L8 – The Natural Logarithm
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Part 1: What is $'e'$?

Example 1: Suppose you invest $1 at 100% interest for 1 year at various compounding levels. What is the highest amount of money you can have after 1 year?

Note: the formula used for compound interest of $1 at 100% interest annually compounded n times during the year is:

$$A = 1\left(1 + \frac{1}{n}\right)^n$$

Compounding Level, n	**Amount, A in dollars**
Annualy (once a year)	$A = 1\left(1 + \frac{1}{1}\right)^1 = 2$
Semi-annually (2-times)	$A = 1\left(1 + \frac{1}{2}\right)^2 = 2.25$
Quarterly (4-times)	$A = 1\left(1 + \frac{1}{4}\right)^4 = 2.4414$
Monthly (12-times)	$A = 1\left(1 + \frac{1}{12}\right)^{12} = 2.61304$
Daily (365-times)	$A = 1\left(1 + \frac{1}{365}\right)^{365} = 2.71457$
Secondly (31 536 000-times)	$A = 1\left(1 + \frac{1}{31536000}\right)^{31536000} = 2.718281785$
Continuously (1 000 000 000-times)	$A = 1\left(1 + \frac{1}{1000000000}\right)^{1000000000} = 2.718281827$

Properties of e:

- $e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n \approx 2.718\ 281\ 828\ 459$
- e is an **irrational** number, similar to π. They are non-terminating and non-repeating.
- $\log_e x$ is known as the **natural logarithm** and can be written as $\mathbf{\ln x}$
- Many naturally occurring phenomena can be modelled using base-e exponential and logarithmic functions.
- $\log_e e = \ln e = 1$

Part 2: Reminder of Log Rules

Power Law of Logarithms	$\log_b x^n = n\log_b x$ for $b > 0, b \neq 1, x > 0$
Product Law of Logarithms	$\log_b(mn) = \log_b m + \log_b n$ for $b > 0, b \neq 1, m > 0, n > 0$
Quotient Law of Logarithms	$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$ for $b > 0, b \neq 1, m > 0, n > 0$
Change of Base Formula	$\log_b m = \frac{\log m}{\log b}, m > 0, b > 0, b \neq 1$
Exponential to Logarithmic	$y = b^x \rightarrow x = \log_b y$
Logarithmic to Exponential	$y = \log_b x \rightarrow x = b^y$
Other useful tips	$\log_a(a^b) = b$ $\log a = \log_{10} a$ $\log_b b = 1$

Part 2: Solving Problems Involving *e*

Example 2: Evaluate each of the following

a) $e^3 \cong 20.086$

b) $\ln 10 \cong 2.303$

c) $\ln e = 1$

Example 3: Solve each of the following equations

a) $20 = 3e^x$

$20 = 3e^x$

$\frac{20}{3} = e^x$

$\ln\left(\frac{20}{3}\right) = \ln(e)^x$

$\ln\left(\frac{20}{3}\right) = x \cdot \ln(e)$

$\ln\left(\frac{20}{3}\right) = x(1)$

$x \doteq 1.897$

b) $e^{1-2x} = 55$

$e^{1-2x} = 55$

$\ln(e)^{1-2x} = \ln(55)$

$(1-2x)(\ln(e)) = \ln(55)$

$(1-2x)(1) = \ln(55)$

$1-2x = \ln(55)$

$1-\ln(55) = 2x$

$\frac{1-\ln(55)}{2} = x$

$x \doteq -1.504$

c) $2\ln(x-3) - 7 = 3$

$2\ln(x-3) - 7 = 3$

$2\ln(x-3) = 10$

$\ln(x-3) = 5$

$e^5 = x-3$

$e^5 + 3 = x$

$x \doteq 151.413$

d) $\ln(4e^x) = 2$

$\ln(4e^x) = 2$

$e^2 = 4e^x$

$\frac{e^2}{4} = e^x$

$\ln\left(\frac{e^2}{4}\right) = \ln(e^x)$

$\ln\left(\frac{e^2}{4}\right) = x \cdot \underbrace{\ln(e)}_{=1}$

$x \doteq 0.614$

<u>: Graphing Functions Involving *e*</u>

<u>Part 3</u>

Example 4: Graph the functions $y = e^x$ and $y = \ln x$

$y = e^x$	
x	y
-1	0.37
0	1
1	2.72
HA	$y = 0$

$y = \ln x$	
x	y
0.37	-1
1	0
2.72	1
VA	$x = 0$

Note: $y = \ln x$ is the inverse of $y = e^x$