L6 – Applications of Exponential & Trigonometric Derivatives

Unit 3

MCV4U

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Part 1: Review of Differentiating Exponential Functions

Example 1: Differentiate $y = 3^x e^{\sin x}$ with respect to x.

$y' = 3^x(\ln 3)\left(e^{\sin x}\right) + e^{\sin x}(\ln e)(\cos x)(3^x)$

$y' = 3^x(\ln 3)\left(e^{\sin x}\right) + e^{\sin x}(\cos x)(3^x)$

$y' = 3^x e^{\sin x}(\ln 3 + \cos x)$

Rule	Derivative
Exponential Functions If $h(x) = b^{g(x)}$	$h'(x) = b^{g(x)} \times \ln b \times g'(x)$
Trig Functions If $f(x) = \sin x$ $g(x) = \cos x$ $h(x) = \tan x$	 $f'(x) = \cos x$ $g'(x) = -\sin x$ $h'(x) = \sec^2 x$
Log Functions If $g(x) = \log_a[f(x)]$	$g'(x) = \frac{f'(x)}{f(x)\ln a}$

Example 2: A radioactive isotope of gold, Au-198, is used in the diagnosis and treatment of liver disease. Suppose that a 6.0 mg sample of Au-198 is injected into a liver, and that this sample decays to 4.6 mg after 1 day. Assume the amount of Au-198 reaming after t days is given by $N(t) = N_0 e^{-\lambda t}$.

a) Determine the disintegration constant for Au-198

The Greek letter λ, lambda, indicates the disintegration constant, which is related to how fast a radioactive substance decays.

$4.6 = 6e^{-\lambda(1)}$

$\frac{4.6}{6} = e^{-\lambda}$

$-\lambda = \ln\left(\frac{4.6}{6}\right)$

$\lambda = -\ln\left(\frac{4.6}{6}\right)$

$\lambda \cong 0.266$ days

b) Determine the half-life of Au-198

$3 = 6e^{-0.266t}$

$0.5 = e^{-0.266t}$

$\ln(0.5) = -0.266t$

$t = \dfrac{\ln(0.5)}{-0.266}$

$t \cong 2.6$ days

c) Write the equation that gives the amount of Au-198 remaining as a function of time in terms of its half-life.

$N(t) = 6\left(\frac{1}{2}\right)^{\frac{t}{2.6}}$

d) How fast is the sample decaying after 3 days?

$N(t) = 6e^{-0.266t}$

$N'(t) = 6(e^{-0.266t})(\ln e)(-0.266)$

$N'(t) = -1.596(e^{-0.266t})$

$N'(3) = -1.596\left[e^{-0.266(3)}\right]$

$N'(3) = -0.72$ mg/day

The amount of gold is decreasing by about 0.72 mg/day.

Example 3: A power supply delivers a voltage signal that consists of an alternating current (AC) component and a direct current (DC) component. Where t is the time, in seconds, the voltage, in volts, at time t is given by the function $V(t) = 5\sin t + 12$.

a) What are the max and min voltages? At which times do these values occur?

$Max = c + |a| = 12 + 5 = 17$ V

$Min = c - |a| = 12 - 5 = 7$ V

Determine when they occur by solving the following equations:

$17 = 5\sin t + 12$

$5 = 5\sin t$

$1 = \sin t$

$t = \dfrac{\pi}{2}$

There is a max of 17 V when $t = \frac{\pi}{2} + 2\pi k, k \in \mathbb{Z}$

$7 = 5\sin t + 12$

$-5 = 5\sin t$

$-1 = \sin t$

$t = \dfrac{3\pi}{2}$

There is a min of 17 V when $t = \frac{3\pi}{2} + 2\pi k, k \in \mathbb{Z}$

b) Determine the period, T, in seconds, frequency, f, in hertz, and amplitude, A, in volts, for this signal.

Period:

$T = \frac{2\pi}{|k|} = \frac{2\pi}{1} = 2\pi$ seconds

Frequency:

The frequency is the reciprocal of the period.

$f = \frac{1}{2\pi}$ Hz

Amplitude:

$A = |a| = 5$ V

Example 4: For small amplitudes, and ignoring the effects of friction, a pendulum is an example of simple harmonic motion. Simple harmonic motion is motion that can be modelled by a sinusoidal function, and the graph of a function modelling simple harmonic motion has a constant amplitude.

The period of a simple pendulum depends only on its length and can be found using the relation $T = 2\pi\sqrt{\frac{l}{g}}$, where T is the period, in seconds, l is the length of the pendulum, in meters, and g is the acceleration due to gravity. On or near the surface of Earth, g has a constant value of 9.8 m/s^2.

Under these conditions, the horizontal position of the bob as a function of time can be described by the function $h(t) = A\cos\left(\frac{2\pi t}{T}\right)$, where A is the amplitude of the pendulum, t is time, in seconds, and T is the period of the pendulum, in seconds.

Find the max speed of the bob and the time at which that speed first occurs for a pendulum having a length of 1.0 meters and an amplitude of 5 cm.

Start by finding the period:

$$T = 2\pi\sqrt{\frac{1.0}{9.8}} \cong 2.0 \text{ seconds.}$$

Therefore, the position equation is:

$$h(t) = 5\cos\left(\frac{2\pi t}{2}\right)$$

$$h(t) = 5\cos(\pi t)$$

To find the max speed, we need to find the max OR min value of the velocity equation (derivative of the position equation).

$$v(t) = h'(t) = -5\sin(\pi t)\,(\pi)$$

$$v(t) = -5\pi\sin(\pi t)$$

Min Velocity:

$$-5\pi = -5\pi\sin(\pi t)$$
$$\sin(\pi t) = 1$$
$$\pi t = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$$
$$t = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots$$
$$t = \frac{1+4k}{2}; \{k \in \mathbb{Z} | k \geq 0\}$$

Max Velocity:

$$5\pi = -5\pi\sin(\pi t)$$
$$\sin(\pi t) = -1$$
$$\pi t = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \ldots$$
$$t = \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \ldots$$
$$t = \frac{3+4k}{2}; \{k \in \mathbb{Z} | k \geq 0\}$$

The first max speed occurs at 0.5 seconds.

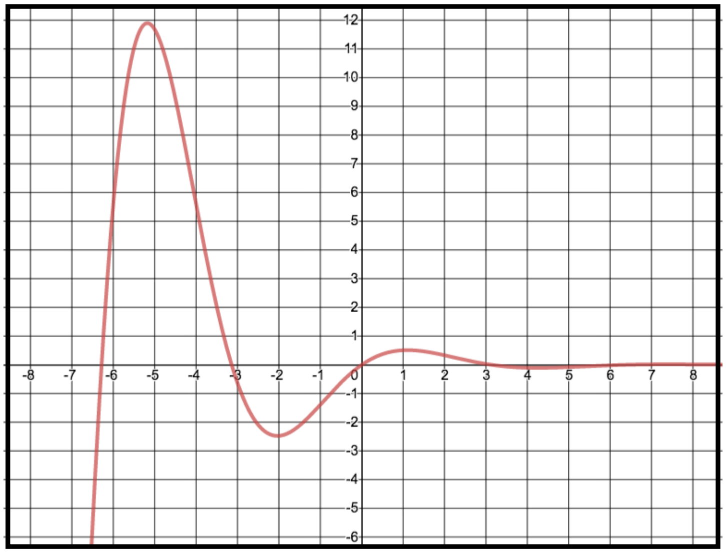
$$v(0.5) = -5\pi\sin[\pi(0.5)] = -5\pi \cong -15.7 \text{ cm/s.}$$

Therefore, the max speed is 15.7 cm/s and it occurs after 0.5 seconds.

Example 5: The vertical displacement of a SUV's body after passing over a bump is modelled by the function $h(t) = e^{-0.5t}\sin t$, where h is the vertical displacement, in meters, at time t, in seconds.

a) Use technology to generate a rough sketch of the graph of the function.

Notice that the amplitude diminishes over time. This motion is called DAMPED HARMONIC MOTION.



b) Determine when the max displacement of the sport utility vehicle's body occurs.

$h'(t) = e^{-0.5t}(-0.5)(\sin t) + \cos t\,(e^{-0.5t})$

$h'(t) = e^{-0.5t}(-0.5\sin t + \cos t)$

$0 = e^{-0.5t}(-0.5\sin t + \cos t)$

$e^{-0.5t} = 0$

No solutions

$0 = -0.5\sin t + \cos t$

$0.5\sin t = \cos t$

$0.5\tan t = 1$

$\tan t = 2$

$t = \tan^{-1}(2)$

$t \cong 1.1$ seconds

There are an infinite number of solutions to this but we only want the first positive answer as the vertical displacement decreases as time increases.

From the graph we know this is a max and not a min but we could verify using the first or second derivative tests.

c) Determine the maximum displacement.

$h(1.1) = e^{-0.5(1.1)}\sin(1.1)$

$h(1.1) \cong 0.51$ m