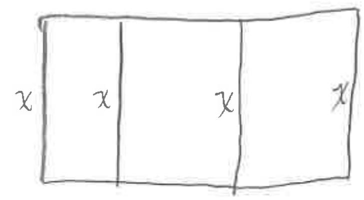
W6 – Optimization Problems Unit 2

MCV4U

Jensen

SOLUTIONS

1) A rectangular pen is to be built with 1200 m of fencing. The pen is to be divided into three parts using two parallel partitions. Find the max possible area of the pen.



width $= x$

$1200 = 4x + 2l$

$\frac{1200-4x}{2} = l$

length $= 600 - 2x$

$A(x) = x(600 - 2x)$

$A(x) = 600x - 2x^2$

Critical Points:

$A'(x) = 600 - 4x$

$0 = 600 - 4x$

$x = 150$

$A(150) = 45\,000 \text{ m}^2$

2nd Derivative Test:

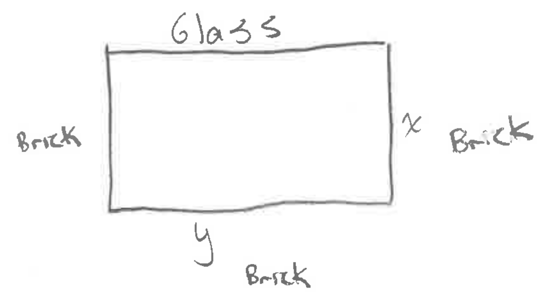
$A''(x) = -4$

$A''(150) = -4$; concave down

$\therefore$ (150, 45000) is a max point.

$\boxed{\text{The max area is } 45000 \text{ m}^2}$

2) A showroom for a car dealership is to be built in the shape of a rectangle with brick on the back and sides, and glass on the front. The floor of the showroom is to have an area of 500 m^2. If a brick wall costs \$1200/m while a glass wall costs \$600/m, what dimensions would minimize the cost of the showroom? What is the min cost?



$xy = 500$

$y = \frac{500}{x}$

$C(x) = 600\left(\frac{500}{x}\right) + 1200\left(\frac{500}{x}\right) + 1200(2x)$

$C(x) = \frac{300\,000}{x} + \frac{600\,000}{x} + 2400x$

$C(x) = \frac{900\,000}{x} + 2400x$

Critical Points:

$C'(x) = -900000x^{-2} + 2400$

$0 = \frac{-900000}{x^2} + 2400$

$-2400x^2 = -900\,000$

$x^2 = 375$

$x \approx 19.36 \text{ m}$

$C(19.36) \simeq \$92951.6$

2nd Derivative Test:

$C''(x) = \frac{1800\,000}{x^3}$

$C''(19.36) \approx 248$

$\therefore$ concave up and (19.36, 92951.6) is a min point. The dimensions that obtain a min cost of \$92951.6 are 19.36 m by 25.83 m.

3) A soup can is to have a capacity of 250 cm^3 and the diameter of the can must be no less than 4 cm and no greater than 8 cm. Wat are the dimensions of the can that can be constructed using the LEAST amount of material?

Domain: $2 \le r \le 4$

$$SA = 2\pi r^2 + 2\pi rh$$

$$SA(r) = 2\pi r^2 + 2\pi r\left(\frac{250}{\pi r^2}\right)$$

$$SA(r) = 2\pi r^2 + \frac{500}{r}$$

$$SA'(r) = 4\pi r - \frac{500}{r^2}$$

$$0 = 4\pi r - \frac{500}{r^2}$$

$$r^3 = \frac{500}{4\pi}$$

$$r = 3.41$$

From Volume Equation:

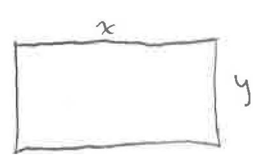
$$V = \pi r^2 h$$
$$250 = \pi r^2 h$$
$$h = \frac{250}{\pi r^2}$$

Test endpoints of domain and critical number:

$SA(2) = 275.1$ cm³
$SA(3.41) = 219.7$ cm³
$SA(2) = 225.5$ cm³

Therefore a $radius = 3.41$ cm and a $height = \frac{250}{\pi(3.41)^2} = 6.84$ cm will minimize the amount of material needed to make the can.

4) A rectangular piece of paper with perimeter 100 cm is to be rolled to form a cylindrical tube. Find the dimensions of the paper that will produce a tube with maximum volume. What is the max volume?



$x = 2\pi r$

$r = \frac{x}{2\pi}$

$100 = 2x + 2y$

$\frac{100 - 2x}{2} = y$

$V(x) = \pi\left(\frac{x}{2\pi}\right)^2(50 - x)$

$V(x) = \pi\left(\frac{x^2}{4\pi^2}\right)(50 - x)$

$V(x) = \frac{x^2}{4\pi}(50 - x)$

$V(x) = \frac{25x^2}{2\pi} \quad \frac{x^3}{4\pi}$

$V(x) - \frac{25}{2\pi}x^2 - \frac{1}{4\pi}x^3$

$V'(x) = \frac{25}{\pi}x - \frac{3}{4\pi}x^2$

$0 = x\left(\frac{25}{\pi} - \frac{3}{4\pi}x\right)$

$x_1 = 0 \qquad 0 = \frac{25}{\pi} - \frac{3x}{4\pi}$

$\frac{3x}{4\pi} = \frac{25}{\pi}$

$3x = 100$

$x = \frac{100}{3}$

$V''(x) = \frac{25}{\pi} - \frac{3}{2\pi}x$

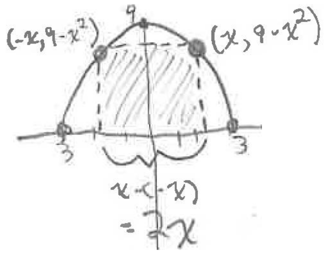
$V''(0) = \frac{25}{\pi}$ ∴ concave up (min)

$V''\left(\frac{100}{3}\right) \approx -8$ ∴ concave down (max)

$V\left(\frac{100}{3}\right) \approx 1473.66\ cm^3$

A max volume of 1473.66 cm³ can be obtained with a length of $\frac{100}{3}$ cm and width of $\frac{50}{3}$ cm

5) Find the area of the largest rectangle that can be inscribed between the x-axis and the graph defined by $y = 9 - x^2$. $= (3-x)(3+x)$



$A(x) = 2x(9-x^2)$

$A(x) = 18x - 2x^3$

$A'(x) = 18 - 6x^2$

$0 = 18 - 6x^2$

$x^2 = 3$

$x = \pm\sqrt{3}$

$A(\sqrt{3}) \simeq 20.78$

$\cancel{A(-\sqrt{3}) \simeq 20.78}$

Can't have $x < 0$

2nd derivative test

$A''(x) = -12x$

$A''(\sqrt{3}) \simeq -20.8$ ∴ concave down (max)

∴ $(\sqrt{3}, 20.78)$ is a max point.

The max area is about 20.78 units^2.

6) For an outdoor concert, a ticket price of \$30 typically attracts 5000 people. For each \$1 increase in the ticket price, 100 fewer people will attend. The revenue, R, is the product of the number of people attending and the price per ticket. Let x equal the number of \$1 increases in price. Find the ticket price that maximizes the revenue. What is the max revenue?

$R(x) = (30+x)(5000-100x)$

$R'(x) = 1(5000-100x) + (-100)(30+x)$

$R'(x) = 2000 - 200x$

$0 = 2000 - 200x$

$x = 10$

2nd derivative test:

$R''(x) = -200$

$R''(10) = -200$ ∴ concave down

∴ 10 price increases will maximize revenue.

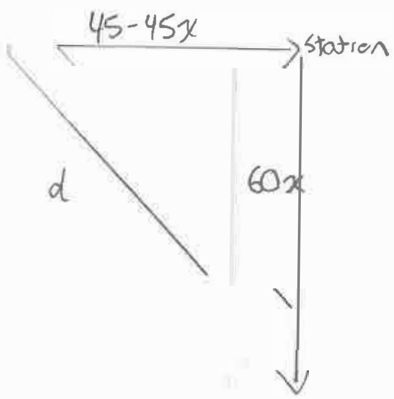
$R(10) = (30+10)(5000-100(10))$

$R(10) = (40)(4000)$

$R(10) = 160\,000$

A \$40 ticket will maximize the revenue to \$160 000.

7) A train leaves the station at 10:00 am and travels SOUTH at a speed of 60 km/h. Another train heading WEST at 45 km/h reaches the same station at 11:00 am. At what time were the two trains closest together?



$$d(x) = \sqrt{(45-45x)^2 + (60x)^2}$$

$$d(x) = \left(2025 - 4050x + 2025x^2 + 3600x^2\right)^{1/2}$$

$$d(x) = \left(5625x^2 - 4050x + 2025\right)^{1/2}$$

$$d'(x) = \frac{1}{2}\left(5625x^2 - 4050x + 2025\right)^{-1/2}(11250x - 4050)$$

$$d'(x) = \frac{5625x - 2025}{\sqrt{5625x^2 - 4050x + 2025}}$$

$$0 = 5625x - 2025$$

$$x = 0.36$$

1st. derivative test:

	$-\infty$	0	0.36	∞
$d'(x)$		$-$		$+$
$d(x)$		decreasing ↘		increasing ↗
		local min		

∴ they are closest together 0.36 hours after 10:00 a.m.

(10:22 am)

Answers:

1) 45000 m^2
2) 19.4 m by 25.8 m; min cost is $92952
3) $r = 3.41$ cm and $h = 6.83$ cm
4) $\frac{50}{3}$ cm by $\frac{100}{3}$ cm; volume is 1473.7 cm^3
5) $12\sqrt{3}$ units^2
6) $40; max revenue is $160 000
7) 0.36 hours after the first train left the station (10:22 am)