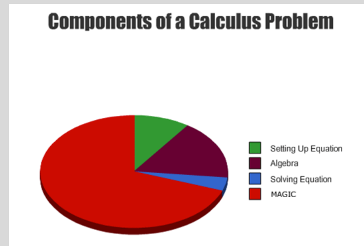
**L6 – Optimization Problems** Unit 2
MCV4U
Jensen

Tips for Optimization Problems:

- Diagrams can be helpful
- Identify the independent variable and express all other variables in terms of it
- Define a function in terms of the independent variable
- Identify any restriction on the variable
- Solve for $f'(x) = 0$ to identify critical points
- Check critical points and endpoints




**Optimization Warm Up:**

A lifeguard has 200 meters of rope and some buoys with which she intends to enclose a rectangular area at a lake for swimming. The beach will form one side of the rectangle, with the rope forming the other 3 sides. Find the dimensions that will produce the maximum enclosed area.

$width = x$
$length = 200 - 2x$

$A = (length)(width)$
$A = x(200 - 2x)$
$A = 200x - 2x^2$

Note: The domain of this function is restricted to values $0 < x < 100$ because there is only 200 meters of rope to use.

To determine the max area, test endpoints of the interval as well as any critical numbers.

Critical Number(s):

$A'(x) = 200 - 4x$
$0 = 200 - 4x$
$4x = 200$
$x = 50$ is a critical number

Tests:

$A(0) = 0[200 - 2(0)]$
$A(0) = 0$ m$^2$

$A(50) = 50[200 - 2(50)]$
$A(50) = 5000$ m$^2$

$A(100) = 100[200 - 2(100)]$
$A(100) = 0$ m$^2$

Therefore, the max area of 5000 m$^2$ occurs when the width is 50m and the length is 100m.

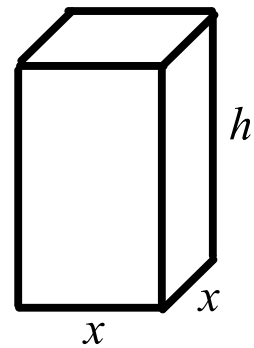
**Example 1:** A cardboard box with a square base is to have a volume of 8 Liters (1 L = 1000 $cm^3$)
Find the dimensions that will minimize the amount of cardboard to be used. What is the minimum surface area?

$SA = 2x^2 + 4xh$

Express $h$ in terms of $x$ using the volume equation

From Volume Equation:

$V = (area\ of\ base)(height)$
$8000 = x^2h$
$h = \frac{8000}{x^2}$




$SA = 2x^2 + 4x\left(\frac{8000}{x^2}\right)$

$SA = 2x^2 + 32000x^{-1}$

Find min value by finding zero(s) of the first derivative:

$SA'(x) = 4x - 32000x^{-2}$

$0 = 4x - 32000x^{-2}$

$32000x^{-2} = 4x$

$32000 = 4x^3$

$8000 = x^3$

$x = 20$ cm

Verify it is a min using the second derivative test:

$SA''(x) = 4 + 64000x^{-3}$

$SA''(20) = 4 + 64000(20)^{-3}$

$SA''(20) = 12$, therefore $SA$ is concave up at $x = 20$, and there is a local min point.

Solve for $h$ when $x = 20$:

$h = \frac{8000}{(20)^2}$

$h = 20$

Find The minimum surface area:

$SA(20) = 2(20)^2 + 32000(20)^{-1}$

$SA(20) = 2400$ cm$^3$

Therefore, a minimum surface area of 2400 cm$^3$ can be obtained when the dimensions of the box are 20 by 20 by 20 cm.

**Example 2:** A soup can of volume 500 $cm^3$ is to be constructed. The material for the top costs 0.4¢/$cm^2$ while the material for the bottom and sides costs 0.2¢/$cm^2$. Find the dimensions that will minimize the cost of producing the can. What is the min cost?

From Volume Equation:

$$V = \pi r^2 h$$
$$500 = \pi r^2 h$$
$$h = \frac{500}{\pi r^2}$$

$$SA = 2\pi r^2 + 2\pi r h$$

$$SA = \pi r^2 + \pi r^2 + 2\pi r h$$

$$SA(r) = \pi r^2 + \pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right)$$

$$SA(r) = \pi r^2 + \pi r^2 + \frac{1000}{r}$$

$$C(r) = 0.4(\pi r^2) + 0.2(\pi r^2) + 0.2\left(\frac{1000}{r}\right)$$

$$C(r) = 0.6\pi r^2 + \frac{200}{r}$$

$$C'(r) = 1.2\pi r - \frac{200}{r^2}$$

$$0 = 1.2\pi r - \frac{200}{r^2}$$

$$\frac{200}{r^2} = 1.2\pi r$$

$$\frac{200}{1.2\pi} = r^3$$

$$r = \sqrt[3]{\frac{200}{1.2\pi}}$$

$$r = 3.76\ cm$$

2nd Derivative Test

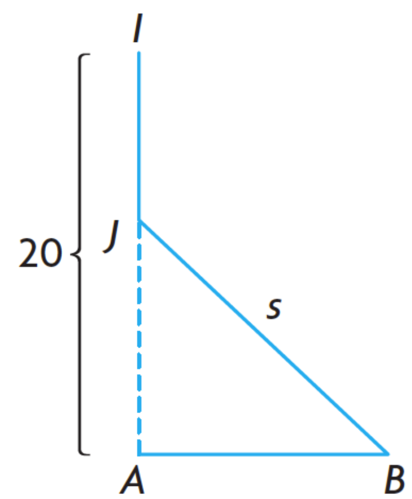
$$C''(r) = 1.2\pi + \frac{400}{r^3}$$

$C''(3.76) = 11.29$; therefore at the point $(3.76, C(3.76))$, $C(r)$ is concave up and the point is a MIN point.

$C(3.76) = 79.84$ cents

A min cost of 79.84 cents can be obtained by using dimensions $r = 3.76$ cm and $h = \frac{500}{\pi(3.76)^2} = 11.26$ cm

**Example 3:** Ian and Ada are both training for a marathon. Ian's house is located 20 km north of Ada's house. At 9:00 am one Saturday, Ian leaves his house and jogs south at 8 km/h. At the same time, Ada leaves her house and jogs east at 6 km/h. When are Ian and Ada closest together, given that they both run for 2.5 hours?




$s^2 = JA^2 + AB^2$

$s = \sqrt{JA^2 + AB^2}$

Express $s$ in terms of time $(t)$

$JA = 20 - 8t$

$AB = 6t$

$s = \sqrt{(20-8t)^2 + (6t)^2}$

$s = [400 - 320t + 64t^2 + 36t^2]^{\frac{1}{2}}$

$s = (100t^2 - 320t + 400)^{\frac{1}{2}}$

Find Critical Number(s):

$s'(t) = \frac{1}{2}(100t^2 - 320t + 400)^{-\frac{1}{2}}(200t - 320)$

$s'(t) = \frac{200t - 320}{2\sqrt{100t^2 - 320t + 400}}$

$s'(t) = \frac{100t - 160}{\sqrt{100t^2 - 320t + 400}}$

$0 = \frac{100t - 160}{\sqrt{100t^2 - 320t + 400}}$

$0 = 100t - 160$

$160 = 100t$

$t = 1.6$

Check endpoints of interval and critical number to determine minimum value:

$s(0) = \sqrt{[20 - 8(0)]^2 + [6(0)]^2} = 20$

$s(1.6) = \sqrt{[20 - 8(1.6)]^2 + [6(1.6)]^2} = 12$

$s(2.5) = \sqrt{[20 - 8(2.5)]^2 + [6(2.5)]^2} = 15$

Therefore, the minimum distance between Ada and Ian occurs after 1.6 hours (10:36 am).