L5 – Cross Product of Vectors Unit 5

MCV4U

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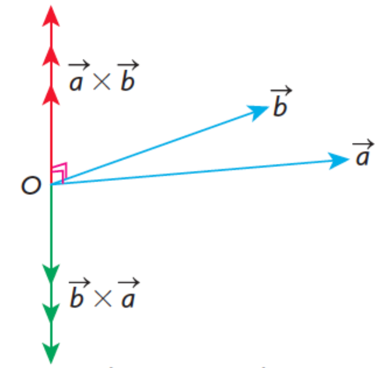
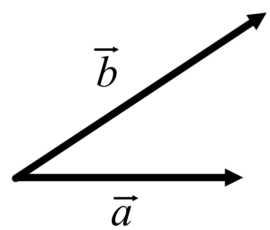
The cross product of two vectors $\vec{a}$ and $\vec{b}$ in R^3 is the vector that is **perpendicular** to these vectors such that the vectors $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right-handed system.

Right handed system tells you to point your hand along vector $\vec{a}$ and curl your fingers towards vector $\vec{b}$. Your thumb will be pointing in the direction of $\vec{a} \times \vec{b}$. Notice that $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ point in opposite directions.



The Cross Product (Vector Product)

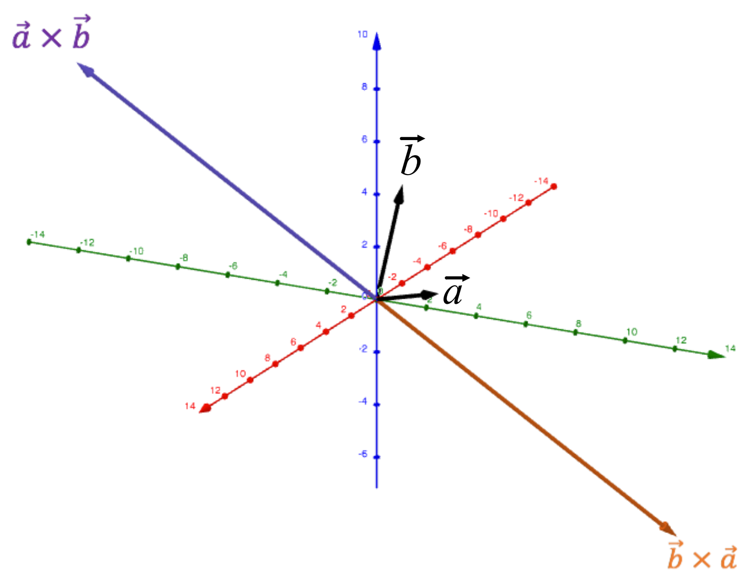
Sometimes the direction of the cross product is defined by either 'in to the page' or 'out of the page':



$\vec{a} \times \vec{b}$	$\vec{b} \times \vec{a}$
"Out of the page"	"In to the page"

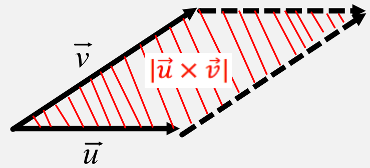
3D visualization:

https://www.geogebra.org/3d/jyqcr3bf



Properties of Cross Product:

- $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
- If $\vec{u} \times \vec{v} = 0$, and $\vec{u}$ and $\vec{v}$ are non-zero, then $\vec{u}$ and $\vec{v}$ are collinear.
- $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$
- $|\vec{u} \times \vec{v}| =$ the area of the parallelogram defined by $\vec{u}$ and $\vec{v}$



Part 2: Cross Product of Geometric Vectors

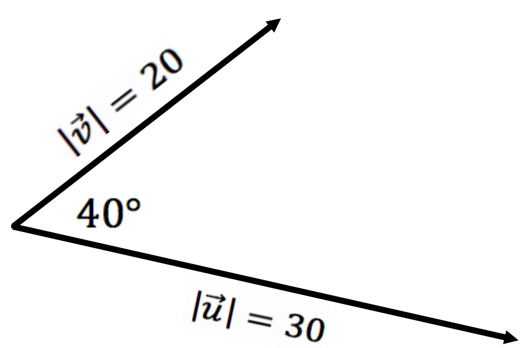
Formula: $\vec{a} \times \vec{b} = \left(|\vec{a}||\vec{b}| \sin\theta\right)\hat{n}$

θ is the angle between the vectors
$\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

Example 1: If $|\vec{u}| = 30$, $|\vec{v}| = 20$, the angle between $\vec{u}$ and $\vec{v}$ is $40°$, and $\vec{u}$ and $\vec{v}$ are in the plane of the page, find…

a) $\vec{u} \times \vec{v}$

$\vec{u} \times \vec{v} = (|\vec{u}||\vec{v}| \sin\theta)\hat{n}$
$\vec{u} \times \vec{v} = [(30)(20) \sin 40]\,\hat{n}$
$\vec{u} \times \vec{v} \cong 385.7\hat{n}$
$\vec{u} \times \vec{v} \cong 385.7$ out of the page



b) $\vec{v} \times \vec{u}$

$\vec{v} \times \vec{u} = (|\vec{v}||\vec{u}| \sin\theta)\hat{n}$
$\vec{v} \times \vec{u} = [(20)(30) \sin 40]\,(-\hat{n})$
$\vec{v} \times \vec{u} \cong -385.7\hat{n}$
$\vec{v} \times \vec{u} \cong 385.7$ in to the page

Part 3: Cross Product of Algebraic Vectors

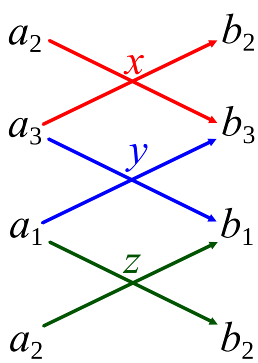
Formula:

If $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$

$$\vec{a} \times \vec{b} = [\, a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1]$$

$$= (a_2b_3 - a_3b_2)\vec{\imath} + (a_3b_1 - a_1b_3)\vec{\jmath} + (a_1b_2 - a_2b_1)\vec{k}$$

How to set it up:



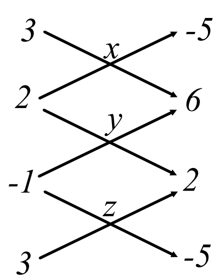
$x = a_2b_3 - a_3b_2$

$y = a_3b_1 - a_1b_3$

$z = a_1b_2 - a_2b_1$

Example 2: If $\vec{p} = [-1,3,2]$ and $\vec{q} = [2,-5,6]$, calculate each of the following:

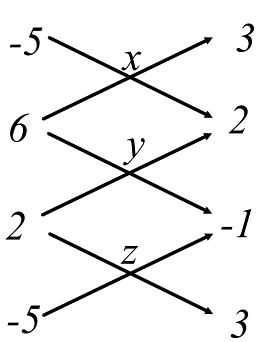
a) $\vec{p} \times \vec{q}$



$\vec{p} \times \vec{q} = [3(6) - 2(-5), 2(2) - (-1)(6), (-1)(-5) - (3)(2)]$

$\vec{p} \times \vec{q} = [28,10,-1]$

b) $\vec{q} \times \vec{p}$

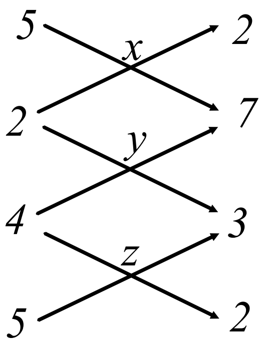


$\vec{q} \times \vec{p} = [(-5)(2) - 6(3), 6(-1) - 2(2), 2(3) - (-5)(-1)]$

$\vec{q} \times \vec{p} = [-28,-10,1]$

Notice $\vec{q} \times \vec{p} = -1(\vec{p} \times \vec{q})$

Example 3a: Determine the area of the parallelogram defined by the vectors $\vec{u} = [4,5,2]$ and $\vec{v} = [3,2,7]$.



$\vec{u} \times \vec{v} = [5(7) - 2(2), 2(3) - 4(7), 4(2) - 5(3)]$

$\vec{u} \times \vec{v} = [31, -22, -7]$

$|\vec{u} \times \vec{v}| = \sqrt{(31)^2 + (-22)^2 + (-7)^2}$

$|\vec{u} \times \vec{v}| = \sqrt{1494}$

$|\vec{u} \times \vec{v}| \cong 38.7$ units^2

Example 3b: Determine the angle between the vectors $\vec{u}$ and $\vec{v}$.

$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$

$$\sin\theta = \frac{|\vec{u} \times \vec{v}|}{|\vec{u}||\vec{v}|}$$

$$\sin\theta = \frac{\sqrt{1494}}{\left(\sqrt{4^2 + 5^2 + 2^2}\right)\left(\sqrt{3^2 + 2^2 + 7^2}\right)}$$

$$\sin\theta = \frac{\sqrt{1494}}{\left(\sqrt{45}\right)\left(\sqrt{62}\right)}$$

$\theta \cong 47°$ OR $\theta \cong 180 - 47 = 133°$

From an inspection of the vectors we can tell that $\theta \cong 47°$.