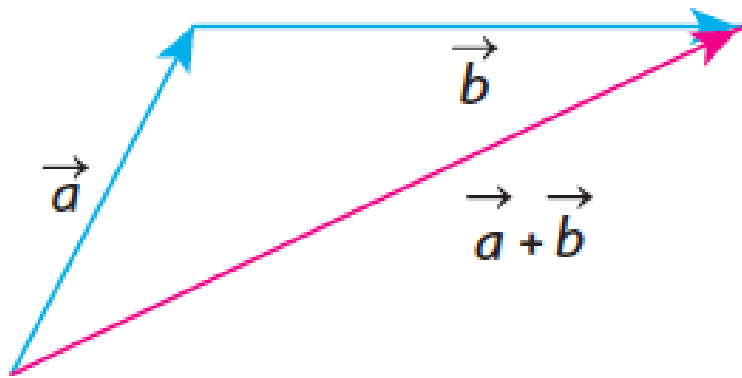


Name:

# *Unit 4- Geometric Vectors*

## *Lesson Package*

*MCV4U*



## Unit 4 Outline

**Unit Goal:** By the end of this unit, you will be able to demonstrate an understanding of vectors in two-space by representing them geometrically and by recognizing their applications.

Section	Subject	Learning Goals	Curriculum Expectations
L1	Intro to Vectors	- use Newton's Quotient to calculate instantaneous rates of change - determine derivatives of polynomial functions	C1.1, C1.2
L2	Vector Addition	- find the derivative of a product of functions using the product rule	C2.1, C2.2
L3	Vector Multiplication	- make connections between the concept of motion and derivatives	C2.1, C2.2
L4	Force, Velocity, and Tension	- find the derivative of a quotient of functions using the quotient rule	C2.3
L5	Resolution of Vector Components	- Find the derivative of a function using the chain rule	C2.3

Assessments	F/A/O	Ministry Code	P/O/C	KTAC
Note Completion	A		P	
Practice Worksheet Completion	F/A		P	
Quiz – Geometric Vectors	F		P	
PreTest Review	F/A		P	
Test – Geometric Vectors	O	C1.1, C1.2, C2.1, C2.2, C2.3	P	K(25%), T(20%), A(30%), C(25%)

**Part 1: What is a Vector?**

A \_\_\_\_\_ quantity describes magnitude or size only. It does NOT include a direction.

Examples: temperature ( $-5^{\circ}\text{C}$ ), distance (5 km), speed (100 km/h), mass (10 kg)

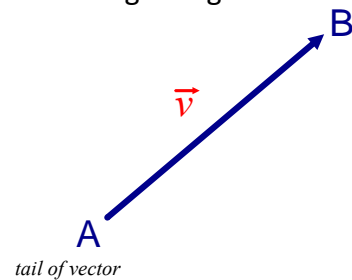
A \_\_\_\_\_ is a mathematical quantity having both MAGNITUDE and DIRECTION

Examples: velocity (80 km/h west), force (10 N downward)

Vectors are represented with directed line segments. A directed line segment has a \_\_\_\_\_, called its magnitude, and a \_\_\_\_\_ indicated by an arrowhead.

Vector  $\overrightarrow{AB}$  has a starting point at A and ends at point B. It could also be expressed using a single letter  $\vec{v}$

head of vector



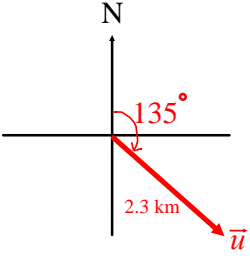
The magnitude, or size, of a vector is designated using absolute value brackets. The magnitude of vector  $\overrightarrow{AB}$  or  $\vec{v}$  is written as  $|\overrightarrow{AB}|$  or  $|\vec{v}|$ . Magnitude is always a non-negative value.

A vectors direction can be expressed in several different ways:

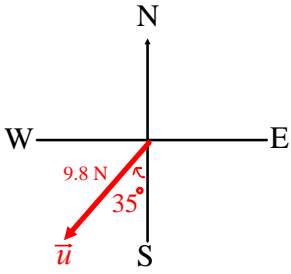
- i) As an angle moving counter-clockwise with respect to a horizontal line

Diagram	Description of Direction
	<p>5 km at <math>30^{\circ}</math> to the horizontal</p>

- ii) A True Bearing is a compass measurement where the angle is measured from North in a clockwise direction.

Diagram	Description of Direction
	
	2 km at a true bearing of 060°

- iii) A Quadrant Bearing is a measurement between 0° and 90° east or west of the north-south line.

Diagram	Description of Direction
	
	25 km/h at a quadrant bearing of N80°W

**Example 1:** Convert the following

- a) Write the true bearing 150° as a quadrant bearing.

b) Write the quadrant bearing N50°W as a true bearing.

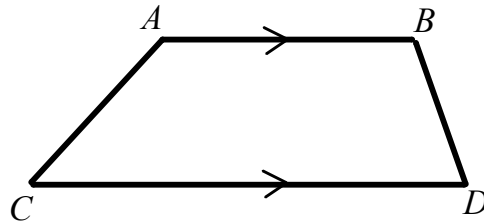
**Part 2: Equivalent and Opposite Vectors**

\_\_\_\_\_ : Vectors that have the same OR opposite direction, but not necessarily the same magnitude.

$$\vec{AB} \parallel \vec{DC}$$

And

$$\vec{AB} \parallel \vec{CD}$$



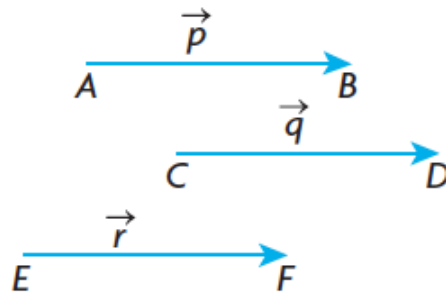
\_\_\_\_\_ : Vectors that have the same magnitude AND direction. The location of the vectors does NOT matter.

Notice that any of these vectors could be translated to be coincident with either of the other two.

$$\vec{p} = \vec{q} = \vec{r}$$

Or

$$\vec{AB} = \vec{CD} = \vec{EF}$$

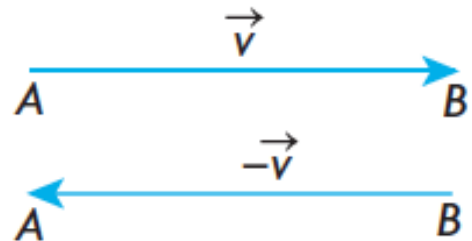


\_\_\_\_\_ : Vectors that have the same magnitude but point in opposite directions.

Notice that  $|\vec{AB}| = |\vec{BA}|$  but they point in opposite Directions. Therefore  $\vec{AB} \neq \vec{BA}$ .

You can write an expression for an opposite vector by Placing a negative sign in front of it or by reversing the order of the letters. The opposite of  $\vec{AB}$  can be written as  $-\vec{AB}$  or  $\vec{BA}$

An equivalent expression between the two vectors shown could be  $\vec{AB} = -\vec{BA}$

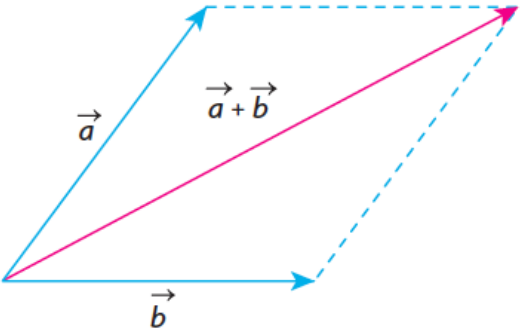
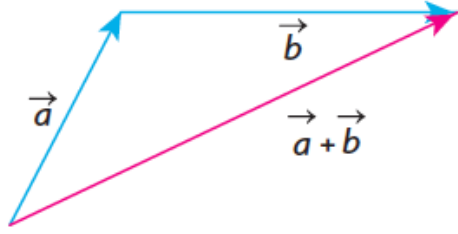


**Example 2:** Given  $\overrightarrow{AB}$ , draw an equivalent vector  $\overrightarrow{CD}$  and an opposite vector  $\overrightarrow{EF}$ . Write equations to show the relationship between the vectors.

**Part 1: Adding Vectors**

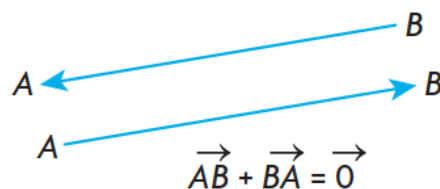
When you add two or more vectors, you are finding a single vector, called the \_\_\_\_\_, that has the same effect as the original vectors applied one after the other.

Two methods:

Parallelogram	Tip to Tail (triangle)
To determine the sum of any two vectors $\vec{a}$ and $\vec{b}$ , arranged tail-to-tail, complete the parallelogram formed by the two vectors. Their sum is the vector that is the diagonal of the constructed parallelogram.	The sum of vectors $\vec{a}$ and $\vec{b}$ can also be found by translating the tail of vector $\vec{b}$ to the head of vector $\vec{a}$ . The resultant is the vector from the tail of $\vec{a}$ to the head of $\vec{b}$ .
	

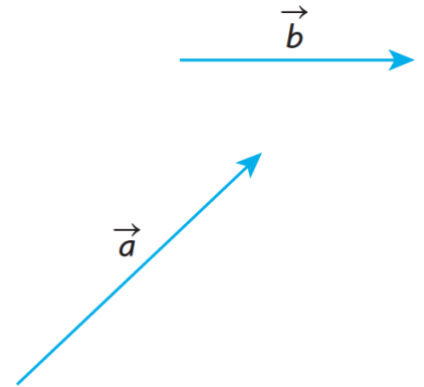
What if we add opposite vectors?

When two opposite vectors are added, the resultant is the zero vector. This means that the combined effect of a vector and its opposite is the zero vector.



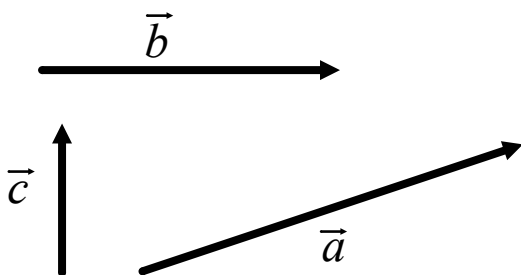
## Part 2: Difference of 2 Vectors

If you want to determine the difference between two vectors,  $\vec{a} - \vec{b}$ , there are two options:



Adding the Opposite	Tail to Tail
<p>The difference between <math>\vec{a}</math> and <math>\vec{b}</math> is found by adding the opposite of vector <math>\vec{b}</math> to <math>\vec{a}</math> using the triangle law of addition.</p>	<p>Another way to think about <math>\vec{a} - \vec{b}</math> is to arrange the vectors tail to tail. In this case, <math>\vec{a} - \vec{b}</math> is the vector that must be added to <math>\vec{b}</math> to get <math>\vec{a}</math></p>
<p><math>\vec{a} + (-\vec{b}) = \vec{a} - \vec{b}</math></p>	<p><math>\vec{a} - \vec{b}</math></p>

**Example 1:** Suppose you are given the vectors  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$  as shown below. Using these three vectors, sketch  $\vec{a} - \vec{b} + \vec{c}$





**Example 2:** In the rectangular box shown below,  $\overrightarrow{OA} = \vec{a}$ ,  $\overrightarrow{OC} = \vec{b}$ , and  $\overrightarrow{OD} = \vec{c}$ . Express each of the following vectors in terms of  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$ .

a)  $\overrightarrow{BC}$

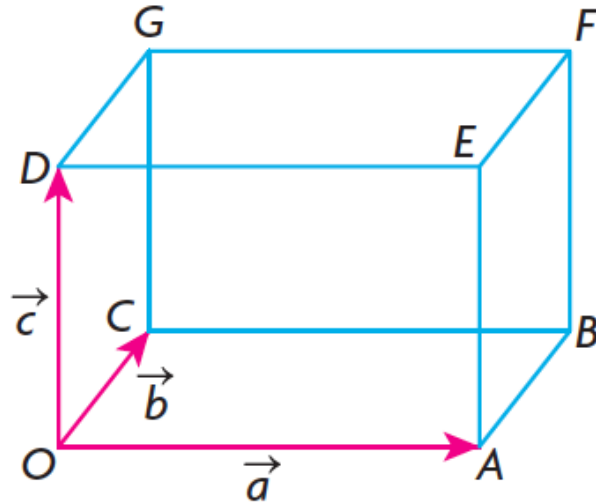
b)  $\overrightarrow{GF}$

c)  $\overrightarrow{OB}$

d)  $\overrightarrow{AC}$

e)  $\overrightarrow{BG}$

f)  $\overrightarrow{OF}$



**Part 3: Properties of Vector Addition**

Commutative Property	$\vec{u} + \vec{v} = \vec{v} + \vec{u}$
Associative Property	$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
Identity Property	$\vec{v} + \vec{0} = \vec{v} = \vec{0} + \vec{v}$

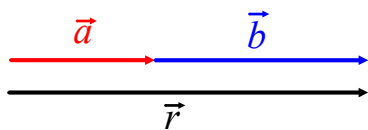
**Example 3:** Simplify each of the following

a)  $(\vec{u} + \vec{v}) - \vec{u}$

b)  $[(\vec{p} + \vec{q}) - \vec{p}] - \vec{q}$

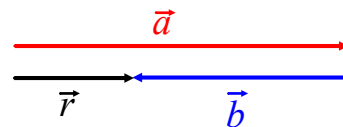
**Part 4: Solving Problems involving Vector Addition and Subtraction**

If you have two vectors acting in the same direction, the overall magnitude is equal to the sum of the two individual magnitudes.



$$|\vec{r}| = |\vec{a}| + |\vec{b}|$$

If you have two vectors acting in opposite directions, the overall magnitude is equal to the difference of the two individual magnitudes.

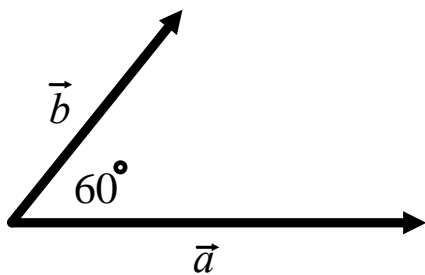


$$|\vec{r}| = |\vec{a}| - |\vec{b}|$$

However, not all forces act in the same or opposite direction. Therefore, we will need some trigonometry to determine the magnitude of resultant vectors.

Rule	When to Use It	
<b>Pythagorean Theorem</b> $a^2 + b^2 = c^2$	Right Triangle Know: 2 sides Want: 3 <sup>rd</sup> side	
$S \frac{O}{H} C \frac{A}{H} T \frac{O}{a}$	Right Triangle Know: 2 sides Want: Angle (use inverse ratio)	Right Triangle Know: 1 side, 1 angle Want: Side
<b>Sine Law</b> $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	Oblique Triangle (no right angle) Know: 2 sides and opposite angle Want: Angle	Oblique Triangle (no right angle) Know: 1 side and all angles Want: Side
<b>Cosine Law</b> $a^2 = b^2 + c^2 - 2bc(\cos A)$ $\cos A = \frac{a^2 - b^2 - c^2}{-2bc}$	Oblique Triangle Know: 2 sides and contained angle Want: 3 <sup>rd</sup> side (use top formula)	Oblique Triangle Know: All 3 sides Want: Angle (use bottom formula)

**Example 4:** Given vectors  $\vec{a}$  and  $\vec{b}$  such that the angle between the two vectors is  $60^\circ$ ,  $|\vec{a}| = 3$ , and  $|\vec{b}| = 2$ , determine  $|\vec{a} + \vec{b}|$ .



**Note:** "angle between vectors" means the angle between the vectors when placed tail to tail.

Translate them tip to tail to determine the resultant vector.

**Example 5:** An airplane heads due south at a speed of 300 km/h and meets a wind from the west at 100 km/h. What is the resultant velocity of the airplane (relative to the ground)?

**Example 6:** In an orienteering race, you walk 100 m due east and then walk  $N70^\circ E$  for 60 m. How far are you from your starting position, and at what bearing?

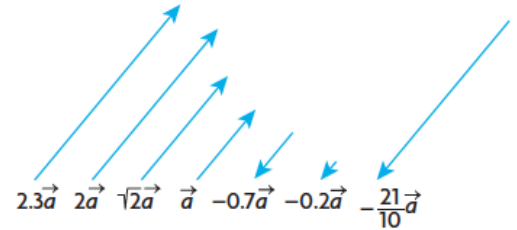
**Part 1: Scalar Multiplication**

In this section we will multiply a vector,  $\vec{a}$ , by a number  $k$  to produce a new vector  $k\vec{a}$ . The number  $k$  used for multiplication is called a SCALAR and can be any real number.

**Reminder:** Scalars have magnitude but no direction. Vectors have both magnitude and direction.

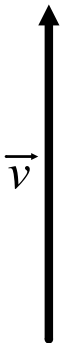
Multiplying a vector by different values of a scalar  $k$  can affect the \_\_\_\_\_ AND \_\_\_\_\_ of the vector.

When two vectors are parallel, these vectors are described as being colinear. They are described this way because they can be translated to be on the same straight line. Scalar multiples of vectors are colinear.

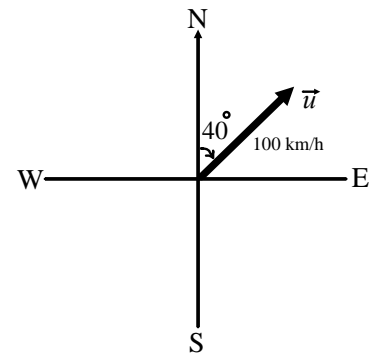
**Rules:**

- If  $k > 0$ , then  $k\vec{a}$  is in the same direction as  $\vec{a}$  with magnitude  $k|\vec{a}|$ 
  - If  $0 < k < 1$ ,  $\vec{a}$  is shortened  $\rightarrow \left|\frac{1}{2}\vec{a}\right| = \frac{1}{2}|\vec{a}|$
  - If  $k > 1$ ,  $\vec{a}$  is lengthened  $\rightarrow |2\vec{a}| = 2|\vec{a}|$
- If  $k < 0$ , then  $k\vec{a}$  is in the opposite direction as  $\vec{a}$  with magnitude  $|k||\vec{a}|$ 
  - If  $-1 < k < 0$ ,  $\vec{a}$  is shortened and changes to the opposite direction  $\rightarrow \left|-\frac{1}{2}\vec{a}\right| = \frac{1}{2}|\vec{a}|$
  - If  $k < -1$ ,  $\vec{a}$  is lengthened and changes to the opposite direction  $\rightarrow |-2\vec{a}| = 2|\vec{a}|$
- If  $k = 0$ , the result is the zero vector  $\vec{0}$
- The distributive property for vector multiplication is  $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

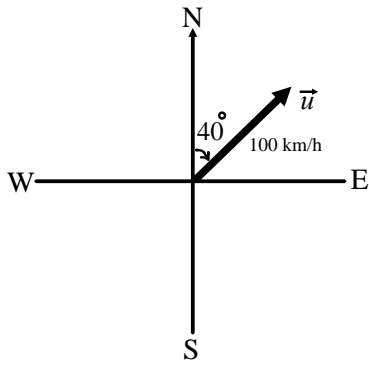
**Example 1:** An airplane is heading due north at 1000 km/h. The airplane's velocity is represented by  $\vec{v}$ . Draw the vectors  $-\vec{v}$ ,  $\frac{1}{2}\vec{v}$ , and  $-\frac{1}{2}\vec{v}$  and give an interpretation for each.



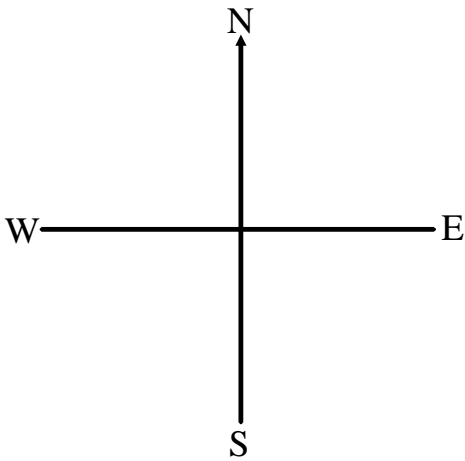
**Example 2:** Consider vector  $\vec{u}$  with magnitude  $|\vec{u}| = 100 \text{ km/h}$ , at a quadrant bearing of  $N40^\circ E$ . Draw a vector with each scalar multiplication. Describe the resulting vector.



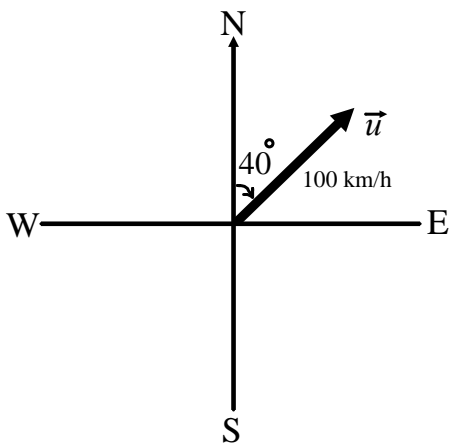
a)  $3\vec{u}$



b)  $0.5\vec{u}$



c)  $-2\vec{u}$



### Vector Properties for Scalar Multiplication

**Distributive Property:** For any scalar  $k \in \mathbb{R}$  and any vectors  $\vec{u}$  and  $\vec{v}$ ,  $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

**Associative Property:** For any scalars  $a, b \in \mathbb{R}$  and any vector  $\vec{v}$ ,  $(ab)\vec{v} = a(b\vec{v})$

**Identity Property:** For any vector  $\vec{v}$ ,  $1\vec{v} = \vec{v}$

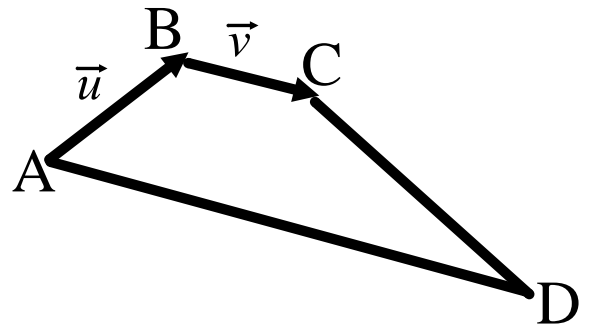
**Example 3:** If  $\vec{x} = 3\vec{i} - 4\vec{j} + \vec{k}$ ,  $\vec{y} = \vec{j} - 5\vec{k}$ , and  $\vec{z} = -\vec{i} - \vec{j} + 4\vec{k}$ , determine each of the following in terms of  $\vec{i}$ ,  $\vec{j}$ , and  $\vec{k}$ .

a)  $\vec{x} + \vec{y}$

b)  $\vec{x} - \vec{y}$

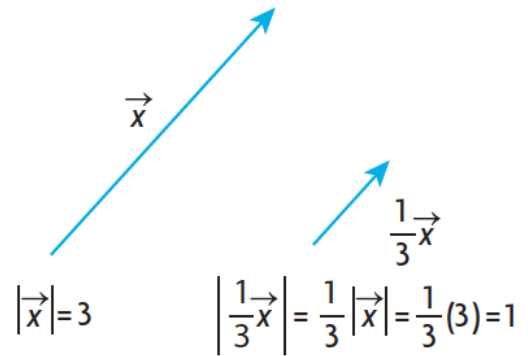
c)  $\vec{x} - 2\vec{y} + 3\vec{z}$

**Example 4:** In trapezoid ABCD,  $BC \parallel AD$  and  $AD = 3BC$ . Let  $\overrightarrow{AB} = \vec{u}$  and  $\overrightarrow{BC} = \vec{v}$ . Express  $\overrightarrow{AD}$ ,  $\overrightarrow{BD}$ , and  $\overrightarrow{CD}$  as linear combinations of  $\vec{u}$  and  $\vec{v}$ .



## Part 2: Creating Unit Vectors

It is sometimes useful to multiply the nonzero vector  $\vec{x}$  by the scalar  $\frac{1}{|\vec{x}|}$ . When we multiply  $\vec{x}$  by  $\frac{1}{|\vec{x}|}$ , we get the vector  $\frac{1}{|\vec{x}|}\vec{x}$ . This is a vector of length one and is called a unit vector, which points in the same direction as  $\vec{x}$ . The concept of unit vectors will prove to be very useful when we discuss applications of vectors.



**Example 5:** Given that  $|\vec{u}| = 4$  and  $|\vec{v}| = 5$  and the angle between  $\vec{u}$  and  $\vec{v}$  is  $120^\circ$ , determine the unit vector in the same direction as  $\vec{u} + \vec{v}$ .

**Part 1: Resultant Force**

When two vectors act on an object, you can use vector addition, the Pythagorean theorem, and trigonometry to find the resultant.

We have shown that if we take any two forces that act at the same point, acting at an angle of  $\theta$  to each other, the forces may be composed to obtain the resultant of these two forces. Furthermore, the resultant of any two forces is unique because there is only one parallelogram that can be formed with these two forces.

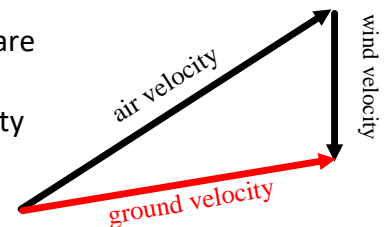
**Example 1:** A sailboat's destination is 8 km east and 6 km north. Find the magnitude and direction, in bearing notation, of the resultant.

**Part 2: Velocity**

The resultant velocity of any two velocities is their sum. In all calculations involving resultant velocities, it is necessary to draw a triangle showing the velocities so there is a clear recognition of the resultant and its relationship to the other two velocities.

For velocity questions involving airplanes:

- When the velocity of the airplane is mentioned, it is understood that we are referring to its air speed (speed relative to the air it is flying in).
- When the velocity of the wind is mentioned, we are referring to its velocity relative to the ground.
- The resultant velocity of the airplane is the velocity of the airplane relative to the ground.
- $\vec{v}_{air} + \vec{w} = \vec{v}_{ground}$



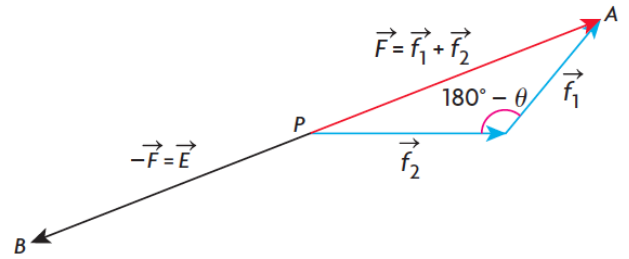


**Example 2:** A plane travels  $N40^\circ E$  at an airspeed of 1000 km/h. Measurement on the ground indicates that the plane is traveling  $N45^\circ E$  at a speed of 1050 km/h. What is the velocity of the wind?

### Part 3: Equilibrant Vector

An equilibrant vector is one that balances another vector or a combination of vectors. It is equal in magnitude but opposite in direction to the resultant vector. If the equilibrant is added to a given system of vectors, the sum of all vectors, including the equilibrant, is  $\vec{0}$ .

The equilibrant of a number of forces is the single force that opposes the resultant of the forces acting on an object. When the equilibrant is applied to the object, this force maintains the object in a state of equilibrium.



**Example 3:** A clown with mass 80 kg is shot out of a cannon with a horizontal force of 2000 N. The vertical force is the acceleration due to gravity, which is  $9.8 \text{ m/s}^2$ , times the mass of the clown.

a) Find the magnitude and direction of the resultant force on the clown.

b) Find the magnitude and direction of the equilibrant force on the clown.

**Example 4:** A mass of 20 kg is suspended from a ceiling by two lengths of rope that make angles of  $60^\circ$  and  $45^\circ$  with the ceiling. Determine the tension in each of the ropes.

**Example 5:** A video projector of mass 2 kg is hung by a 5 m rope from the ceiling. The projector is pulled back 2 m (measured horizontally) by a horizontal force. Find the magnitude of the horizontal force and the tension in the rope.

### Resolving a Vector into its Components

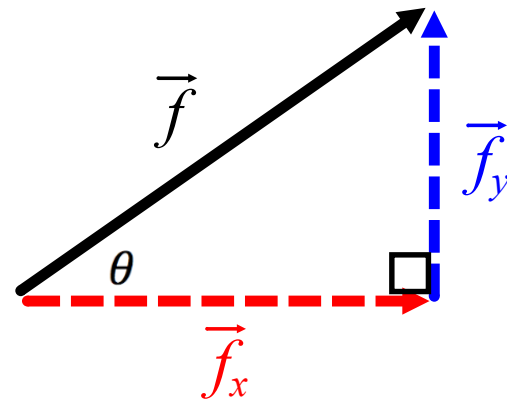
In many situations involving forces, we are interested in a process that is the opposite of composition. This process is called \_\_\_\_\_, which means taking a single force and decomposing it into two components. When we resolve a force into two components, it is possible to do this in an infinite number of ways because there are infinitely many parallelograms having a particular single force as the diagonal. However, the most useful and important way to resolve a force vector occurs when this vector is resolved into two components that are at right angles to each other. These components are usually referred to as the horizontal and vertical components or \_\_\_\_\_.

If  $\vec{f}$  is resolved into its respective horizontal and vertical components,  $\vec{f}_x$  and  $\vec{f}_y$ , then:

$$|\vec{f}_x| =$$

$$|\vec{f}_y| =$$

Where  $\theta$  is the angle that  $\vec{f}$  makes with the  $x$ -axis.



**Example 1:** A tow truck is pulling a car from a ditch. The tension in the cable is 15 000 N at an angle of  $40^\circ$  to the horizontal.

a) Draw a diagram showing the resolution of the force into its rectangular vector components.

b) Determine the magnitudes of the horizontal and vertical vector components of the force.

**Example 2:** Kayla pulls on a rope attached to her sleigh with a force of 200 N. If the rope makes an angle of  $20^\circ$  with the horizontal, determine:

a) the force that pulls the sleigh forward

b) the force that tends to lift the sleigh

**Example 3:** A box weighting 140 N is resting on a ramp that is inclined at an angle of  $20^\circ$ . Resolve the weight into rectangular vector components that keep the box at rest.

**Position Diagram:**

**Vector Diagram:**

The force acting perpendicular to the ramp,  $\vec{n}$ , and the force of friction on the ramp,  $\vec{f}$  are holding the box in a state of equilibrium. Therefore, the resultant must be equal to the opposite of the force of the box.