L5 – Curve Sketching Unit 2

MCV4U

Jensen

SOLUTIONS

1) For each function, determine the coordinates of the local extrema. Classify each point as a local max or min.

a) $y = 2x - 3x^2$

$y' = 2 - 6x$

$0 = 2 - 6x$

$x = \frac{1}{3}$

$y(\frac{1}{3}) = \frac{1}{3}$

critical point: $(\frac{1}{3}, \frac{1}{3})$

2nd derivative test:

$y'' = -6$

$y''(\frac{1}{3}) = -6$; concave down

$\therefore (\frac{1}{3}, \frac{1}{3})$ is a local MAX

b) $y = 2t^3 + 6t^2 + 6t + 4$

$y' = 6t^2 + 12t + 6$

$0 = 6(t^2 + 2t + 1)$

$0 = 6(t+1)^2$

CN: $t = -1$

CP: $(-1, 2)$

2nd DT

$y'' = 12t + 12$

$y''(-1) = 0$

$\therefore$ 2nd DT fails; check 1st DT.

	$-\infty$ to -1	-1 to ∞
	-2	0
y'	$+$	$+$
y	incr. ↗	incr. ↗

$(-1, 2)$ is NOT a local max or min

2) For each function, determine the coordinates of any points of inflection.

a) $f(x) = 2x^3 - 4x^2$

$f'(x) = 6x^2 - 8x$

$f''(x) = 12x - 8$

$0 = 12x - 8$

$x = \frac{2}{3}$

$f(\frac{2}{3}) = \frac{-32}{27}$

Possible PoI is $(\frac{2}{3}, \frac{-32}{27})$

Test:

	$-\infty$ to $2/3$	$2/3$ to ∞
	0	1
$f''(x)$	$-$	$+$
$f(x)$	con. down ∩	con. up ∪

$\boxed{\text{PoI: } (\frac{2}{3}, \frac{-32}{27})}$

b) $f(x) = 3x^5 - 5x^4 - 40x^3 + 120x^2$

$f'(x) = 15x^4 - 20x^3 - 120x^2 + 240x$

$f''(x) = 60x^3 - 60x^2 - 240x + 240$

$0 = 60(x^3 - x^2 - 4x + 4)$

$0 = x^2(x-1) - 4(x-1)$

$0 = (x-1)(x^2-4)$

$0 = (x-1)(x-2)(x+2)$

$x_1 = 1$ $x_2 = 2$ $x_3 = -2$

$f(1) = 78$ $f(2) = 176$ $f(-2) = 624$

Test:

	$-\infty$ to -2	-2 to 1	1 to 2	2 to ∞
	-3	0	1.5	3
$f''(x)$	$-$	$+$	$-$	$+$
$f(x)$	down ∩	up ∪	down ∩	up ∪

$\boxed{\text{PoI's: } (-2, 624), (1, 78), \text{ and } (2, 176)}$

3) Use the algorithm for curve sketching to analyze the key features of each of the following functions and to sketch a graph of them.

a) $f(x) = x^4 - 8x^3$

1. No domain restrictions; no asymptotes

2. $0 = x^3(x-8)$ x-int: $(0,0)$, $(8,0)$ $f(0) = 0$ y-int: $(0,0)$

$x_1 = 0$ $\boldsymbol{x_2 = 8}$

3. $f'(x) = 4x^3 - 24x^2$

$0 = 4x^2(x-6)$

$x_1 = 0$ $x_2 = 6$

$f(0) = 0$ $f(6) = -432$

critical points: $(0,0)$, $(6,-432)$

4. $f''(x) = 12x^2 - 48x$

$0 = 12x(x-4)$

$x_1 = 0$ $x_2 = 4$

$f(0) = 0$ $f(4) = -256$

possible points of inflection: $(0,0)$, $(4,-256)$

5/6.

	$-\infty$ to 0	0 to 4	4 to 6	6 to ∞
Test	-1	1	5	7
$f'(x)$	$-$	$-$	$-$	$+$
$f''(x)$	$+$	$-$	$+$	$+$
$F(x)$	Con. UP Decreasing	Con. Down Decreasing	Con. UP Decreasing	Con. UP Increasing

POI (at 0) POI (at 4) local min (at 6)

7. Local min: $(6,-432)$

Local max: NONE

Points of inflection: $(0,0)$ and $(4,-256)$

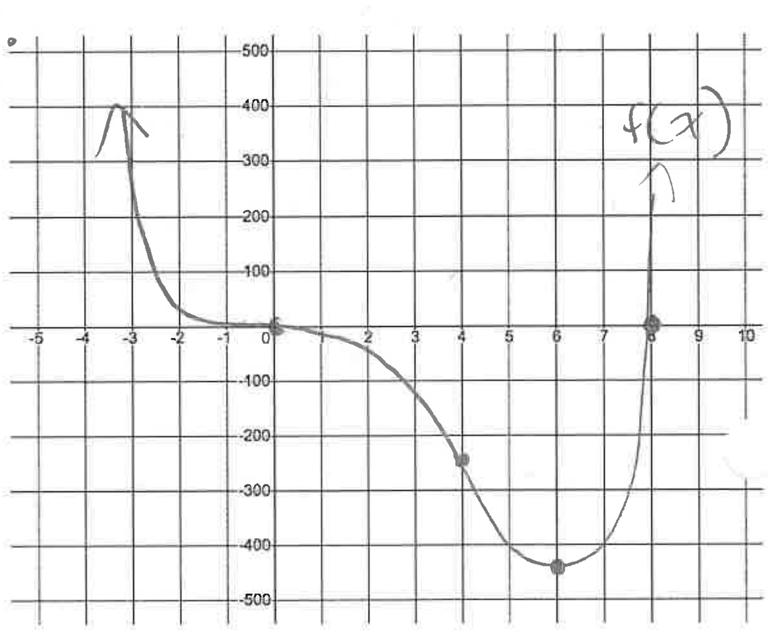
increasing: $x > 6$

decreasing: $x < 6$

C.U.: $x < 0$, $x > 4$

C.D.: $0 < x < 4$

8.



b) $g(x) = 3x^3 + 7x^2 + 3x - 1$

① No restrictions on the domain ; no asymptotes

② x-int

Test: $g(-1) = 0$ $\therefore x+1$ is a Factor

$0 = 3x^3 + 7x^2 + 3x - 1$

$0 = (x+1)(3x^2 + 4x - 1)$

$x_1 = -1$ $\quad x = \dfrac{-4 \pm \sqrt{(4)^2 - 4(3)(-1)}}{2(3)}$

$x = \dfrac{-4 \pm \sqrt{28}}{6}$

$x_2 \simeq 0.215$ $\quad x_3 \simeq -1.549$

$$\begin{array}{r|rrrr} -1 & 3 & 7 & 3 & -1 \\ & \downarrow & -3 & -4 & 1 \\ \hline \times & 3 & 4 & -1 & 0 \\ & x^2 & x & * & R \end{array}$$

x-int: $(-1, 0)$, $(0.215, 0)$, and $(-1.549, 0)$

y-int:

$g(0) = -1$

$(0, -1)$

③ $g'(x) = 9x^2 + 14x + 3$

$0 = 9x^2 + 14x + 3$

$x = \dfrac{-14 \pm \sqrt{(14)^2 - 4(9)(3)}}{2(9)}$

$x = \dfrac{-14 \pm \sqrt{88}}{18}$

$x_1 \simeq -0.26$ $\quad x_2 \simeq -1.30$

$g(-0.26) = -1.36$ $\quad g(-1.3) = 0.34$

Critical Points: $(-0.26, -1.36)$, $(-1.3, 0.34)$

④ $g''(x) = 18x + 14$

$0 = 18x + 14$

$x = -\frac{7}{9} \simeq -0.78$

$g\left(-\frac{7}{9}\right) = \dfrac{-124}{243} \simeq -0.51$

Possible PoI: $(-0.78, -0.51)$

⑦ Local min: $(-0.26, -1.36)$

Local max: $(-1.3, 0.34)$

Point of inflection: $(-0.78, -0.51)$

5/6

$-\infty$ … ∞	$-\infty$ to -1.3	-1.3 to -0.78	-0.78 to -0.26	-0.26 to ∞
Test	-2	-1	-0.5	0
$g'(x)$	$+$	$-$	$-$	$+$
$g''(x)$	$-$	$-$	$+$	$+$
	Con. DOWN Increasing	Con. DOWN Decreasing	Con. UP decreasing	Con. UP increasing
$g(x)$				

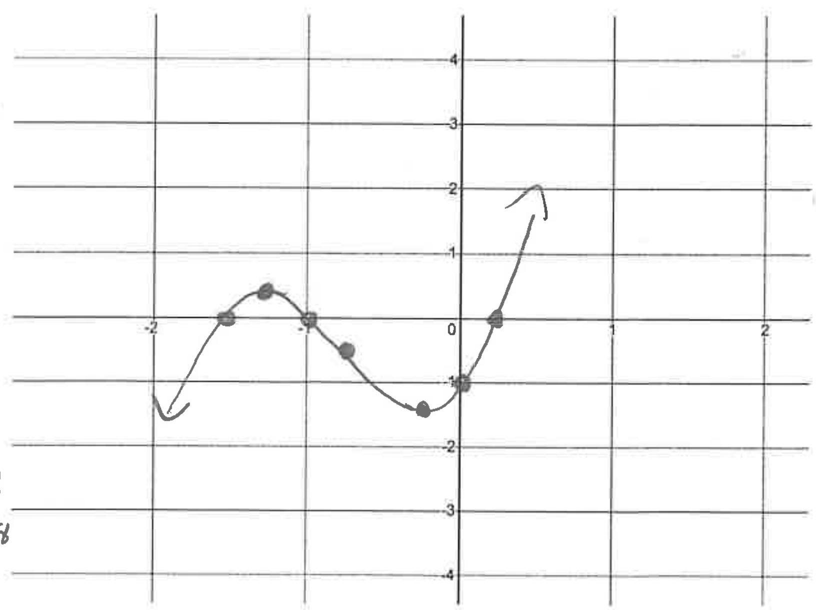
local max (at -1.3) $\quad$ POI (at -0.78) $\quad$ local min (at -0.26)

increasing: $x < -1.3$, $x > -0.26$

decreasing: $-1.3 < x < -0.26$

C.U.: $x > -0.78$

C.D.: $x < -0.78$



c) $h(x) = 2x^4 - 26x^2 + 72$

① no restrictions ; no asymptotes

② x-int

$0 = 2x^4 - 26x^2 + 72$

$0 = x^4 - 13x^2 + 36$

$0 = (x^2)^2 - 13(x^2) + 36$

$0 = (x^2-9)(x^2-4)$

$0 = (x-3)(x+3)(x-2)(x+2)$

$x_1 = -3 \quad x_2 = -2 \quad x_3 = 2 \quad x_4 = 3$

x-int: $(-3,0), (-2,0), (2,0), (3,0)$

y-int

$h(0) = 72$

$(0, 72)$

③ $h'(x) = 8x^3 - 52x$

$0 = 4x(2x^2 - 13)$

$x_1 = 0 \qquad 2x^2 - 13 = 0$

$h(0) = 72 \qquad x = \pm\sqrt{13/2}$

$x_2 \simeq 2.55 \quad x_3 \simeq -2.55$

$h(2.55) \simeq -12.5 \quad h(-2.55) \simeq -12.5$

critical points: $(0,72), (2.55,-12.5), (-2.55,-12.5)$

④ $h''(x) = 24x^2 - 52$

$0 = 24x^2 - 52$

$x = \pm\sqrt{13/6}$

$x_1 \simeq 1.47 \quad x_2 \simeq -1.47$

$h(1.47) \simeq 25.16 \quad h(-1.47) \simeq 25.16$

Possible POI's: $(1.47, 25.16), (-1.47, 25.16)$

5/6

	$-\infty$ to -2.55	-2.55 to -1.47	-1.47 to 0	0 to 1.47	1.47 to 2.55	2.55 to ∞
	-3	-2	-1	1	2	3
$h'(x)$	−	+	+	−	−	+
$h''(x)$	+	+	−	−	+	+
$h(x)$	CU Decreasing	CU Increasing	CD incr.	CD decr.	CU decr.	CU incr.

min $\quad$ POI $\quad$ max $\quad$ POI $\quad$ min

increasing: $-2.55 < x < 0, \; x > 2.55$

decreasing: $x < -2.55, \; 0 < x < 2.55$

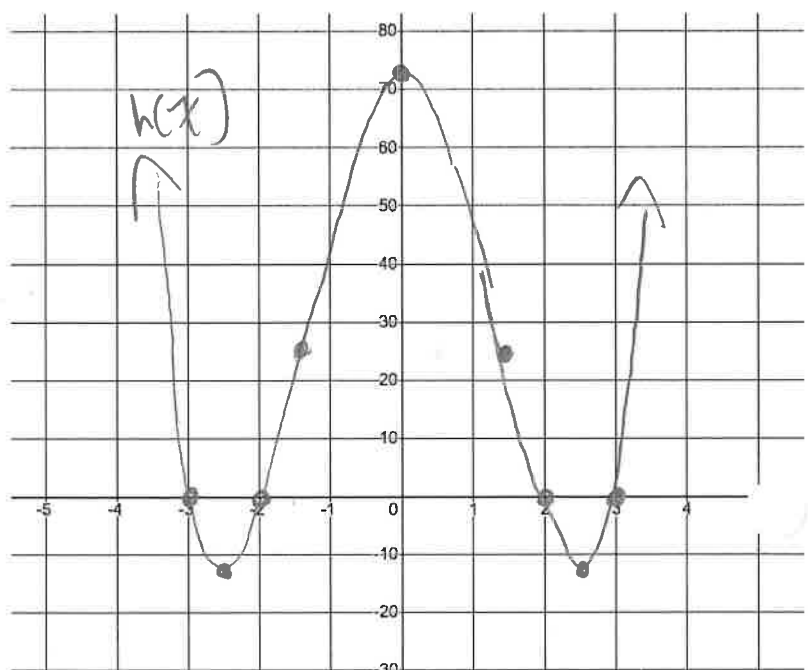
CU: $x < -1.47, \; x > 1.47$

CD: $-1.47 < x < 1.47$

⑦ Local min: $(-2.55, -12.5)$ and $(2.55, -12.5)$

Local max: $(0, 72)$

POI's: $(-1.47, 25.16)$ and $(1.47, 25.16)$



d) $j(x) = \frac{x^2+2x-4}{x^2}$

① $x \neq 0$; VA at $x=0$
HA at $y=1$

② x-int

$x = \frac{-2 \pm \sqrt{(2)^2-4(1)(-4)}}{2(1)}$

$x = \frac{-2 \pm \sqrt{20}}{2}$

$x = \frac{-2 \pm 2\sqrt{5}}{2}$

$x = -1 \pm \sqrt{5}$

$(1.24, 0)$
$(-3.24, 0)$

y-int:

$j(0) = \frac{-4}{0}$ = undefined

∴ no y-intercept.

③ $j'(x) = \frac{(2x+2)(x^2) - 2x(x^2+2x-4)}{x^4}$

$j'(x) = \frac{\cancel{x}[(2x+2)(x) - 2(x^2+2x-4)]}{x^{\cancel{4}3}}$

$j'(x) = \frac{2x^2+2x-2x^2-4x+8}{x^3}$

$j'(x) = \frac{-2x+8}{x^3}$

$0 = -2x+8$
$x = 4$
$j(4) = 1.25$

$x=0$ is not a critical # because it is NOT in the domain of $j'(x)$

Critical #: $(4, 1.25)$

④ $j''(x) = \frac{-2(x^3) - 3x^2(-2x+8)}{x^6}$

$j''(x) = \frac{-2x^3+6x^3-24x^2}{x^6}$

$j''(x) = \frac{4x^3-24x^2}{x^6}$

$j''(x) = \frac{4x^2(x-6)}{x^6}$

$j''(x) = \frac{4(x-6)}{x^4}$

$0 = 4(x-6)$
$x = 6$
$j(6) = 1.22$

possible POI is $(6, 1.22)$

5/6

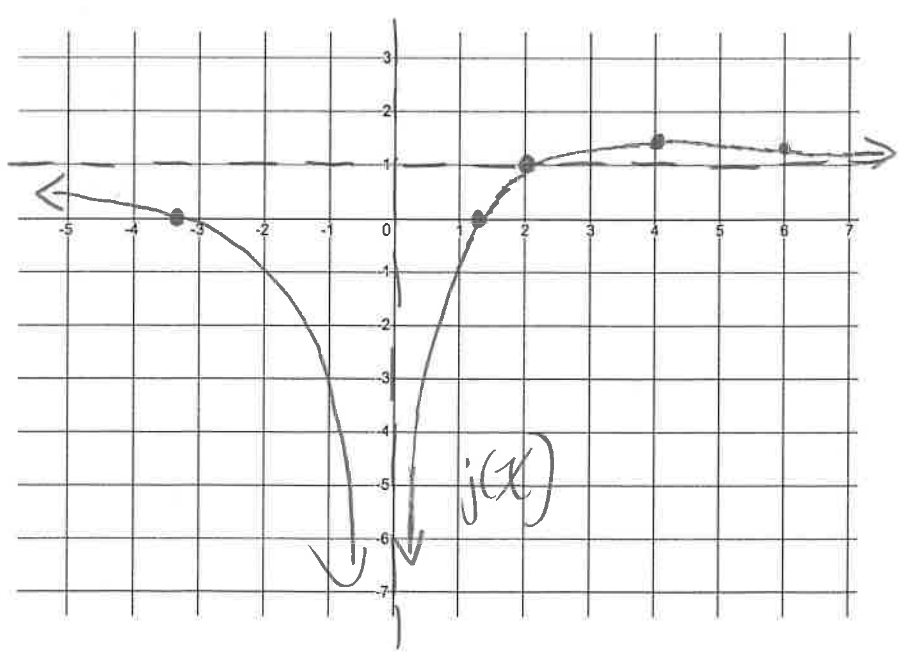
	$-\infty$ … 0	0 … 4	4 … 6	6 … ∞
	-1	1	5	7
$j'(x)$	−	+	−	−
$j''(x)$	−	−	−	+
$j(x)$	CD decrease	CD increase	CD decrease	CU decrease
	VA	Max	POI	

increasing: $0<x<4$

decreasing: $x<0$, $x>4$

CU: $x>6$

CD: $x<0$, $0<x<6$

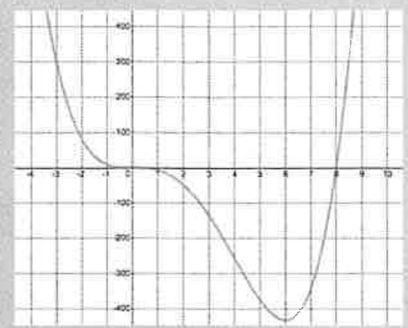


Answers:

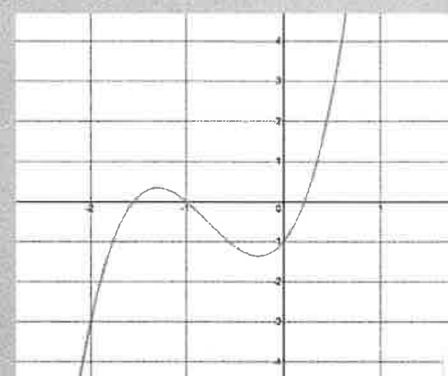
1)a) max: $\left(\frac{1}{3},\frac{1}{3}\right)$ **b)** no local extrema; $(-1,2)$ is an inflection point NOT a max or min

2)a) $\left(\frac{2}{3},-\frac{32}{27}\right)$ **b)** $(-2,624)$, $(2,176)$, and $(0,0)$

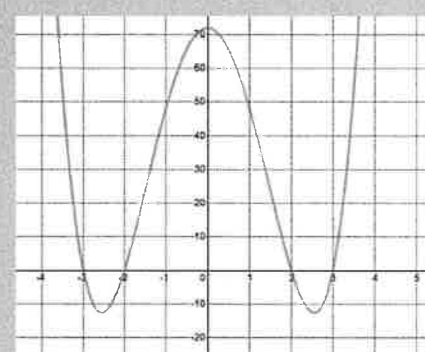
3)a) x-int: $(0,0)$ and $(8,0)$; y-int: $(0,0)$; local max: none; local min: $(6,-432)$; POI: $(0,0)$ and $(4,-256)$; increasing: $x>6$; decreasing: $x<0$ and $0<x<6$; concave up: $x<0$ and $x>4$; concave down: $0<x<4$



b) x-int: $(-1,0)$, $(0.215,0)$, and $(-1.549,0)$; y-int: $(0,-1)$; local max: $(-1.3,0.34)$; local min: $(-0.26,-1.36)$; POI: $(-0.78,-0.51)$; increasing: $x<-1.3$ and $x>-0.26$; decreasing: $-1.3<x<-0.26$; concave up: $x>-0.78$; concave down: $x<-0.78$



c) x-int: $(-3,0)$, $(-2,0)$, $(2,0)$ and $(3,0)$; y-int: $(0,72)$; local max: $(0,72)$; local min: $(-2.55,-12.5)$ and $(2.55,-12.5)$; POI: $(-1.47,25.16)$ and $(1.47,25.16)$; increasing: $-2.55<x<0$ and $x>2.55$; decreasing: $x<-2.55$, and $0<x<2.55$; concave up: $x<-1.47$ and $x>1.47$; concave down: $-1.47<x<1.47$



d) VA: $x=0$; HA: $y=1$; x-int: $(-3.24,0)$, and $(1.24,0)$; y-int: none; local max: $(4,1.25)$; local min: none; POI: $(6,1.22)$; increasing: $0<x<4$ and; decreasing: $x<0$, and $x>4$; concave up: $x>6$; concave down: $x<0$ and $0<x<6$