L2 –Solving Quadratics by Completing the Square Unit 5
MPM2D
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Part 1: Investigation

Example 1: For the quadratic $y = x^2 + 6x + 8$

$\underline{2} \times \underline{4} = 8$
$\underline{2} + \underline{4} = 6$

a) Determine the x-intercepts by factoring

$0 = x^2 + 6x + 8$
$0 = (x+2)(x+4)$
$x + 2 = 0 \qquad x + 4 = 0$
$\boxed{x_1 = -2} \qquad \boxed{x_2 = -4}$

b) Determine the vertex using the x-intercepts

$x\text{-vertex} = \frac{-2+(-4)}{2} = \frac{-6}{2} = -3$

$y\text{-vertex} = (-3)^2 + 6(-3) + 8 = -1$

$\boxed{\text{the vertex is } (-3, -1)}$

c) Convert to vertex form by completing the square

$y = (x^2 + 6x) + 8$
$y = (x^2 + 6x + 9 - 9) + 8$
$y = (x^2 + 6x + 9) - 9 + 8$
$y = (x+3)^2 - 1$

d) Solve for the x-intercepts by rearranging the vertex form equation to isolate for x.

$0 = (x+3)^2 - 1$
$1 = (x+3)^2$
$\pm\sqrt{1} = x + 3$
$-3 \pm \sqrt{1} = x$

$x_1 = -3 + 1 \qquad x_2 = -3 - 1$
$\boxed{x_1 = -2} \qquad \boxed{x_2 = -4}$

A quadratic equation in standard form, $0 = ax^2 + bx + c$, can be solved for x by first converting into vertex form, $0 = a(x-h)^2 + k$, by completing the square, then rearranging to isolate for x.

Part 2: Rational Solutions

Example 2: Solve each of the following equations by completing the square.

a) $0 = x^2 - 12x + 20$

$0 = (x^2 - 12x + 36 - 36) + 20$

$0 = (x^2 - 12x + 36) - 36 + 20$

$0 = (x-6)^2 - 16$

$16 = (x-6)^2$

$\pm\sqrt{16} = x - 6$

$x = 6 \pm 4$

$x_1 = 6+4 \quad x_2 = 6-4$

$\boxed{x_1 = 10} \quad \boxed{x_2 = 2}$

b) $10 = 2x^2 - x$

$0 = 2x^2 - x - 10$

$0 = 2(x^2 - \frac{1}{2}x) - 10$

$0 = 2(x^2 - \frac{1}{2}x + \frac{1}{16} - \frac{1}{16}) - 10$

$0 = 2(x^2 - \frac{1}{2}x + \frac{1}{16}) - \frac{1}{8} - \frac{80}{8}$

$0 = 2(x - \frac{1}{4})^2 - \frac{81}{8}$

$\frac{\left(\frac{81}{8}\right)}{2} = (x - \frac{1}{4})^2$

$\pm\sqrt{\frac{81}{16}} = x - \frac{1}{4}$

$\frac{1}{4} \pm \frac{9}{4} = x$

$x_1 = \frac{1}{4} + \frac{9}{4} \quad x_2 = \frac{1}{4} - \frac{9}{4}$

$x_1 = \frac{10}{4} \quad x_2 = \frac{-8}{4}$

$\boxed{x_1 = \frac{5}{2}} \quad \boxed{x_2 = -2}$

Notice for each of the equations in this question there were **RATIONAL** solutions. That means that each of the equations could have been solved by **factoring**.

Part 3: Irrational Solutions

Example 3: Solve each of the following quadratic equations by completing the square.

a) $-3x^2 + 6x + 7 = 0$

$-3(x^2 - 2x) + 7 = 0$

$-3(x^2 - 2x + 1 - 1) + 7 = 0$

$-3(x^2 - 2x + 1) + 3 + 7 = 0$

$-3(x-1)^2 + 10 = 0$

$(x-1)^2 = \frac{-10}{-3}$

$x - 1 = \pm\sqrt{\frac{10}{3}}$

$x = 1 \pm \sqrt{\frac{10}{3}}$

$\boxed{x_1 = 1 + \sqrt{\frac{10}{3}} \quad x_2 = 1 - \sqrt{\frac{10}{3}}}$ Exact

$\boxed{x_1 \simeq 2.83 \quad x_2 \simeq -0.83}$ Approximate

b) $x^2 - 6x - 8 = 0$

$(x^2 - 6x) - 8 = 0$

$(x^2 - 6x + 9 - 9) - 8 = 0$

$(x^2 - 6x + 9) - 9 - 8 = 0$

$(x-3)^2 - 17 = 0$

$(x-3)^2 = 17$

$x - 3 = \pm\sqrt{17}$

$x = 3 \pm \sqrt{17}$

$\boxed{x_1 = 3 + \sqrt{17} \quad x_2 = 3 - \sqrt{17}}$ Exact

$\boxed{x_1 \simeq 7.12 \quad x_2 \simeq -1.12}$ Approximate

Notice for each of the equations in this question there were only **IRRATIONAL** solutions. That means that each of the equations could **NOT** have been solved by **factoring**. This is why solving by completing the square is a useful strategy.

Part 4: Non-Real Solutions

Example 4: Solve the following quadratic equation

$$x^2 - 6x + 10 = 0$$

$$(x^2 - 6x + 9 - 9) + 10 = 0$$

$$(x^2 - 6x + 9) - 9 + 10 = 0$$

$$(x-3)^2 + 1 = 0$$

$$(x-3)^2 = -1$$

$$x - 3 = \pm\sqrt{-1}$$

$$x = 3 \pm \sqrt{-1}$$

not a real number

$\boxed{\therefore \text{ no real solutions.}}$

Notice that not all quadratics have **REAL solutions**.

Part 5: Application

Example 5: For the quadratic $y = 2x^2 - 5x + 3$, determine the vertex, x-intercepts, and then sketch a graph of the function.

$$0 = 2x^2 - 5x + 3 \qquad -2 \times -3 = 6$$

$$0 = 2x^2 - 2x - 3x + 3 \qquad -2 + -3 = -5$$

$$0 = 2x(x-1) - 3(x-1)$$

$$0 = (x-1)(2x-3)$$

$$x - 1 = 0 \qquad 2x - 3 = 0$$

$$\boxed{x_1 = 1} \qquad 2x = 3$$

$$\boxed{x_2 = \frac{3}{2} = 1.5}$$

$$x\text{-vertex} = \frac{1 + 1.5}{2} = 1.25$$

$$y\text{-vertex} = 2(1.25)^2 - 5(1.25) + 3$$
$$= -0.125$$

$\boxed{\text{vertex: } (1.25, -0.125)}$