L5 – The Chain Rule **Unit 1**
MCV4U
Jensen

A composite function consists of an outer function, $g(x)$, and an inner function, $h(x)$. The chain rule says to differentiate outer function with respect to the inner function, and then multiply by the derivative of the inner function.

Given two differentiable functions $g(x)$ and $h(x)$, the derivative of the composite function $f(x) = g[h(x)]$ is $f'(x) = g'[h(x)] \times h'(x)$

A special case of the chain rule is the 'Power of a Function Rule'. This occurs when the outer function is a power function:

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$$

Summary of Derivative Rules:

Rule	Derivative
Power Rule If $f(x) = x^n$	$f'(x) = nx^{n-1}$
Constant Multiple Rule If $f(x) = c \cdot g(x)$ where c is a constant	$f'(x) = c \cdot g'(x)$
Sum Rule If $h(x) = f(x) + g(x)$	$h'(x) = f'(x) + g'(x)$
Difference Rule If $h(x) = f(x) - g(x)$	$h'(x) = f'(x) - g'(x)$
Product Rule If $h(x) = f(x)g(x)$	$h'(x) = f'(x)g(x) + f(x)g'(x)$
Quotient Rule If $h(x) = f(x) \div g(x)$	$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$
Power of a Function Rule If $h(x) = [f(x)]^n$	$h'(x) = n[f(x)]^{n-1} \times f'(x)$
Chain Rule If $h(x) = f[g(x)]$	$h'(x) = f'[g(x)] \times g'(x)$

Example 1: Differentiate each function using the chain rule. Express in simplified factored form.

a) $f(x) = (3x - 5)^4$

$f'(x) = 4(3x - 5)^3(3)$

$f'(x) = 12(3x - 5)^3$

b) $g(x) = \sqrt{4 - x^2}$

$g'(x)$
$= \frac{1}{2}(4 - x^2)^{-\frac{1}{2}}(-2x)$

$g'(x) = \frac{-x}{\sqrt{4 - x^2}}$

c) y=$\left(2\sqrt{x}\right)^3 - 4\sqrt{x} + 1$

$$\frac{dy}{dx} = 3\left(2\sqrt{x}\right)^2(x)^{-\frac{1}{2}} - 2x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = 3(4x)(x)^{-\frac{1}{2}} - 2x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = (12x)(x)^{-\frac{1}{2}} - 2x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{2(6x-1)}{\sqrt{x}}$$

d) $h(x) = (x^2+3)^4(4x-5)^3$

$$h'(x) = 4(x^2+3)^3(2x)(4x-5)^3 + 3(4x-5)^2(4)(x^2+3)^4$$

$$h'(x) = 4(x^2+3)^3(4x-5)^2[2x(4x-5) + 3(x^2+3)]$$

$$h'(x) = 4(x^2+3)^3(4x-5)^2[11x^2 - 10x + 9]$$

Example 2: Determine an equation for the tangent to $f(x) = 3x(1-x)^2$ at $x = 0.5$

Point on Tangent Line:

$$f(0.5) = 3(0.5)(1-0.5)^2$$

$$f(0.5) = \frac{3}{8}$$

Equation of line:

$$y = mx + b$$

$$\frac{3}{8} = \left(-\frac{3}{4}\right)\left(\frac{1}{2}\right) + b$$

$$b = \frac{3}{4}$$

$$y = -\frac{3}{4}x + \frac{3}{4}$$

Slope of tangent line:

$$f'(x) = 3(1-x)^2 + 2(1-x)(-1)(3x)$$

$$f'(x) = 3(1-x)^2 - 6x(1-x)$$

$$f'(x) = 3(1-x)[(1-x) - 2x]$$

$$f'(x) = 3(1-x)(1-3x)$$

$$f'(0.5) = 3(1-0.5)[1-3(0.5)]$$

$$f'(0.5) = -\frac{3}{4}$$