L6 – Implicit Differentiation and Derivatives of Log Functions Unit 3
MCV4U
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Part 1: Chain Rule Using Leibniz Notation

If y is a function of u and u is a function of x, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

Example 1: Suppose we wish to differentiate $y = (5 + 2x)^{10}$ in order to calculate $\frac{dy}{dx}$. We make a substitution and let $u = 5 + 2x$ so that $y = u^{10}$

The chain rule states:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

If $y = u^{10}$ then $\frac{dy}{du} = 10u^9$

If $u = 5 + 2x$ then $\frac{du}{dx} = 2$

Therefore:

$$\frac{dy}{dx} = 10u^9 \times 2$$

$$\frac{dy}{dx} = 20u^9$$

$$\frac{dy}{dx} = 20(5 + 2x)^9$$

Part 2: Implicit Differentiation

Up until now, most functions were written in the form in which y was defined explicitly as a function of$x$, such as $y = x^3 - 4x$. In that equation, y is isolated and is expressed EXPLICITLY as a function of x.

Functions can also be defined implicitly by relations, such as a circle $x^2 + y^2 = 16$. In this case, y is not isolated or explicitly defined in terms of x. You could rearrange to isolate for y but often this is very difficult or impossible. It will be necessary to use the chain rule to implicitly differentiate when it is in implicit form.

$$\frac{d}{dx}f(y) = \frac{d}{dy}f(y) \times \frac{dy}{dx}$$

Example 2: Differentiate each of the following using implicit differentiation

a) $x^2 + y^2 = 16$

$$\frac{d}{dx}x^2 + \frac{d}{dy}y^2 \times \frac{dy}{dx} = \frac{d}{dx}16$$

$$2x + 2y \times \frac{dy}{dx} = 0$$

$$2y \times \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

b) $y^2 + x^3 - y^3 + 6 = 3y$

$$2y(y') + 3x^2 - 3y^2(y') = 3(y')$$

$$2y(y') - 3y^2(y') - 3(y') = -3x^2$$

$$y'(2y - 3y^2 - 3) = -3x^2$$

$$y' = \frac{-3x^2}{2y - 3y^2 - 3}$$

Part 3: Derivative of Logarithms

Proof of the derivative of $y = \log_a x$

Start by writing in inverse form:

$$a^y = x$$

Now use implicit differentiation to differentiate with respect to x

$$\ln a\,(a^y)y' = 1$$

$$y' = \frac{1}{\ln a\,(a^y)}$$

$$y' = \frac{1}{x \ln a}$$

Rule:

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$$

If you need chain rule:

$$\frac{d}{dx}\log_a[f(x)] = \frac{1}{f(x)\ln a}f'(x) = \frac{f'(x)}{f(x)\ln a}$$

Example 3: Differentiate each of the following with respect to x.

a) $y = 2\ln(1 + x^2)$

$$\frac{dy}{dx} = 2\frac{1}{(1 + x^2)\ln e}(2x)$$

$$\frac{dy}{dx} = \frac{4x}{1 + x^2}$$

b) $f(x) = 1 - \log_4(2x - 1)$

$$\frac{dy}{dx} = 0 - \frac{1}{(2x - 1)\ln 4}(2)$$

$$\frac{dy}{dx} = -\frac{2}{(2x - 1)\ln 4}$$