L3 –Solving Quadratics using the Quadratic Formula Unit 5

MPM2D

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Part 1: Proof of Quadratic Formula

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

is a formula that can be used to find rational, irrational, and non-real solutions to any quadratic equation that is in the form $ax^2 + bx + c = 0$. It is a formula that is derived from completing the square of a general standard form quadratic and then rearranging to isolate for x. Using the formula is just a shortcut to avoid having to complete the square and rearrange like we did last lesson.

Proof:

$$ax^2 + bx + c = 0$$

$$a\left(x^2 + \frac{b}{a}x\right) + c = 0$$

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c = 0$$

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{\cancel{a}b^2}{4a^{\cancel{2}\,1}} + c = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - \frac{c}{1} \times \frac{4a}{4a}$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Part 2: Rational Solutions

Example 2: Solve each of the following equations using the Quadratic Formula.

a) $0 = \overset{a}{1}x^2 \overset{b}{- 7}x \overset{c}{+ 12}$

$$x = \frac{7 \pm \sqrt{(-7)^2 - 4(1)(12)}}{2(1)}$$

$$x = \frac{7 \pm \sqrt{1}}{2}$$

$$x_1 = \frac{7+1}{2} \qquad x_2 = \frac{7-1}{2}$$

$$\boxed{x_1 = 4} \qquad \boxed{x_2 = 3}$$

b) $0 = \overset{a}{3}x^2 \overset{b}{+ 11}x \overset{c}{- 4}$

$$x = \frac{-11 \pm \sqrt{(11)^2 - 4(3)(-4)}}{2(3)}$$

$$x = \frac{-11 \pm \sqrt{169}}{6}$$

$$x_1 = \frac{-11+13}{6} \qquad x_2 = \frac{-11-13}{6}$$

$$x_1 = \frac{2}{6} \qquad x_2 = \frac{-24}{6}$$

$$\boxed{x_1 = \frac{1}{3}} \qquad \boxed{x_2 = -4}$$

c) $-9 = x^2 - 6x$

$$0 = \overset{a}{1}x^2 \overset{b}{- 6}x \overset{c}{+ 9} = 0$$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{6 \pm \sqrt{0}}{2}$$

$$x = \frac{6}{2}$$

$$\boxed{x = 3}$$

Notice for each of the equations in this question there were **RATIONAL** solutions. That means that each of the equations could have been solved by factoring. But quadratic formula also works!

Part 3: Irrational Solutions

Example 3: Solve each of the following quadratic equations

a) $0 = \overset{a}{3}x^2 + \overset{b}{8}x - \overset{c}{5}$

$x = \frac{-8 \pm \sqrt{(8)^2 - 4(3)(-5)}}{2(3)}$

$x = \frac{-8 \pm \sqrt{124}}{6}$

$\boxed{x_1 = \frac{-8+\sqrt{124}}{6} \qquad x_2 = \frac{-8-\sqrt{124}}{6}}$ exact

$\boxed{x_1 \simeq 0.52 \qquad x_2 \simeq -3.19}$ approximate

b) $-5 = -x^2 - 10x$

$\overset{a}{1}x^2 + \overset{b}{10}x - \overset{c}{5} = 0$

$x = \frac{-10 \pm \sqrt{(10)^2 - 4(1)(-5)}}{2(1)}$

$x = \frac{-10 \pm \sqrt{120}}{2}$

$\boxed{x_1 = \frac{-10+\sqrt{120}}{2} \qquad x_2 = \frac{-10-\sqrt{120}}{2}}$ exact

$\boxed{x_1 \simeq 0.48 \qquad x_2 \simeq -10.48}$ approx.

Notice for each of the equations in this question there were only **IRRATIONAL** solutions. That means that each of the equations could NOT have been solved by factoring. This is why the Quadratic formula is so useful!

Part 4: Non-Real Solutions

Example 4: Solve each of the following quadratic equations

a) $0 = \overset{a}{1}x^2 - \overset{b}{5}x + \overset{c}{11}$

$x = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(11)}}{2(1)}$

$x = \frac{5 \pm \sqrt{-19}}{2}$

$\therefore$ no real solutions

b) $-7 = 2x^2 - 4x$

$0 = \overset{a}{2}x^2 - \overset{b}{4}x + \overset{c}{7}$

$x = \frac{4 \pm \sqrt{(-4)^2 - 4(2)(7)}}{2(2)}$

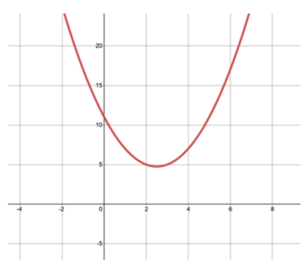
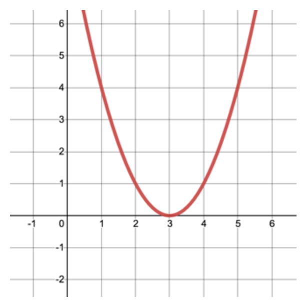
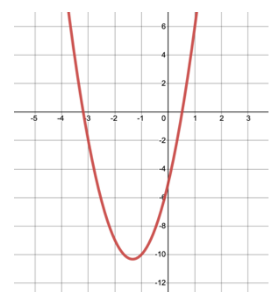
$x = \frac{4 \pm \sqrt{40}}{4}$

$\therefore$ no real solutions

There are no real solutions when under the square root (the $\underbrace{\text{discriminant}}_{b^2-4ac}$) is **less than zero**. You can't square root a negative!

Part 5: The Discriminant

Looking back through the previous parts of the lesson, you can see that quadratic equations can have either zero, one, or two solutions. The discriminant, $b^2 - 4ac$, is the part of the equation that tells you how many solutions a quadratic equation will have.

# of solutions	0	1	2
Discriminant	$b^2 - 4ac < 0$	$b^2 - 4ac = 0$	$b^2 - 4ac > 0$
Example Equation	$0 = x^2 - 5x + 11$ $x = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(11)}}{2(1)}$ $x = \frac{5 \pm \sqrt{-19}}{2}$ No real solutions Notice $b^2 - 4ac = -19$ which is less than zero	$0 = x^2 - 6x + 9$ $x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(9)}}{2(1)}$ $x = \frac{6 \pm \sqrt{0}}{2(1)}$ $x = 3$ Notice $b^2 - 4ac = 0$	$0 = 3x^2 + 8x - 5$ $x = \frac{-8 \pm \sqrt{(8)^2 - 4(3)(-5)}}{2(3)}$ $x = \frac{-8 \pm \sqrt{124}}{6}$ $x = -3.19, 0.52$ Notice $b^2 - 4ac = 124$ which is greater than zero
Example Graph	 Notice the graph never crosses the x-axis	 Notice the vertex is ON the x-axis	 Notice the graph crosses through the x-axis twice

Example 5: How many real solutions does each quadratic equation have?

a) $0 = \overset{a}{1}x^2 + \overset{b}{10}x + \overset{c}{25}$

$b^2 - 4ac = 100 - 4(1)(25)$
$= 0$

1 real solution

b) $0 = \overset{a}{-2}x^2 + \overset{b}{2}x + \overset{c}{7}$

$b^2 - 4ac = (2)^2 - 4(-2)(7)$
$= 60$

2 real solutions

c) $0 = \overset{a}{2}x^2 + \overset{b}{2}x + \overset{c}{7}$

$b^2 - 4ac = (2)^2 - 4(2)(7)$
$= -52$

0 real solutions

Another piece of useful information about the discriminant:

If $b^2 - 4ac$ is a **PERFECT SQUARE** number, you will get rational solutions which means the quadratic is factorable. If you aren't sure if a quadratic is factorable, just check to see if $b^2 - 4ac$ is a perfect square number $(0,1,4,9,16,25,36, \ldots)$