W5 – Differentiation Rules for Exponential Functions Unit 3

MCV4U

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SOLUTIONS

1)a) Rewrite the function $y = b^x$ with base e.

$y = e^{\ln(b^x)}$

$y = e^{x\ln(b)}$

b) Find the derivative of your function in part a) and simplify.

$y' = e^{x\ln(b)}\,[\ln(b)]$

$y' = e^{\ln(b^x)}\,[\ln(b)]$

$y' = b^x \ln(b)$

2) Differentiate with respect to x.

a) $y = e^{-3x}$

$y' = e^{-3x}(-3)$

$y' = -3e^{-3x}$

b) $f(x) = e^{4x-5}$

$f'(x) = e^{4x-5}(4)$

$f'(x) = 4e^{4x-5}$

c) $y = e^{2x} - e^{-2x}$

$y' = e^{2x}(2) - e^{-2x}(-2)$

$y' = 2e^{2x} + 2e^{-2x}$

d) $y = 2^x + 3^x$

$y' = 2^x \ln(2) + 3^x \ln(3)$

e) $f(x) = 3e^{2x} - 2^{3x}$

$f'(x) = 3e^{2x}(2) - 2^{3x}\ln(2)(3)$

$= 6e^{2x} - 3\ln(2)(2^{3x})$

f) $y = 4xe^x$

$y' = 4(e^x) + e^x(4x)$

$y' = 4e^x + 4xe^x$

g) $y = 5^x e^{-x}$

$y' = 5^x(\ln 5)(e^{-x}) + (e^{-x})(-1)(5^x)$

$y' = 5^x(\ln 5)(e^{-x}) - e^{-x}(5^x)$

$y' = 5^x(e^{-x})(\ln 5 - 1)$

h) $f(x) = xe^{2x} + 2e^{-3x}$

$f'(x) = 1(e^{2x}) + (e^{2x})(2)x - 6e^{-3x}$

$f'(x) = e^{2x} + 2xe^{2x} - 6e^{-3x}$

$f'(x) = e^{2x}(1 + 2x - 6e^{-5x})$

3) Determine the derivative with respect to x for each function.

a) $y = e^{-x} \sin x$

$y' = -e^{-x} \sin x + \cos x (e^{-x})$

$y' = e^{-x}(-\sin x + \cos x)$

$y' = e^{-x}(\cos x - \sin x)$

b) $y = e^{\cos x}$

$y' = e^{\cos x}(-\sin x)$

$y' = -\sin x (e^{\cos x})$

c) $f(x) = e^{2x}(x^2 - 3x + 2)$

$f'(x) = 2e^{2x}(x^2 - 3x + 2) + (2x - 3)(e^{2x})$

$f'(x) = e^{2x}[2x^2 - 6x + 4 + 2x - 3]$

$f'(x) = e^{2x}(2x^2 - 4x + 1)$

d) $g(x) = 2x^2 e^{\cos(2x)}$

$g'(x) = 4x[e^{\cos(2x)}] + e^{\cos(2x)}[-\sin(2x)](2)(2x^2)$

$g'(x) = 4xe^{\cos(2x)}[1 - x\sin(2x)]$

4) Identify the coordinates of any local extrema of the function $y = e^x - e^{2x}$

$y' = e^x - 2e^{2x}$

$0 = e^x - 2e^{2x}$

$2e^{2x} = e^x$

$2e^x = 1$

$e^x = \frac{1}{2}$

$x = \ln(\frac{1}{2})$

$y[\ln(\frac{1}{2})] = e^{\ln(\frac{1}{2})} - e^{2[\ln(\frac{1}{2})]}$

$= \frac{1}{2} - \frac{1}{4}$

$= \frac{1}{4}$

2nd deriv. test:

$y'' = e^x - 4e^{2x}$

$y''[\ln(\frac{1}{2})] = \frac{1}{2} - 4(\frac{1}{4})$

$= -\frac{1}{2}$

$\therefore$ concave down

$\boxed{\text{Max at } [\ln(\tfrac{1}{2}), \tfrac{1}{4}]}$

5) Find an equation for the tangent to the curve $y = 2e^{2x} + 2x + 1$ when $x = 0$.

Point:

$y(0) = 2e^{2(0)} + 2(0) + 1$

$y(0) = 2 + 0 + 1$

$y(0) = 3$

$(0, 3)$

Slope:

$y' = 4e^{2x} + 2$

$y'(0) = 4e^{2(0)} + 2$

$y'(0) = 4 + 2$

$y'(0) = 6$

$m = 6$

Eq^n

$y = mx + b$

$3 = 6(0) + b$

$b = 3$

$\boxed{y = 6x + 3}$

6) Find the equation of the tangent to $y = x \ln x$ that is parallel to $y = 3x + 7$.

$m = 3$

$y' = (1)\ln(x) + \frac{1}{x \ln e}(x)$

$y' = \ln(x) + 1$

$3 = \ln(x) + 1$

$2 = \ln(x)$

$x = e^2$

Point:

$y(e^2) = e^2 \ln(e^2)$

$y(e^2) = 2e^2$

$(e^2, 2e^2)$

Eq^n:

$y = mx + b$

$2e^2 = 3(e^2) + b$

$2e^2 - 3e^2 = b$

$b = -e^2$

$\boxed{y = 3x - e^2}$

7) Find all local extrema for $y = \frac{1}{2}x(2)^{3x+1}$.

1st deriv. test:

	$-\infty$	-0.48	∞
	-1		0
y'	$-$		$+$
y	$\searrow$		$\nearrow$
		min	

$$y' = \frac{1}{2}(2)^{3x+1} + \ln(2)(2)^{3x+1}(3)\left(\frac{1}{2}x\right)$$

$$y' = \frac{1}{2}(2)^{3x+1}\left[1 + 3x\ln(2)\right]$$

$$0 = \frac{1}{2}(2)^{3x+1}\left[1 + 3x\ln(2)\right]$$

$0 = 2^{3x+1}$ No solution

$0 = 1 + 3x\ln(2)$

$x = \frac{-1}{3\ln(2)}$

$x \simeq -0.48$

$y(-0.48) = -0.18$

Min at $(-0.48, -0.18)$

8) Continuous growth or decay follows the formula $A = ce^{kt}$, where *c* is the initial amount, and *k* is a rate factor. The mass of a radioactive substance is 1000 g on day 1, and only 100 g after 100 days. Find ...

a) *k*, then write the equation with *c* and *k*

$100 = 1000e^{k(100)}$

$\frac{1}{10} = e^{100k}$

$\ln(\frac{1}{10}) = 100k$

$k = \frac{\ln(\frac{1}{10})}{100}$

$k \simeq -0.023$

$A(t) = 1000e^{-0.023t}$

b) the half-life,

$500 = 1000e^{-0.023t}$

$\frac{1}{2} = e^{-0.023t}$

$\ln(\frac{1}{2}) = -0.023t$

$t \simeq 30.14$ days

b) the amount that remains after 300 days, and

$A(300) = 1000e^{-0.023(300)}$

$A(300) \simeq 1$ gram.

c) the rate of decay after 50 days.

$A'(t) = 1000e^{-0.023t}(-0.023)$

$A'(t) = -23e^{-0.023t}$

$A'(50) = -23e^{-0.023(50)}$

$A'(50) \simeq -7.28$ g/day.

Answers:

1)a) $y = e^{x \ln b}$ **b)** $\frac{dy}{dx} = \left(e^{x \ln b}\right) \ln b$

2)a) $y' = -3e^{-3x}$ **b)** $f'(x) = 4e^{4x-5}$ **c)** $y' = 2(e^{2x} + e^{-2x})$ **d)** $y' = 2^x(\ln 2) + 3^x(\ln 3)$

e) $f'(x) = 6e^{2x} - 3(2^{3x}) \ln 2$ **f)** $y' = 4xe^x + 4e^x$ **g)** $y' = -(5^x)(e^{-x})(1 - \ln 5)$ **h)** $f'(x) = e^{2x}(2x + 1 - 6e^{-5x})$

3)a) $y' = e^{-x}(\cos x - \sin x)$ **b)** $y' = -\sin x\,(e^{\cos x})$ **c)** $f'(x) = e^{2x}(2x^2 - 4x + 1)$ **d)** $g'(x) = -4xe^{\cos(2x)}[x \sin(2x) - 1]$

4) local max of $y = 0.25$ when $x = \ln(0.5)$

5) $y = 6x + 3$

6) $y = 3x - e^2$

7) CN ~ -0.48, so the point (-0.48,-0.18) is a local minimum

8)a) k ~ -0.023, so the formula is $A = 1000e^{-0.023t}$

b) t ~ 30 days

c) $A(300) \approx 1g$

d) $A'(50) \approx -7.3g/day$