Unit 2 Pretest – Curve Sketching

MCV4U
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SOLUTIONS

1) Find the increasing and decreasing intervals for each function.

a) $f(x) = 7 + 6x - x^2$

$f'(x) = 6 - 2x$

$0 = 6 - 2x$

$x = 3$

$-\infty$ 3 ∞

	2	4
$f'(x)$	+	−
$f(x)$	inc. ↗	dec. ↘

increasing: $x < 3$

decreasing: $x > 3$

b) $y = x^3 - 48x + 5$

$y' = 3x^2 - 48$

$0 = 3x^2 - 48$

$x^2 = 16$

$x = \pm 4$

$-\infty$ -4 4 ∞

	-5	0	5
y'	+	−	+
y	inc. ↗	dec. ↘	inc. ↗

increasing: $x < -4,\ x > 4$

decreasing: $-4 < x < 4$

c) $g(x) = x^4 - 18x^2$

$g'(x) = 4x^3 - 36x$

$0 = 4x(x^2 - 9)$

$0 = 4x(x-3)(x+3)$

$x_1 = 0 \quad x_2 = 3 \quad x_3 = -3$

$-\infty$ -3 0 3 ∞

	-4	-1	1	4
$g'(x)$	−	+	−	+
$g(x)$	dec. ↘	inc. ↗	dec. ↘	inc. ↗

increasing: $-3 < x < 0,\ x > 3$

decreasing: $x < -3,\ 0 < x < 3$

d) $f(x) = x^3 + 10x - 9$

$f'(x) = 3x^2 + 10$

$0 = 3x^2 + 10$

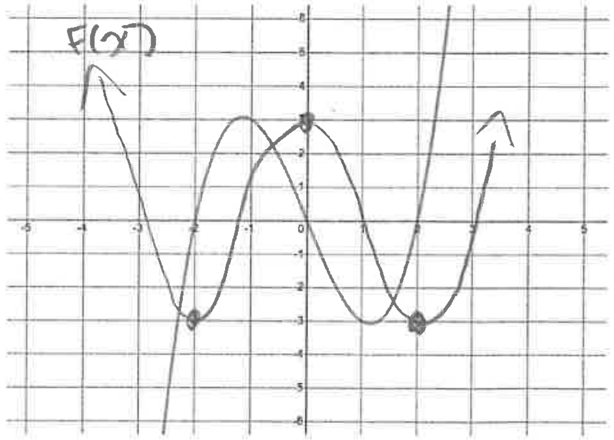
No critical #'s

$f'(x) > 0$ for $x \in \mathbb{R}$, $\therefore$ $f(x)$ is always increasing.

2) Given the graph of $f'(x)$, state the intervals of increase and decrease for the function $f(x)$. Then sketch a possible graph of $f(x)$.

increasing: $-2 < x < 0,\ x > 2$

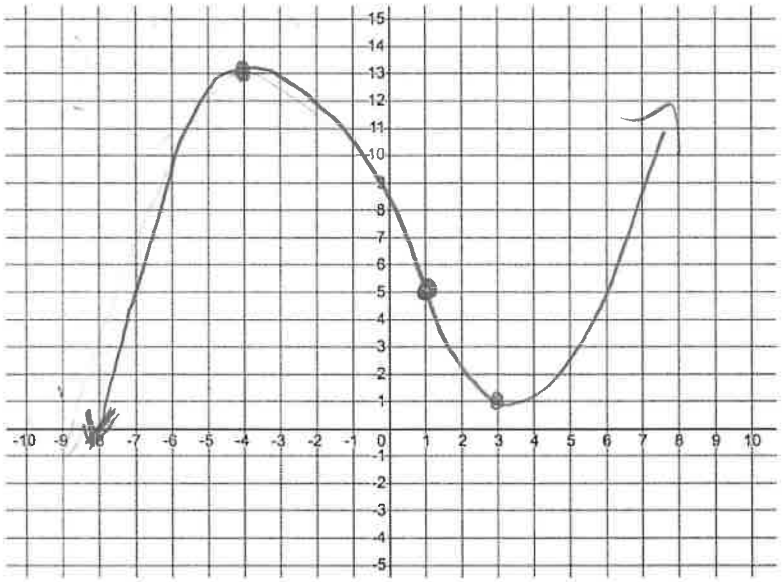
decreasing: $x < -2,\ 0 < x < 2$



increasing

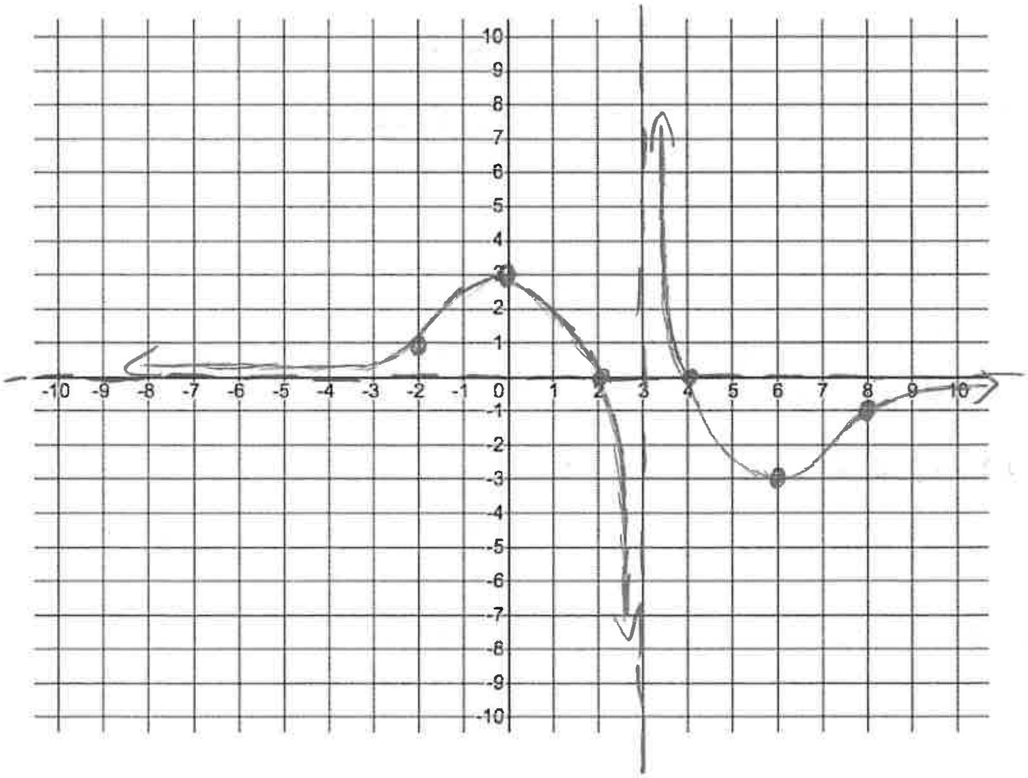
3) Sketch a continuous graph that satisfies the following set of conditions: $f'(x) > 0$ when $x < -4$ and $x > 3$, $f'(x) < 0$ when $-4 < x < 3$ and $f(1) = +5$.

decreasing



4) Given the following information about $y = f(x)$, sketch a graph for the function on the axes provided below. Label an appropriate scale for the sketch.

Local minimum $(6, -3)$. Local maximum $(0,3)$. Points of inflection at $(-2,1)$ and $(8, -1)$. Increasing when $x < 0$ and $x > 6$; decreasing when $0 < x < 3$ and $3 < x < 6$. Concave up when $x < -2$ and $3 < x < 8$; concave down when $-2 < x < 3$ and $x > 8$. HA at $y = 0$. VA at $x = 3$. y-intercept at $(0,3)$. x-intercepts at $(2,0)$ and $(4,0)$.



5) Find the local extrema for each function and classify them as local max or local min.

a) $f(x) = 16 - x^4$

$f'(x) = -4x^3$

$0 = -4x^3$

$x = 0$

$f(0) = 16$

$-\infty \quad 0 \quad \infty$

	-1	1
$f'(x)$	+	−
$f(x)$	inc. ↗	dec. ↘

(0,16)
max

Local max at (0,16)

b) $g(x) = x^3 + 9x^2 - 21x - 12$

$g'(x) = 3x^2 + 18x - 21$

$0 = x^2 + 6x - 7$

$0 = (x+7)(x-1)$

$x_1 = -7 \quad x_2 = 1$

$g(-7) = 233 \quad g(1) = -23$

$-\infty \quad -7 \quad 1 \quad \infty$

	-8	0	2
$g'(x)$	+	−	+
$g(x)$	inc. ↗	dec. ↘	inc. ↗

(-7,233) (1,-23)
max min

Local max: (-7,233)

Local min: (1,-23)

6) The speed, in km/h, of a certain car t seconds after passing a police radar location is given by the function $v(t) = 3t^2 - 24t + 88$.

a) Find the min speed of the car.

$v'(t) = 6t - 24$

$0 = 6(t-4)$

$t = 4$

critical point:

$v(4) = 40$

Second derivative test:

$v''(t) = 6$

$v''(4) = 6$; ∴ concave up
∴ a local min.

The min speed is 40 km/h

b) The radar tracks the car on the interval $2 \leq t \leq 5$. Find the max speed of the car on this interval.

Test endpoints and critical #'s.

$V(2) = 52$ km/h

$v(4) = 40$ km/h

$V(5) = 43$km

Max speed is 52 km/h.

7) Determine the absolute extreme values of each function on the given interval.

a) $y = x^2 - 3x + 2; -4 \le x \le 4$

$y' = 2x - 3$

$0 = 2x - 3$

$x = \frac{3}{2} = 1.5$

Test endpoints and critical #'s:

$y(-4) = 3$[illegible]

$y(1.5) = -0.25$

$y(4) = 6$

absolute may: $(-4, 30)$

absolute min: $(1.5, -0.25)$

b) $g(x) = 2x^3 - 24x + 3; -4 \le x \le 2$

$g'(x) = 6x^2 - 24$

$0 = 6(x^2 - 4)$

$0 = 6(x-2)(x+2)$

$x_1 = 2 \quad x_2 = -2$

Tests:

$g(-4) = -29$

$g(-2) = 35$

$g(2) = -29$

absolute max: $(-2, 35)$

absolute min: $(-4, -29)$ and $(2, -29)$

8) For the function $f(x) = x^4 - 2x^3 - 12x^2 + 3$, determine the points of inflection and the intervals of concavity.

$f'(x) = 4x^3 - 6x^2 - 24x$

$f''(x) = 12x^2 - 12x - 24$

$0 = 12(x^2 - x - 2)$

$0 = 12(x-2)(x+1)$

$x_1 = 2 \quad x_2 = -1$

Possible POI:

$f(2) = -45$

$f(-1) = -6$

$-\infty \quad -1 \quad 2 \quad \infty$

	-2	0	3
$f''(x)$	$+$	$-$	$+$
$f(x)$	C.U. ∪	C.D. ∩	C.U. ∪

Concave up: $x < -1$, $x > 2$

Concave down: $-1 < x < 2$

POI $(-1, -6)$

POI $(2, -45)$

9) For the function $f(x) = 2x^3 - x^4$, determine the critical points and classify them using the second derivative test.

$f'(x) = 6x^2 - 4x^3$

$0 = 2x^2(3 - 2x)$

$x_1 = 0 \qquad x_2 = \frac{3}{2} = 1.5$

Critical Points:

$f(0) = 0 \quad (0, 0)$

$f(1.5) = \frac{27}{16} = 1.6875 \quad (1.5, 1.6875)$

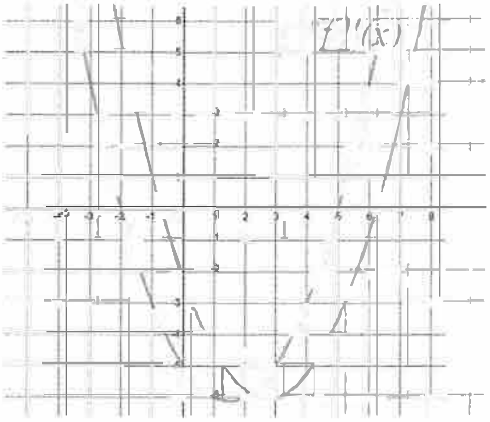
Second Derivative Test:

$f''(x) = 12x - 12x^2$

$f''(0) = 0$; ***2nd derivative test fails; would need to do 1st derivative test to classify critical point.***

$f''(1.5) = -9$; ∴ concave down, local max at $(1.5, 1.6875)$

10) Given the graph of $f''(x)$, state the intervals of concavity for $f(x)$.



concave up: $x<-2$, $x>5$

concave down: $-2<x<5$

11) For each function, state equations for any asymptotes.

a) $f(x)=\frac{x^2-4}{x}$

HA: $y=x$

VA: $x=0$

b) $g(x)=\frac{2x-3}{2x-4}$

HA: $y=1$

VA: $x=2$

c) $y=\frac{x^2+1}{x^2-3x-10}$ $=\frac{x^2+1}{(x-5)(x+2)}$

HA: y = 1
VA: x = 5 and x = -2

d) $\frac{x-1}{x^2+2x+1}$ $=\frac{x-1}{(x+1)^2}$

HA: $y=0$

VA: $x=-1$

12) State the equation of the tangent to the graph of $f(x)=\frac{x+1}{x^2+1}$ at the point where $x=-1$.

Point:

$f(-1)=\frac{(-1)+1}{(-1)^2+1}$

$f(-1)=0$

$(-1,0)$

Slope:

$f'(x)=\frac{1(x^2+1)-2x(x+1)}{(x^2+1)^2}$

$f'(x)=\frac{-1x^2-2x+1}{(x^2+1)^2}$

$f'(-1)=\frac{-1(-1)^2-2(-1)+1}{[(-1)^2+1]^2}$

$f'(-1)=\frac{1}{2}$

$m\ \frac{1}{2}$

Eq^n:

$y=mx+b$

$0=(\frac{1}{2})(-1)+b$

$b=\frac{1}{2}$

$\boxed{y=\frac{1}{2}x+\frac{1}{2}}$

13) Analyze and sketch each function using the algorithm for curve sketching

a) $k(x) = \frac{1}{4}x^4 - \frac{9}{2}x^2$

① No domain restrictions ; no asymptotes

② $0 = \frac{1}{4}x^2(x^2 - 18)$

x-int: (0,0)
(4.24,0)
(-4.24,0)

y-int: (0,0)

$\frac{1}{4}x^2 = 0$ $\quad x^2 - 18 = 0$

$x_1 = 0$ $\quad x_2 = \sqrt{18} \simeq 4.24$

$x_3 = -\sqrt{18} \simeq -4.24$

③ $k'(x) = x^3 - 9x$

$0 = x(x^2 - 9)$

$0 = x(x-3)(x+3)$

$x_1 = 0 \quad x_2 = 3 \quad x_3 = -3$

$k(0) = 0 \quad k(3) = -20.25 \quad k(-3) = -20.25$

critical points: (0,0), (3,-20.25), (-3,-20.25)

④ $k''(x) = 3x^2 - 9$

$0 = 3(x^2 - 3)$

$x = \pm\sqrt{3} \simeq \pm 1.73$

possible PoI: $(\sqrt{3}, -11.25)$ and $(-\sqrt{3}, -11.25)$

(5/6/7)

$-\infty \quad -3 \quad -1.73 \quad 0 \quad 1.73 \quad 3 \quad \infty$

	-4	-2	-1	1	2	4
$k'(x)$	−	+	+	−	−	+
$k''(x)$	+	+	−	−	+	+
	dec- cu	inc cu	inc CD	dec. CD	dec cu	inc cu
$k(x)$						

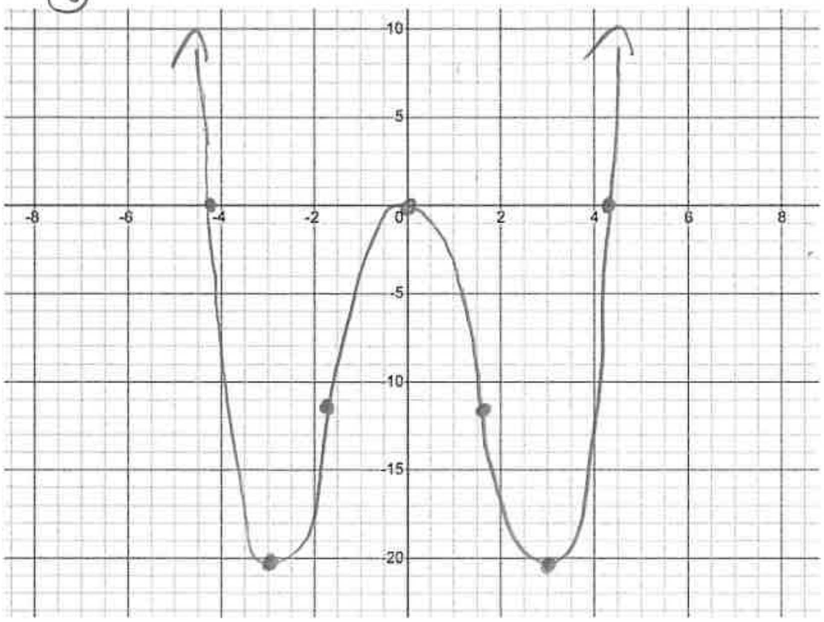
local min   PoI   local max   PoI   local min

Local min: (-3,-20.25) and (3,-20.25)

Local max: (0,0)

PoI: $(-\sqrt{3}, -11.25)$ and $(\sqrt{3}, -11.25)$

⑧



b) $h(x) = 2x^3 - 3x^2 - 3x + 2$

① No domain restrictions ; no asymptotes.

② x-int : (-1,0), (2,0), and (0.5,0) y-int : $h(0) = 2$

(0,2)

-1	2	-3	-3	2
	↓	-2	5	-2
	2	-5	2	0

$0 = (x+1)(2x^2 - 5x + 2)$

$0 = (x+1)(2x^2 - 4x - 1x + 2)$

$0 = (x+1)[2x(x-2) - 1(x-2)]$

$0 = (x+1)(x-2)(2x-1)$

$x_1 = -1 \quad x_2 = 2 \quad x_3 = \frac{1}{2}$

③ $h'(x) = 6x^2 - 6x - 3$

$0 = 3(2x^2 - 2x - 1)$

$x = \frac{2 \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$

$x = \frac{2 \pm 2\sqrt{3}}{4}$

$x = \frac{2(1 \pm \sqrt{3})}{4}$

$x = \frac{1 \pm \sqrt{3}}{2}$

$x_1 \simeq 1.37 \quad x_2 \simeq -0.37$

critical points: (1.37, -2.6), (-0.37, 2.6)

④ $h''(x) = 12x - 6$

$0 = 6(2x - 1)$

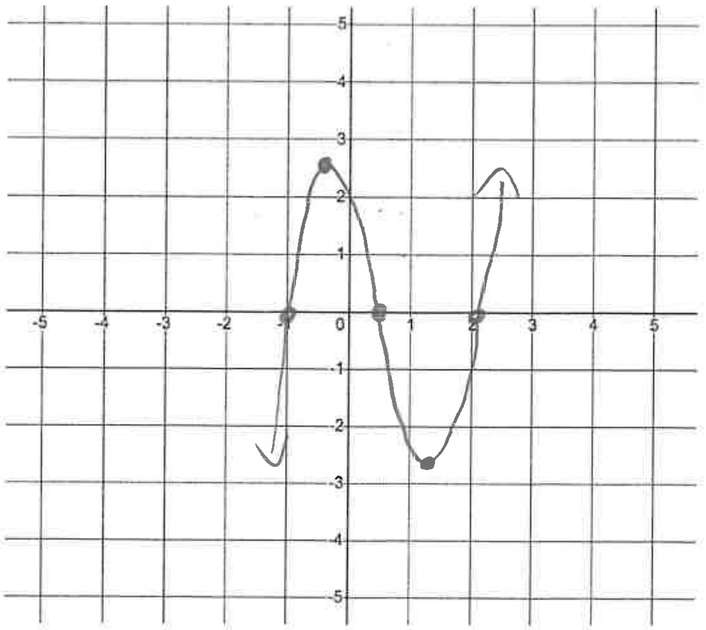
$x = \frac{1}{2}$

Possible PoI (0.5, 0)

-∞   -0.37   0.5   1.37   ∞

5/6/7	-1	0	1	2
h'(x)	+	−	−	+
h''(x)	−	−	+	+
h(x)	inc. CD 	dec. CD 	dec. cu. 	inc. cu. 

local max   PoI   local min



Local min : (1.37, -2.6)

Local max : (-0.37, 2.6)

PoI : (0.5, 0)

c) $f(x) = \frac{x^2+2x-4}{x^2}$

① HA: $y = 1$

VA: $x = 0$

② x-int: $(1.24, 0)$ and $(-3.24, 0)$

$0 = x^2+2x-4$

$x = \frac{-2 \pm \sqrt{20}}{2}$

$x = \frac{-2 \pm 2\sqrt{5}}{2}$

$x = -1 \pm \sqrt{5}$

$x_1 \simeq 1.24 \quad x_2 \simeq -3.24$

y-int: $f(0) = \frac{-4}{0}$

& no y-intercept.

③ $f'(x) = \frac{(2x+2)(x^2) - 2x(x^2+2x-4)}{x^4}$

$f'(x) = \frac{x[x(2x+2) - 2(x^2+2x-4)]}{x^4}$

$f'(x) = \frac{-2x+8}{x^3}$

$0 = -2x+8$

$x = 4$

critical point: $(4, 1.25)$

④ $f''(x) = \frac{-2(x^3) - 3x^2(-2x+8)}{x^6}$

$f''(x) = \frac{x^2[-2x - 3(-2x+8)]}{x^6}$

$f''(x) = \frac{4x-24}{x^4}$

$0 = 4x-24$

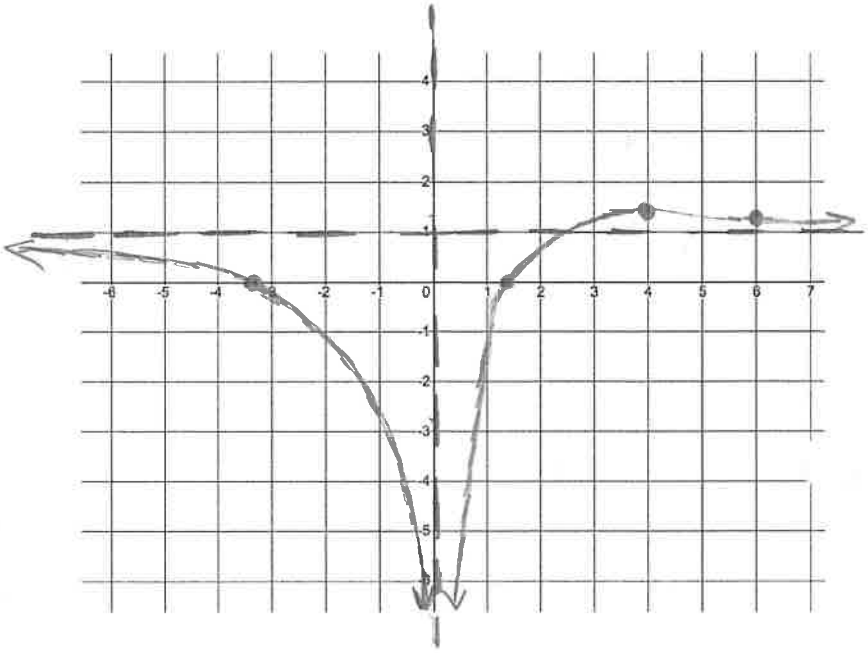
$x = 6$

possible POI: $(6, 1.22)$

$-\infty \quad 0 \quad 4 \quad 6 \quad \infty$

	-1	1	5	7
$f'(x)$	−	+	−	−
$f''(x)$	−	−	−	+
$f(x)$	dec CD	inc. CD	dec CD	dec CU.

VA — local max $(4, 1.25)$ — POI $(6, 1.22)$



14) Consider the graph to the right.

a) How many times does the function have a erivative = 0? How do you know?

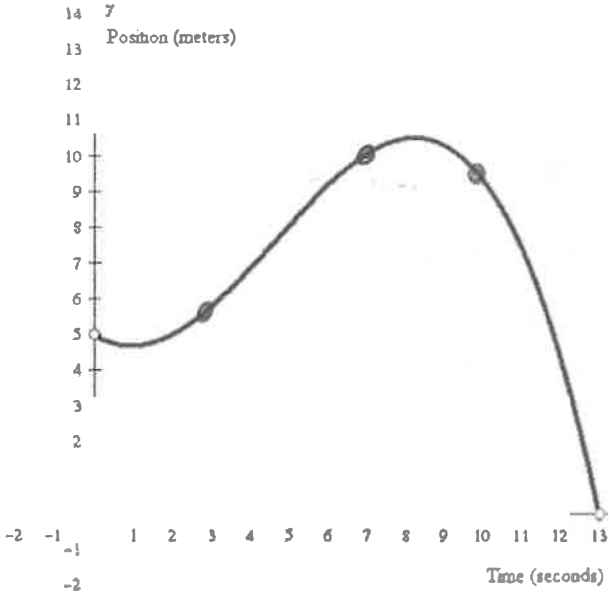
2 times; 2 turning points.

b) State the sign of the first and second derivatives at the following times.

i) t = 3 s **ii)** t = 7 s **iii)** t = 10 s

$y'(3) = +$ $y'(7) = +$ $y'(10) =$

$y''(3) = +$ $y''(7) = -$ $y''(10) = -$



15) A garbage can is in the shape of a cylinder with no lid. It needs to have a volume of 5000 cm³. What will be the radius and height of the can that uses the least amount of material to construct it? (Hint: Surface Area)

$V = \pi r^2 h$

$5000 = \pi r^2$

$= \frac{5000}{\pi r^2}$

$h = \frac{5000}{\pi\left[\left(\frac{5000}{\pi}\right)^{1/3}\right]^2}$

$h \approx 11.7$ cm

$SA = \pi r^2 + 2\pi r h$

$SA(r) = \pi r^2 + 2\pi \cancel{r}\left(\frac{5000}{\cancel{\pi} r^{\cancel{2}1}}\right)$

$SA(r) = \pi r^2 + 10000 r^{-1}$

Verify $r \approx 11.7$ is a max:

$-\infty$ 11.7 ∞

	10	12
SA'(r)	−	+
SA(r)	dec. ↘	inc ↗

MIN

$SA'(r) = 2\pi r - 10000 r^{-2}$

$SA'(r) = 2\pi r \quad \frac{10000}{r^2}$

$0 = 2\pi r \quad \frac{10000}{r^2}$

$\frac{10000}{r^2} = 2\pi r$

$\frac{10000}{2} = r^3$

$r = \left(\frac{5000}{\pi}\right)^{1/3}$

$r \approx 11.7$ cm

Min SA when: $r \approx 11.7$ cm

$h \approx 11.7$ cm

16) A carpenter builds an open box with a square base. She has 8 m^2 of wood available. Find the volume of the largest box she can build.

$V = b^2h$

$SA = b^2 + 4bh$

$8 = b^2 + 4bh$

$\frac{8-b^2}{4b} = h$

$V(b) = b^2\left(\frac{8-b^2}{4b}\right)$

$V(b) = \frac{b(8-b^2)}{4}$

$V(b) = \frac{8b-b^3}{4}$

$V(b) = 2b - \frac{1}{4}b^3$

$V'(b) = 2 - \frac{3}{4}b^2$

$0 = 2 - \frac{3}{4}b^2$

$\frac{3}{4}b^2 = 2$

$b^2 = \frac{8}{3}$

$b = \pm\sqrt{\frac{8}{3}}$

$b \simeq 1.63$

2nd derivative test:

$V''(b) = -\frac{3}{2}b$

$V''(1.63) = -2.445$

$\therefore V(1.63)$ is a MAX.

$V(1.63) \simeq 2.18$ m^3

$\boxed{\text{The max volume is } 2.18 \text{ m}^3.}$

17) Sandy will make a closed rectangular jewellery box with a square base from two different woods. The wood from the top and botton costs \$0.0020/cm^2. The wood for the sides costs \$0.0030/cm^2. Find the dimensions that minimize the cost for a box with volume 4000 cm^3.

$SA = 2x^2 + 4xh$

$SA(x) = 2x^2 + 4x\left(\frac{4000}{x^2}\right)$

$SA(x) = 2x^2 + 16000x^{-1}$

$C(x) = 0.002(2x^2) + 0.003(16000)x^{-1}$

$C(x) = 0.004x^2 + 48x^{-1}$

$C'(x) = 0.008x - 48x^{-2}$

$0 = 0.008x - \frac{48}{x^2}$

$\frac{48}{x^2} = 0.008x$

$6000 = x^3$

$x \cong 18.17$

Constraint:

$V = x^2h$

$4000 = x^2h$

$h = \frac{4000}{x^2}$

2nd derivative test:

$C''(x) = 0.008 + 96x^{-3}$

$C''(18.17) \cong 0.024$

Since $C''(18.17) > 0$, $C(18.17)$ is concave UP and $(18.17, C(18.17))$ is a MIN point.

Answer: The min cost of $C(18.17) = \$3.96$ will occur when the length and width of the box are 18.17 cm and the height is 12.12 cm.

18) A music store sells an average of 120 CDs per week at \$24 each. A market survey indicates that for each \$0.75 decrease in price, five additional CDs will be sold per week. The cost of producing x CDs is $C(x) = -0.003x^2 + 3x + 2000$. What price and quantity of CDs maximizes profit?

sold $= x = 120 + 5n \longrightarrow n = \frac{x-120}{5} = 0.2x - 24$

price $= p = 24 - 0.75n$

$p(x) = 24 - 0.75(0.2x - 24)$

$p(x) = 24 - 0.15x + 18$

$p(x) = 42 - 0.15x$

$R(x) = x(42 - 0.15x)$

$R(x) = 42x - 0.15x^2$

$P(x) = (42x - 0.15x^2) - (-0.003x^2 + 3x + 2000)$

$P(x) = -0.147x^2 + 39x - 2000$

$P'(x) = -0.294x + 39$

$0 = -0.294x + 39$

$x \doteq 132.65 \simeq 133$ CD's.

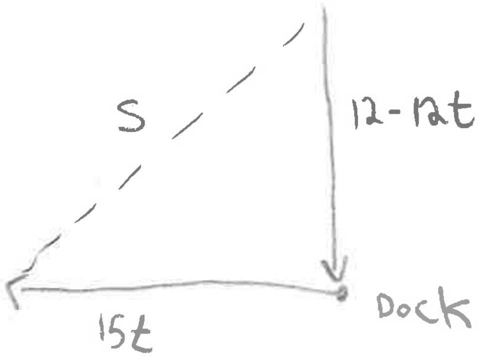
$p(133) = 42 - 0.15(133) = \$22.05$

$P''(133) < 0$; $\therefore$ a max

$\boxed{\text{Selling 133 CD's for \$22.05 each maximizes the profit.}}$

19) A boat leaves a dock at 2:00 p.m., heading west at 15 km/h. Another boat heads south at 12 km/h and reaches the same dock at 3:00 p.m. When were the boats closest to each other?

t = hours after 2pm



$$s(t) = \sqrt{(15t)^2 + (12-12t)^2}$$

$$s(t) = \sqrt{225t^2 + 144 - 288t + 144t^2}$$

$$s(t) = (369t^2 - 288t + 144)^{1/2}$$

$$s'(t) = \frac{1}{2}(369t^2 - 288t + 144)^{-1/2}(738t - 288)$$

$$s'(t) = \frac{369t - 144}{(369t^2 - 288t + 144)^{1/2}}$$

$$0 = 369t - 144$$

$$t \approx 0.39 \text{ hours}$$

1st deriv. test

	-∞	0.39	∞
	0.1	0.5	
s'(t)	−	+	
	dec	inc	
	↘	↗	

min

∴ The boats were closest together 0.39 hours after 2pm. (2:23 pm).

Answers:

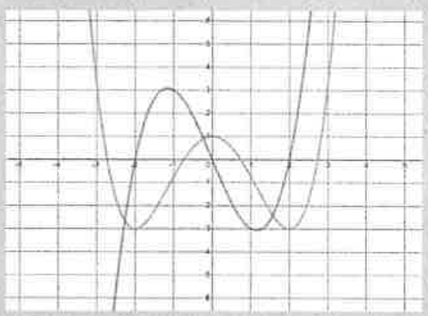
1)a) increasing: $(-\infty, 3)$
decreasing: $(3, \infty)$

b) increasing: $(-\infty, -4) \cup (4, \infty)$
decreasing: $(-4,4)$

c) increasing: $(-3,0) \cup (3, \infty)$
decreasing: $(-\infty, -3) \cup (0,3)$

d) increasing: $(-\infty, \infty)$
decreasing: never

2) increasing: $(-2,0) \cup (2, \infty)$
decreasing: $(-\infty, -2) \cup (0,2)$



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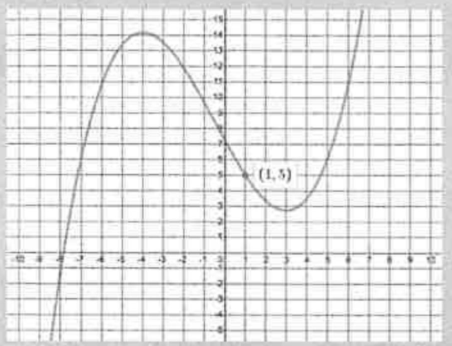
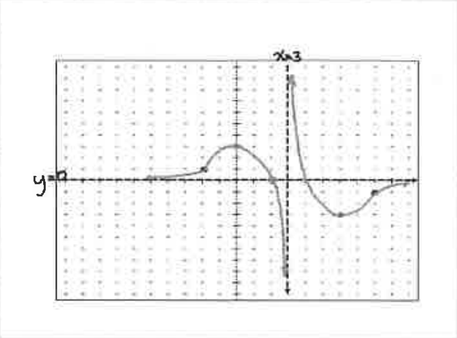


4)



5)a) $(0,16)$ is a local max **b)** local max at $(-2,233)$; local min at $(1,-23)$

6)a) 40 km/h **b)** 52 km/h

7)a) absolute min: $(1.5, -0.25)$
absolute max: $(-4, 30)$

b) absolute min: $(-4, -29)$ and $(2, -29)$
absolute max: $(-2, 35)$

8) concave up: $x < -1$ and $x > 2$
concave down: $-1 < x < 2$

points of inflection: $(-1, -6)$ and $(2, -45)$

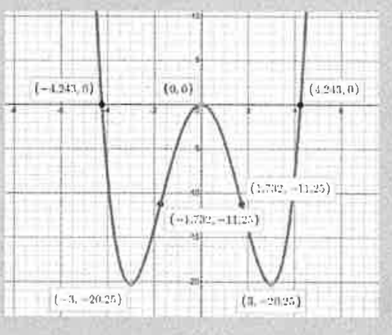
9) $(0,0)$ is a point of inflection NOT a local min or max; $(1.5,1.7)$ is a local max

10) concave up: $x < -2,\ x > 5$
concave down: $-2 < x < 5$

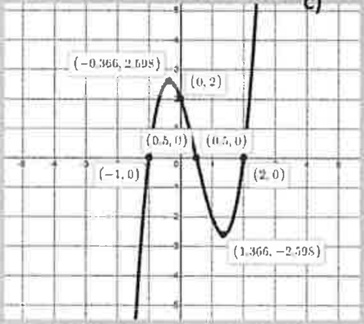
11)a) VA: $x = 0$; SA: $y = x$ **b)** VA: $x = 2$; HA: $y = 1$ **c)** VA: $x = 5$ and $x = -2$; HA: $y = 1$
d) VA: $x = -1$; HA: $y = 0$

12) $y = \frac{1}{2}x + \frac{1}{2}$

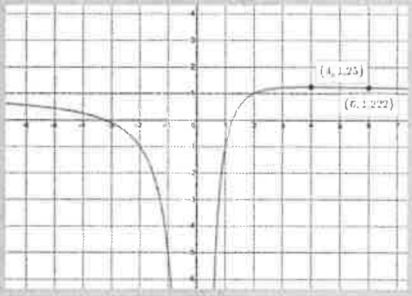
13)a)



b)



c)



14) a) twice – there are two turning points for the graph, so two points with $y' = 0$

b) i) $y' > 0,\ y'' > 0$ **ii)** $y' > 0,\ y'' < 0$ **iii)** $y' < 0,\ y'' < 0$

15) $r = h = 11.7$ cm

16) $V \cong 2.17$ m^3

17) 12.1 cm X 18.2 cm X 18.2 cm

18) approximately 133 CDs at a price of $22.05

19) 2:23 pm