W6 – Implicit Differentiation and Derivatives of Log Functions Unit 3

MCV4U

Jensen

SOLUTIONS

1) For each problem, use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y.

a) $2x^3 = 2y^2 + 5$

$6x^2 = 4y\frac{dy}{dx}$

$\frac{6x^2}{4y} = \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{3x^2}{2y}$

b) $5y^2 = 2x^3 - 5y$

$10y\frac{dy}{dx} = 6x^2 - 5\frac{dy}{dx}$

$10y\frac{dy}{dx} + 5\frac{dy}{dx} = 6x^2$

$\frac{dy}{dx}(10y+5) = 6x^2$

$\frac{dy}{dx} = \frac{6x^2}{10y+5}$

c) $5x^3 = -3xy + 2$

$15x^2 = -3y + \frac{dy}{dx}(-3x)$

$\frac{15x^2+3y}{-3x} = \frac{dy}{dx}$

$\frac{3(5x^2+y)}{-3x} = \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-5x^2-y}{x}$

d) $2x^3 = (3xy+1)^2$

$6x^2 = 2(3xy+1)\left[3y + \frac{dy}{dx}(3x)\right]$

$6x^2 = (6xy+2)\left[3y + \frac{dy}{dx}(3x)\right]$

$6x^2 = 18xy^2 + 18x^2y\frac{dy}{dx} + 6y + 6x\frac{dy}{dx}$

$6x^2 - 18xy^2 - 6y = \frac{dy}{dx}(18x^2y + 6x)$

$\frac{6(x^2-3xy^2-y)}{6(3x^2y+x)} = \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{x^2-3xy^2-y}{3x^2y+x}$

e) $x^3 - 3x^2y + 4xy^2 = 12$

$3x^2 - 6xy + \frac{dy}{dx}(-3x^2) + 4y^2 + 2y\frac{dy}{dx}(4x) = 0$

$\frac{dy}{dx}(8xy - 3x^2) = -3x^2 + 6xy - 4y^2$

$\frac{dy}{dx} = \frac{-3x^2+6xy-4y^2}{8xy-3x^2}$

f) $4\sin(2y)\cos x = 2$

$4\cos(2y)\left(2\frac{dy}{dx}\right)\cos x + (-\sin x)\left[4\sin(2y)\right] = 0$

$8\cos(2y)\cos x\frac{dy}{dx} = 4\sin x\sin(2y)$

$\frac{dy}{dx} = \frac{4\sin x\sin(2y)}{8\cos x\cos(2y)}$

$\frac{dy}{dx} = \frac{\sin x\sin(2y)}{2\cos x\cos(2y)}$

g) $y^2 = \frac{x^2-4}{x^2+4}$

$$2y\frac{dy}{dx} = \frac{2x(x^2+4) - 2x(x^2-4)}{(x^2+4)^2}$$

$$2y\frac{dy}{dx} = \frac{2x(x^2+4-x^2+4)}{(x^2+4)^2}$$

$$2y\frac{dy}{dx} = \frac{16x}{(x^2+4)^2}$$

$$\frac{dy}{dx} = \frac{16x}{2y(x^2+4)^2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{8x}{y(x^2+4)^2}$$

2) Find the equation of the tangent line to $(x+y)^3 = x^3 + y^3$ at the point $(-1,1)$.

Slope:

$$3(x+y)^2\left(1+\frac{dy}{dx}\right) = 3x^2 + 3y^2\frac{dy}{dx}$$

$$3(x^2+2xy+y^2)\left(1+\frac{dy}{dx}\right) = 3x^2 + 3y^2\frac{dy}{dx}$$

$$(3x^2+6xy+3y^2)\left(1+\frac{dy}{dx}\right) = 3x^2 + 3y^2\frac{dy}{dx}$$

$$3x^2+6xy+3y^2+3x^2\frac{dy}{dx}+6xy\frac{dy}{dx}+3y^2\frac{dy}{dx} = 3x^2+3y^2\frac{dy}{dx}$$

$$3x^2\frac{dy}{dx} + 6xy\frac{dy}{dx} = -6xy - 3y^2$$

$$\frac{dy}{dx}(3x^2+6xy) = -3y(2x+y)$$

$$\frac{dy}{dx} = \frac{-3y(2x+y)}{3x(x+2y)}$$

$$\frac{dy}{dx} = \frac{-y(2x+y)}{x(x+2y)}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=-1\\y=1}} = \frac{-1[2(-1)+1]}{-1[-1+2(1)]}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=-1\\y=1}} = \frac{1}{-1} = -1$$

$y = mx+b$

$1 = -1(-1)+b$

$b = 0$

$\boxed{y = -1x}$

3) Differentiate each of the following with respect to x.

$y = \frac{\ln x}{2x+3}$

$y' = \frac{\frac{1}{x}(2x+3) - 2\ln x}{(2x+3)^2} \cdot \frac{x}{x}$

$y' = \frac{2x+3-2x\ln x}{x(2x+3)^2}$

b) $f(x) = e^{x^7}$

$f'(x) = e^{x^7}(7x^6)$

$f'(x) = 7x^6(e)^{x^7}$

c) $f(x) = \ln\left(\frac{x^2+1}{x^3-x}\right)$

$f'(x) = \frac{1}{\left(\frac{x^2+1}{x^3-x}\right)} \cdot \frac{2x(x^3-x) - (3x^2-1)(x^2+1)}{(x^3-x)^2}$

$f'(x) = \frac{x^3-x}{x^2+1} \cdot \frac{2x^4-2x^2-3x^4-3x^2+x^2+1}{(x^3-x)^2}$

$f'(x) = \frac{1}{x^2+1} \cdot \frac{-x^4-4x^2+1}{(x^3-x)^2}$

$f'(x) = \frac{-x^4-4x^2+1}{(x^2+1)(x^3-x)}$

d) $y = \log_2(4x^2)$

$y' = \frac{1}{4x^2\ln(2)} \cdot 8x$

$y' = \frac{2}{x\ln(2)}$

Answers:

1)a) $\frac{dy}{dx} = \frac{3x^2}{2y}$ **b)** $\frac{dy}{dx} = \frac{6x^2}{10y+5}$ **c)** $\frac{dy}{dx} = \frac{-y-5x^2}{x}$ **d)** $\frac{dy}{dx} = \frac{-3y^2x-y+x^2}{3x^2y+x}$ **e)** $\frac{dy}{dx} = \frac{6xy-3x^2-4y^2}{8xy-3x^2}$ **f)** $\frac{dy}{dx} = \frac{\sin(2y)\sin x}{2\cos(2y)\cos x}$

g) $\frac{dy}{dx} = \frac{8x}{y(x^2+4)^2}$

2) $y = -x$

3)a) $y' = \frac{2x+3-2x\ln x}{x(2x+3)^2}$ **b)** $f'(x) = 7x^6e^{x^7}$ **c)** $f'(x) = \frac{2x}{x^2+1} - \frac{3x^2-1}{x^3-x}$ **d)** $y' = \frac{2}{x\ln 2}$