Algorithm for Curve Sketching

1. Determine any restrictions in the domain. State any horizontal and vertical asymptotes or holes in the graph.
2. Determine the intercepts of the graph
3. Determine the critical numbers of the function (where is $f'(x) = 0$ or undefined)
4. Determine the possible points of inflection (where is $f''(x)$=0 or undefined)
5. Create a sign chart that uses the critical numbers and possible points of inflection as dividing points.
6. Use the sign chart to find intervals of increase/decrease and the intervals of concavity. Use all critical numbers, possible points of inflection, and vertical asymptotes as dividing points.
7. Identify local extrema and points of inflection
8. Sketch the function

Example 1: Use the algorithm for curve sketching to analyze the key features of each of the following functions and to sketch a graph of them.

a) $g(x) = x^3 + 6x^2 + 9x$

1. No restrictions on the domain; no asymptotes

2. x-intercepts:

$0 = x^3 + 6x^2 + 9x$

$0 = x(x^2 + 6x + 9)$

$0 = x(x + 3)^2$

$x = 0$ or $x = -3$

$(0, 0)$ and $(-3, 0)$ are x-intercepts

y-intercept:

$g(0) = 0$

$(0, 0)$ is the y-intercept

3. $g'(x) = 3x^2 + 12x + 9$

$0 = 3(x^2 + 4x + 3)$

$0 = 3(x + 3)(x + 1)$

Critical Numbers: $x = -3$ and $x = -1$

Critical Points: $(-3, 0)$ and $(-1, -4)$

4. $g''(x) = 6x + 12$

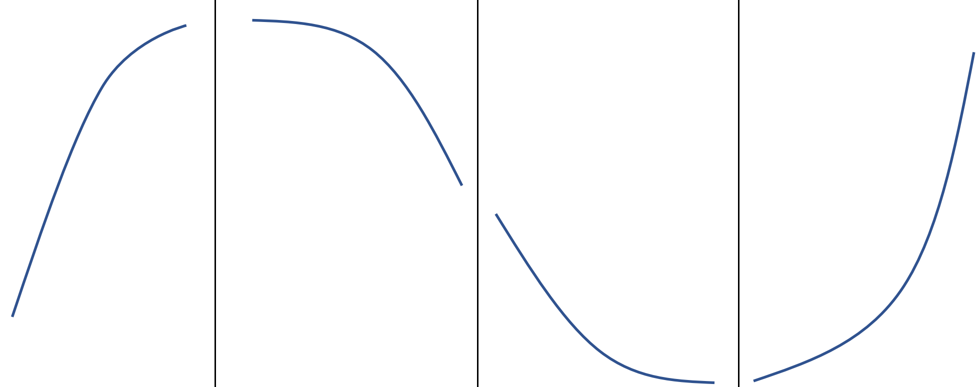
$0 = 6(x + 2)$

$x = -2$

$(-2, -2)$ is a possible point of inflection

5/6/7.

	$-\infty$ to -3	-3 to -2	-2 to -1	-1 to ∞
Test value for x	-4	-2.5	-1.5	0
$g'(x)$	$+$	$-$	$-$	$+$
$g''(x)$	$-$	$-$	$+$	$+$
$g(x)$	Concave Down Increasing	Concave Down Decreasing	Concave UP Decreasing	Concave UP Increasing

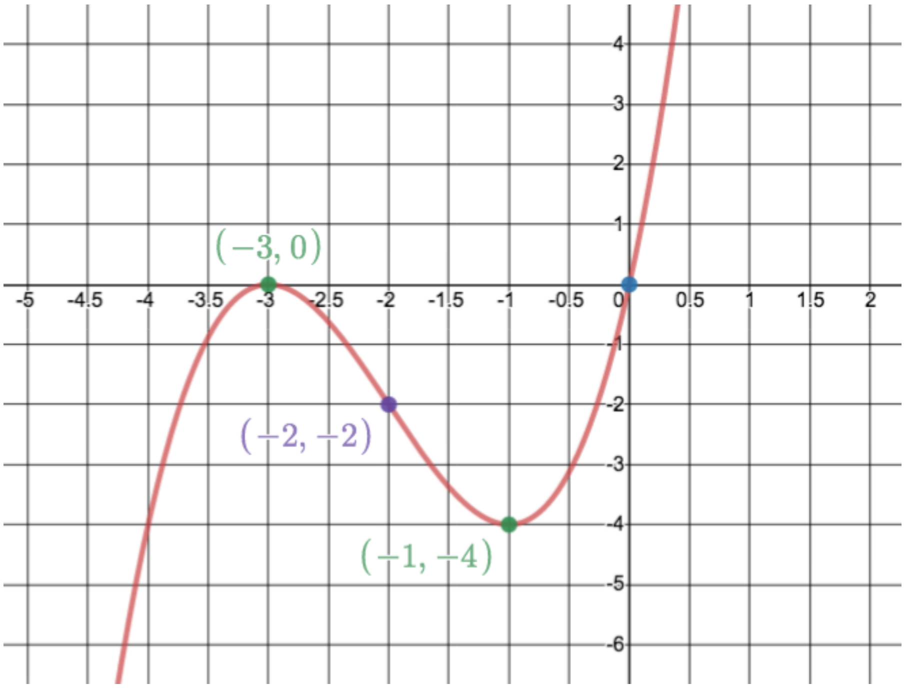


Local max at $(-3, 0)$

POI at $(-2, -2)$

Local min at $(-1, -4)$

8.



b) $f(x) = \frac{1}{(x+1)(x-4)} = \frac{1}{x^2-3x-4}$

1. VA: $x = -1$ and $x = 4$ (from restrictions on denominator)

HA: $y = 0$ (degree in denominator is higher than degree in numerator)

2. x-int: NONE

y-int: $f(0) = \frac{1}{(0+1)(0-4)} = -\frac{1}{4}$ $(0, -0.25)$

3. $f'(x) = \frac{0(x+1)(x-4)-(2x-3)(1)}{(x^2-3x-4)^2}$

$f'(x) = \frac{-2x+3}{(x^2-3x-4)^2}$

Check for zeros:

$0 = -2x + 3$

$x = \frac{3}{2} = 1.5$

Check for restrictions:

$0 = (x^2 - 3x - 4)^2$

$0 = [(x-4)(x+1)]^2$

$x = 4$ and $x = -1$

However, these are not in the domain of $f(x)$, therefore are NOT critical numbers

Critical Number: $x = \frac{3}{2} = 1.5$

Critical Point: $f(1.5) = -0.16 \quad (1.5, -0.16)$

4. $f''(x) = \frac{-2(x^2-3x-4)^2 - 2(x^2-3x-4)(2x-3)(-2x+3)}{(x^2-3x-4)^4}$

$$f''(x) = \frac{(x^2 - 3x - 4)[-2(x^2 - 3x - 4) - 2(2x - 3)(-2x + 3)}{(x^2 - 3x - 4)^4}$$

$$f''(x) = \frac{-2(x^2 - 3x - 4) - 2(2x - 3)(-2x + 3)}{(x^2 - 3x - 4)^3}$$

$$f''(x) = \frac{-2x^2 + 6x + 8 - 2(-4x^2 + 12x - 9)}{(x^2 - 3x - 4)^3}$$

$$f''(x) = \frac{-2x^2 + 6x + 8 + 8x^2 - 24x + 18}{(x^2 - 3x - 4)^3}$$

$$f''(x) = \frac{6x^2 - 18x + 26}{(x^2 - 3x - 4)^3}$$

Check for zeros:

$0 = 6x^2 - 18x + 26$

$b^2 - 4ac = (-18)^2 - 4(6)(26) = -300$

$b^2 - 4ac < 0$ therefore no real solutions

$f(x)$ has not points of inflection

Check for restrictions:

$0 = (x^2 - 3x - 4)^3$

$0 = [(x - 4)(x + 1)]^3$

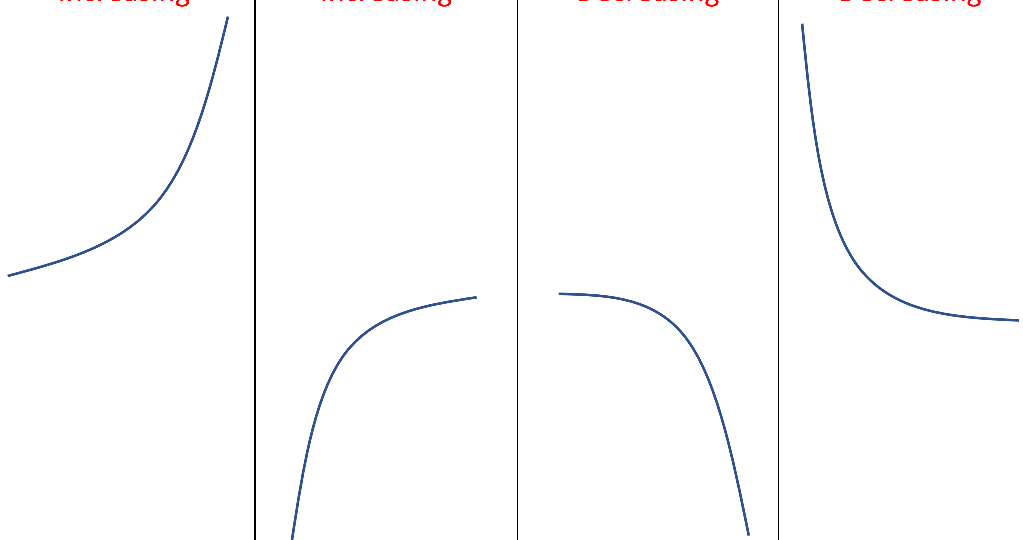
$x = 4$ and $x = -1$

Possible changes in concavity at $x = -1$ and $x = 4$ but $f(x)$ is not defined at these values either, therefore they can NOT be considered points of inflection.

5/6/7.

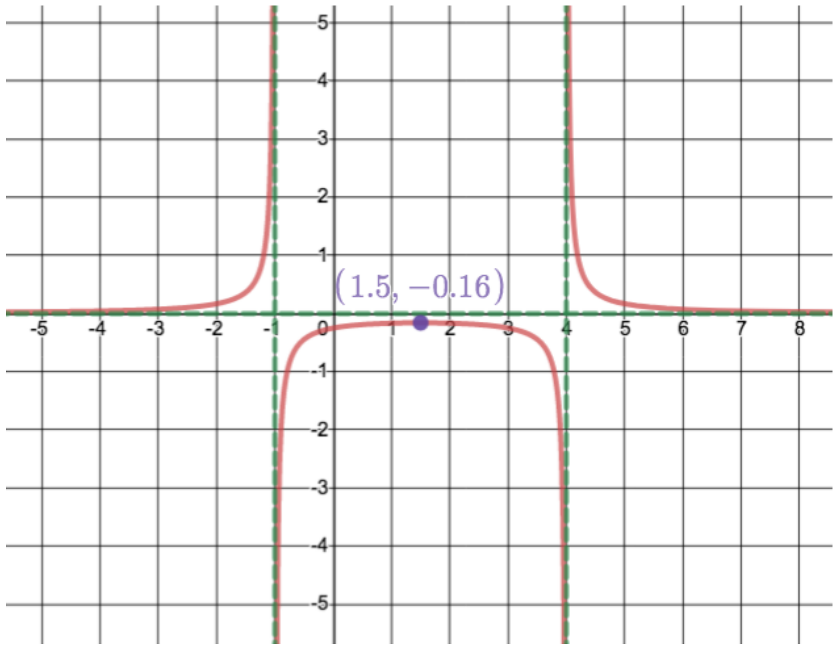
$-\infty$ $\quad -1 \quad 1.5 \quad 4 \quad \infty$

Test value for x	-2	0	2	5
$f'(x)$	$+$	$+$	$-$	$-$
$f''(x)$	$+$	$-$	$-$	$+$
$f(x)$	Concave UP Increasing	Concave DOWN Increasing	Concave DOWN Decreasing	Concave UP Decreasing



Local Max at $(1.5, -0.16)$

8.



c) $h(x) = x^4 - 5x^3 + x^2 + 21x - 18$

1. No restrictions; no asymptotes

2. x-int:

$0 = x^4 - 5x^3 + x^2 + 21x - 18$

$0 = (x-1)(x^3 - 4x^2 - 3x + 18)$

Factor: $x^4 - 5x^3 + x^2 + 21x - 18$

Possible zeros: $\pm 1, 2, 3, 4, 6, 9, 18$

$h(1) = 0$; $x - 1$ is a factor

1	1	-5	1	21	-18
		1	-4	-3	18
	1	-4	-3	18	0

$0 = x^4 - 5x^3 + x^2 + 21x - 18$

$0 = (x-1)(x^3 - 4x^2 - 3x + 18)$

Factor: $x^3 - 4x^2 - 3x + 18$

Possible zeros: $\pm 1, 2, 3, 4, 6, 9, 18$

$h(-2) = 0$; $x + 2$ is a factor

-2	1	-4	-3	18
		-2	12	-18
	1	-6	9	0

$0 = (x-1)(x+2)(x^2 - 6x + 9)$

$0 = (x-1)(x+2)(x-3)^2$

$x = 1 \quad x = -2 \quad x = 3$

$(1, 0)$, $(-2, 0)$ and $(3, 0)$

y-int: $h(0) = -18 \quad (0, -18)$

3. $h'(x) = 4x^3 - 15x^2 + 2x + 21$

$0 = 4x^3 - 15x^2 + 2x + 21$

$0 = (x+1)(4x^2 - 19x + 21)$

$0 = (x+1)[4x^2 - 7x - 12x + 21]$

$0 = (x+1)[x(4x-7) - 3(4x-7)]$

$0 = (x+1)(4x-7)(x-3)$

Factor: $4x^3 - 15x^2 + 2x + 21$

Possible zeros: $\pm 1, \frac{1}{2}, \frac{1}{4}, 3, \frac{3}{2}, \frac{3}{4}, 7, \frac{7}{2}, \frac{7}{4}, 21, \frac{21}{2}, \frac{21}{4}$

$h'(-1) = 0$; $x + 1$ is a factor

-1	4	-15	2	21
		-4	19	-21
	4	-19	21	0

Critical Numbers: $x = -1, \frac{7}{4}, 3$

Critical Points: $(-1, -32)$, $(1.75, 4.39)$, and $(3, 0)$

4. $h''(x) = 12x^2 - 30x + 2$

$0 = 2(6x^2 - 15x + 1)$

$0 = 6x^2 - 15x + 1$

$$x = \frac{15 \pm \sqrt{(-15)^2 - 4(6)(1)}}{2(6)}$$

$$x = \frac{15 \pm \sqrt{201}}{12}$$

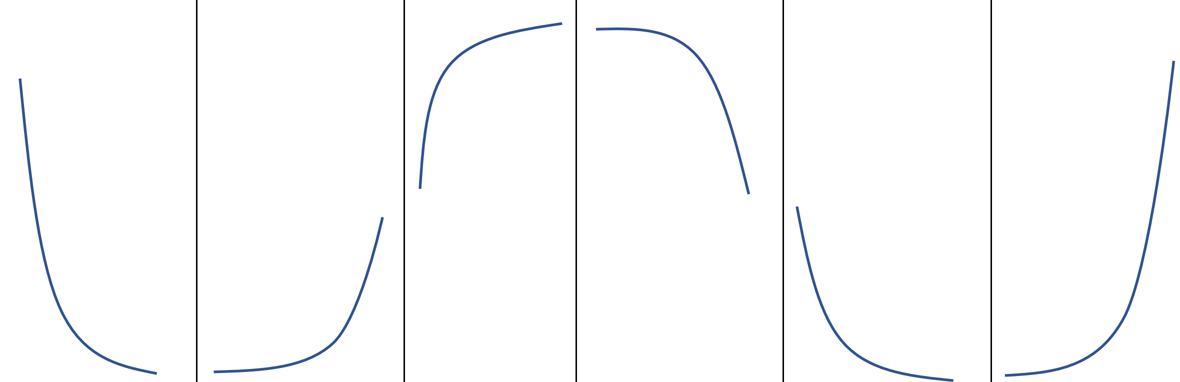
$x \cong 2.43$ and $x \cong 0.07$

Possible points of inflection: $(2.43, 2.05)$ and $(0.07, -16.56)$

5/6/7.

$-\infty$ -1 0.07 1.75 2.43 3 ∞

Test valu for x	-2	0	1	2	2.5	4
$h'(x)$	$-$	$+$	$+$	$-$	$-$	$+$
$h''(x)$	$+$	$+$	$-$	$-$	$+$	$+$
$h(x)$	Decreasing Concave UP	Increasing Concave UP	Increasing Concave DOWN	Decreasing Concave DOWN	Decreasing Concave UP	Increasing Concave UP



Local min at $(-1,-32)$

POI at $(0.07,-16.56)$

Local max at $(1.75, 4.39)$

POI at $(2.43, 2.05)$

Local min at $(3, 0)$