(x)}{g(x)} = \pm\infty$

There is a Horizontal Asymptote at $y = L$ when $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = L$

Note: Horizontal asymptote only exists if the degree of the numerator is **less than or equal to** the degree of the denominator.

Part 3: Apply What You Know to Graph Rational Functions

Example 1: State the Horizontal Asymptotes of the following functions:

a) $y = \frac{3x^2+2}{6x^2-4x-1}$

HA: $y = \frac{3}{6} = \frac{1}{2}$

b) $y = \frac{3x^2+2}{6x^3-4x-1}$

HA: $y = 0$

Example 2: Consider the function $f(x) = \frac{1}{(x+2)(x-3)}$

a) Find the asymptotes

HA: $y = 0$

VA: $x = -2$ and $x = 3$

b) Find the one-sided limits as the x-values approach the vertical asymptotes (sub values very close to the limit for x, and find what the value of the function is approaching)

$$\lim_{x \to -2^-} \frac{1}{(x+2)(x-3)} = \infty$$

Test: $f(-2.00001) = 19999.96$; therefore going towards $+\infty$

$$\lim_{x \to -2^+} \frac{1}{(x+2)(x-3)} = -\infty$$

Test: $f(-1.9999) = -2000.04$; therefore going towards $-\infty$

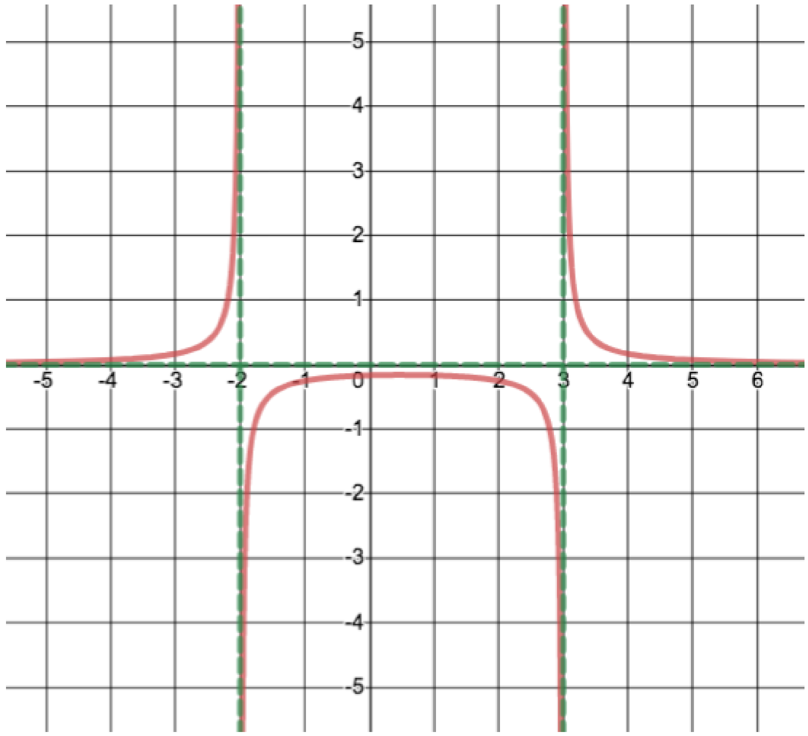
$$\lim_{x \to 3^-} \frac{1}{(x+2)(x-3)} = -\infty$$

Test: $f(2.9999) = -2000.04$; therefore going towards $-\infty$

$$\lim_{x \to 3^+} \frac{1}{(x+2)(x-3)} = \infty$$

Test: $f(3.00001) = 19999.96$; therefore going towards $+\infty$

c) Sketch the graph



Example 3: Consider the function $f(x) = \frac{1}{x^2+1}$

a) Where are the vertical and horizontal asymptotes?

HA: $y = 0$
VA: none $\quad x^2 + 1 \neq 0$

b) Find any local max/min points and the intervals of increase/decrease

$$f'(x) = \frac{0(x^2+1) - 2x(1)}{(x^2+1)^2}$$

$$f'(x) = \frac{-2x}{(x^2+1)^2}$$

Find Critical Numbers:

$$0 = \frac{-2x}{(x^2+1)^2}$$

$$0 = -2x$$

$$x = 0$$

$$x = 0$$

Critical Point: $(0,1)$

$$f(0) = 1$$

$-\infty \qquad 0 \qquad \infty$

Test value for x	-1	1
$f'(x)$	$+$	$-$
$f(x)$	Increasing ↗	Decreasing ↘

Local max at $(0, 1)$

Increasing: $(-\infty, 0)$

Decreasing: $(0, \infty)$

c) Find the points of inflection

$$f'(x) = \frac{-2x}{(x^2+1)^2}$$

$$f''(x) = \frac{-2(x^2+1)^2 - 2(x^2+1)(2x)(-2x)}{(x^2+1)^4}$$

$$f''(x) = \frac{-2(x^2+1) - 2(2x)(-2x)}{(x^2+1)^3}$$

$$f''(x) = \frac{-2x^2 - 2 + 8x^2}{(x^2+1)^3}$$

$$f''(x) = \frac{6x^2 - 2}{(x^2+1)^3}$$

Find Possible POI's:

$$0 = \frac{6x^2 - 2}{(x^2+1)^3}$$

$$0 = 6x^2 - 2$$

$$x^2 = \frac{2}{6}$$

$$x = \pm\frac{1}{\sqrt{3}} \cong 0.577$$

$(0.577, 0.75)$ and $(-0.577, 0.75)$

$-\infty$ $\quad -0.577 \quad 0.577 \quad \infty$

Test value for x	-1	0	1
$f''(x)$	$+$	$-$	$+$
$f(x)$	Concave UP	Concave DOWN	Concave UP

POI at $(-0.577, 0.75)$

POI at $(0.577, 0.75)$

d) Sketch a graph of the function