W5 – Cross Product of Vectors Unit 5

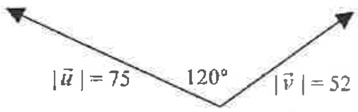
MCV4U

Jensen

SOLUTIONS

1) Determine $\vec{u} \times \vec{v}$.

a)

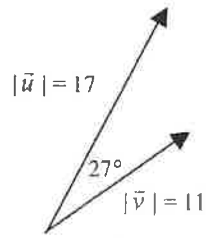


$\vec{u} \times \vec{v} = [75(52)\sin(120)](-\hat{n})$

$= -3377.5\hat{n}$

OR 3377.5 into the page.

b)



$\vec{u} \times \vec{v} = [17(11)\sin(27)](-\hat{n})$

$= -84.9\hat{n}$

OR 84.9 into the page

c) $\vec{u} = [2, -1, 7]$, $\vec{v} = [2, 1, 3]$

$u_1 \; u_2 \; u_3 \qquad v_1 \; v_2 \; v_3$

$-1 \quad 1$
$7 \quad 3$
$2 \quad 2$
$-1 \quad 1$

$\vec{u} \times \vec{v} = [-1(3) - 7(1), 7(2) - 2(3), 2(1) - (-1)(2)]$

$= [-10, 8, 4]$

d) $\vec{u} = [-3, 4, 7]$, $\vec{v} = [4, 3, -5]$

$4 \quad 3$
$7 \quad -5$
$-3 \quad 4$
$4 \quad 3$

$\vec{u} \times \vec{v} = [4(-5) - 7(3), 7(4) - (-3)(-5), -3(3) - 4(4)]$

$= [-41, 13, -25]$

e) $\vec{u} = 3\hat{\imath} + 4\hat{\jmath} - \hat{k}$ $\quad \vec{v} = 5\hat{\imath} + \hat{\jmath} - 2\hat{k}$

$= [3, 4, -1] \qquad = [5, 1, -2]$

$4 \quad 1$
$-1 \quad -2$
$3 \quad 5$
$4 \quad 1$

$\vec{u} \times \vec{v} = [4(-2) - (-1)(1), -1(5) - 3(-2), 3(1) - 4(5)]$

$= [-7, 1, -17]$

f) $\vec{u} = 2\hat{\imath} - 3\hat{\jmath} + 7\hat{k}$ $\quad \vec{v} = -\hat{\imath} + \hat{\jmath}$

$= [2, -3, 7] \qquad = [-1, 1, 0]$

$-3 \quad 1$
$7 \quad 0$
$2 \quad -1$
$-3 \quad 1$

$\vec{u} \times \vec{v} = [-3(0) - 7(1), 7(-1) - 2(0), 2(1) - (-3)(-1)]$

$= [-7, -7, -1]$

2) Find a vector perpendicular to each of the following pairs of vectors. Use the dot product to check your answer.

a) [5, 0, 1] and [–2, 5, 8]

$$\begin{array}{cc} 0 & 5 \\ 1 & 8 \\ 5 & -2 \\ 0 & 5 \end{array}$$

$[0(8) - 1(5), 1(-2) - 5(8), 5(5) - 0(-2)]$

$= [-5, -42, 25]$

$[-5, -42, 25] \cdot [5, 0, 1] = -25 + 0 + 25 = 0$

$[-5, -42, 25] \cdot [-2, 5, 8] = 10 - 210 + 200 = 0$

$\therefore [-5, -42, 25]$ is orthogonal to both.

b) [1, 4, –2] and [–4, 9, 0]

$$\begin{array}{cc} 4 & 9 \\ -2 & 0 \\ 1 & -4 \\ 4 & 9 \end{array}$$

$[4(0) - (-2)(9), -2(-4) - 1(0), 1(9) - 4(-4)]$

$= [18, 8, 25]$

$[18, 8, 25] \cdot [1, 4, -2] = 18 + 32 - 50 = 0$

$[18, 8, 25] \cdot [-4, 9, 0] = -72 + 72 + 0 = 0$

$\therefore [18, 8, 25]$ is orthogonal to both.

3) Find a unit vector perpendicular to $\vec{a}$= [6, –2, –3] and $\vec{b}$ = [5, 1, –4].

$\vec{a} \times \vec{b}$

$$\begin{array}{cc} -2 & 1 \\ -3 & -4 \\ 6 & 5 \\ -2 & 1 \end{array}$$

$\vec{a} \times \vec{b} = [-2(-4) - (-3)(1), -3(5) - 6(-4), 6(1) - (-2)(5)]$

$= [11, 9, 16]$

$|\vec{a} \times \vec{b}| = \sqrt{458}$

unit vector $= \dfrac{1}{|\vec{a} \times \vec{b}|} [\vec{a} \times \vec{b}]$

$= \dfrac{1}{\sqrt{458}} [11, 9, 16]$

$= \left[\dfrac{11}{\sqrt{458}}, \dfrac{9}{\sqrt{458}}, \dfrac{16}{\sqrt{458}}\right]$

4) Given $\vec{a}$ = [1, –2, –1], $\vec{b}$ = [2, 2, –1] and $\vec{c}$ = [2, –3, –4], evaluate each of the following:

a) $\vec{a} \times (\vec{b} \times \vec{c})$

$\vec{b} \times \vec{c}$

$$\begin{array}{cc} 2 & -3 \\ -1 & -4 \\ 2 & 2 \\ 2 & -3 \end{array}$$

$\vec{a} \times (\vec{b} \times \vec{c})$

$$\begin{array}{cc} -2 & 6 \\ -1 & -10 \\ 1 & -11 \\ -2 & 6 \end{array}$$

$\vec{b} \times \vec{c} = [2(-4) - (-1)(-3), -1(2) - 2(-4), 2(-3) - 2(2)]$

$= [-11, 6, -10]$

$\vec{a} \times (\vec{b} \times \vec{c}) = [-2(-10) - (-1)(6), -1(-11) - (1)(-10), 1(6) - (-2)(-11)]$

$= [26\ 21, -16]$

b) $(\vec{a} \times \vec{b}) \times \vec{c}$

$\vec{a} \times \vec{b}$

$$\begin{array}{cc} -2 & 2 \\ -1 & -1 \\ 1 & 2 \\ -2 & 2 \end{array}$$

$\vec{a} \times \vec{b} \times \vec{c}$

$$\begin{array}{cc} -1 & -3 \\ 6 & -4 \\ 4 & 2 \\ -1 & -3 \end{array}$$

$\vec{a} \times \vec{b} = [-2(-1) - (-1)(2), -1(2) - 1(-1), 1(2) - (-2)(2)]$

$= [4, -1, 6]$

$(\vec{a} \times \vec{b}) \times \vec{c}$

$= [-1(-4) - 6(-3), 6(2) - 4(-4), 4(-3) - (-1)(2)]$

$= [22, 28, -10]$

$\vec{a} = [1,-2,-1]$ $\quad \vec{b} = [2,2,-1]$ $\quad \vec{c} = [2,-3,-4]$

c) $\vec{a} \times \vec{c} - \vec{a} \times \vec{b}$

$\vec{a} \times \vec{c}$

$$\begin{array}{cc} -2 & -3 \\ -1 & -4 \\ 1 & 2 \\ -2 & -3 \end{array}$$

$\vec{a}\times\vec{c} = [-2(-4) - (-1)(-3), (-1)(2) - 1(-4), 1(-3) - 2(-2)]$
$= [5,2,1]$

$\vec{a}\times\vec{b} = [4,-1,6]$

$(\vec{a}\times\vec{c}) - (\vec{a}\times\vec{b}) = [1,3,-5]$

d) $\vec{b} \times 3\vec{c}$

$\vec{b} \times 3\vec{c}$

$$\begin{array}{cc} 2 & -9 \\ -1 & -12 \\ 2 & 6 \\ 2 & -9 \end{array}$$

$\vec{b}\times 3\vec{c} = [2(-12) - (-1)(-9), (-1)(6) - 2(-12), 2(-9) - 2(6)]$
$= [-33, 18, -30]$

e) $(\vec{a} \times \vec{c}) \cdot \vec{b}$

$\vec{a}\times\vec{c} = [5,2,1]$

$(\vec{a}\times\vec{c}) \cdot \vec{b} = [5,2,1] \cdot [2,2,-1]$
$= 5(2) + 2(2) + 1(-1)$
$= 13$

f) $(\vec{a} \times \vec{b}) \cdot \vec{c}$

$= [4,-1,6] \cdot [2,-3,-4]$
$= 4(2) + (-1)(-3) + 6(-4)$
$= -13$

g) $|\vec{a} \times \vec{b}|$

$= \sqrt{(4)^2 + (-1)^2 + (6)^2}$

$= \sqrt{53}$

h) $|\vec{a} \times (\vec{b} - \vec{c})|$

$\vec{b} - \vec{c} = [2,2,-1] - [2,-3,-4]$
$= [0,5,3]$

$\vec{a} \times (\vec{b} - \vec{c})$

$$\begin{array}{cc} -2 & 5 \\ -1 & 3 \\ 1 & 0 \\ -2 & 5 \end{array}$$

$\vec{a}\times(\vec{b}-\vec{c}) = [-2(3) - (-1)(5), -1(0) - 1(3), 1(5) - (-2)(0)]$
$= [-1,-3,5]$

$|\vec{a}\times(\vec{b}-\vec{c})| = \sqrt{35}$

5) Use the cross product to determine the angles between the vectors $\vec{a} = [2, 1, -3]$ and $\vec{b} = [5, -4, 3]$. Consider ambiguous case. Use dot product to confirm or use graphing software to inspect.

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} \qquad \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

$$\begin{array}{cc} \vec{a} & \times \quad \vec{b} \\ 1 & -4 \\ -3 & 3 \\ 2 & 5 \\ 1 & -4 \end{array}$$

$$\sin\theta = \frac{\sqrt{691}}{(\sqrt{14})(\sqrt{50})} \qquad \cos\theta = \frac{2(5) + 1(-4) + (-3)(3)}{(\sqrt{14})(\sqrt{50})}$$

$$\vec{a} \times \vec{b} = [1(3) - (-3)(-4), -3(5) - 2(3), 2(-4) - 1(5)]$$

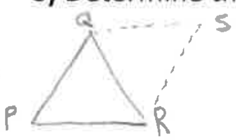
$$\vec{a} \times \vec{b} = [-9, -21, -13]$$

$$\theta_1 \simeq 83.5^\circ \qquad \cos\theta = \frac{-3}{\sqrt{700}}$$

$$\boxed{\begin{aligned} \theta_2 &= 180 - \theta_1 \\ &= 96.5^\circ \end{aligned}} \qquad \theta = 96.5^\circ$$

verified using dot product.

6) Determine the area of ΔPQR with vertices of $P(3, -2, 7)$, $Q(2, 2, -3)$, and $R(1, 1, 2)$.



$$\vec{PQ} = [-1, 4, -10] \qquad \vec{PR} = [-2, 3, -5]$$

$$\vec{PQ} \times \vec{PR} = [4(-5) - (-10)(3), -10(-2) - (-1)(-5), -1(3) - 4(-2)]$$

$$\begin{array}{cc} 4 & 3 \\ -10 & -5 \\ -1 & -2 \\ 4 & 3 \end{array} \qquad = [10, 15, 5]$$

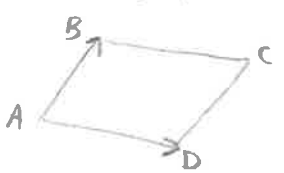
$$\text{Area of } \Delta PQR = \frac{1}{2}|\vec{PQ} \times \vec{PR}| = \frac{1}{2}\sqrt{350}$$

$$= \frac{1}{2}(5)\sqrt{14}$$

$$= 2.5\sqrt{14} \text{ units}^2$$

(2, −1, −1) (−4, −2, 3) (2, 3, 2)

7) Determine the area of the parallelogram ABCD defined by the vertices A(~~2, 1, 5~~), B(~~−2, 7, 8~~), C(~~1, 3, 8~~), and D(~~4, −3, 7~~). (8, 4, −2)



$$\vec{AB} = [-6, -1, 4] \qquad \vec{AD} = [6, 5, -1]$$

$$\vec{AB} \times \vec{AD} = [-1(-1) - 4(5), 4(6) - (-6)(-1), -6(5) - (-1)(6)]$$

$$\begin{array}{cc} -1 & 5 \\ 4 & -1 \\ -6 & 6 \\ -1 & 5 \end{array} \qquad = [-19, 18, -24]$$

$$\text{Area} = |\vec{AB} \times \vec{AD}| = \sqrt{1261} \text{ units}^2$$

ANSWER KEY:

1)a) $-3377.5\hat{n}$ or 3377.5 in to the page **b)** $-84.9\hat{n}$ or 84.9 in to the page **c)** [−10, 8, 4] **d)** [−41, 13, −25] **e)** [−7, 1, −17] **f)** [−7, −7, −1]

2)a) [−5, −42, 25] **b)** [18, 8, 25]

3) $\frac{1}{\sqrt{458}}[11, 9, 16]$

4)a) [26, 21, −16] **b)** [22, 28, −10] **c)** [1, 3, −5] **d)** [−33, 18, −30] **e)** 13 **f)** −13 **g)** $\sqrt{53}$ **h)** $\sqrt{35}$

5) 96.5°

6) $2.5\sqrt{14}$ units²

7) ~~√1556 units²~~ $\sqrt{1261}$ units²