Unit 6 Pre-Test Review – Rational Inequalities and Rates of Change
MHF4U
Jensen

SOLUTIONS

Section 1: Rational Equations and Inequalities

1) Solve the following equations

a) $\frac{10}{x+1} = x - 2$

$\cancel{(x+1)}\left(\frac{10}{\cancel{x+1}}\right) = (x+1)(x-2)$

$10 = x^2 - x - 2$

$0 = x^2 - x - 12$

$0 = (x-4)(x+3)$

$x - 4 = 0$ $\quad$ $x + 3 = 0$

$\boxed{x_1 = 4}$ $\quad$ $\boxed{x_2 = -3}$

b) $\frac{x+3}{x-1} = 2x + 1$

$\cancel{(x-1)}\left(\frac{x+3}{\cancel{x-1}}\right) = (x-1)(2x+1)$

$x + 3 = 2x^2 - x - 1$

$0 = 2x^2 - 2x - 4$

$0 = 2(x^2 - x - 2)$

$0 = x^2 - x - 2$

$0 = (x-2)(x+1)$

$\boxed{x_1 = 2}$ $\quad$ $\boxed{x_2 = -1}$

c) $\frac{3}{3x+2} = \frac{6}{5x}$

$3(5x) = 6(3x+2)$

$15x = 18x + 12$

$-3x = 12$

$\boxed{x = -4}$

d) $\frac{x-2}{x+3} = \frac{x-4}{x+5}$

$(x+5)(x-2) = (x-4)(x+3)$

$x^2 + 3x - 10 = x^2 - x - 12$

$3x + x = -12 + 10$

$4x = -2$

$x = \frac{-1}{2}$

e) $\frac{2}{x} + \frac{5}{3} = \frac{7}{x}$

$3x\left(\frac{2}{x} + \frac{5}{3}\right) = 3x\left(\frac{7}{x}\right)$

$3(2) + 5(x) = 3(7)$

$6 + 5x = 21$

$5x = 15$

$\boxed{x = 3}$

f) $\frac{10}{x+3} + \frac{10}{3} = 6$

$3(x+3)\left(\frac{10}{x+3} + \frac{10}{3}\right) = 3(x+3)(6)$

$3(10) + 10(x+3) = 18(x+3)$

$30 + 10x + 30 = 18x + 54$

$6 = 8x$

$\boxed{x = \frac{3}{4}}$

g) $\frac{3}{x} + \frac{4}{x+1} = 2$

$x(x+1)\left(\frac{3}{x} + \frac{4}{x+1}\right) = x(x+1)(2)$

$3(x+1) + 4(x) = 2x(x+1)$

$3x + 3 + 4x = 2x^2 + 2x$

$0 = 2x^2 - 5x - 3$

$0 = (x-3)(2x+1)$

P: -6, S: -5; $\frac{-3}{1}$, $\frac{-6}{2}$, $\frac{1}{2}$

$\boxed{x_1 = 3}$ $\boxed{x_2 = -\frac{1}{2}}$

h) $3 = \frac{6}{2x^2 - x - 4}$

$3(2x^2 - x - 4) = \frac{6}{\cancel{2x^2 - x - 4}}\cancel{(2x^2 - x - 4)}$

$2x^2 - x - 4 = 2$

$2x^2 - x - 6 = 0$

$2x^2 - 4x + 3x - 6 = 0$

$2x(x-2) + 3(x-2) = 0$

$(x-2)(2x+3) = 0$

$\boxed{x_1 = 2}$ $\boxed{x_2 = -\frac{3}{2}}$

2) Solve the following rational inequalities using a factor table

a) $\frac{6x}{x+1} > 4$

$\frac{6x}{x+1} - 4 > 0$

$\frac{6x}{x+1} - \frac{4(x+1)}{x+1} > 0$

$\frac{6x - 4x - 4}{x+1} > 0$

$\frac{2x-4}{x+1} > 0$

$\frac{2(x-2)}{x+1} > 0$

x-int: $x = 2$

restrictions: $x \neq -1$

	$-\infty$ to -1 (-2)	-1 to 2 (0)	2 to ∞ (3)
2	+	+	+
$x-2$	−	−	+
$x+1$	−	+	+
overall	+	−	+

Solution: $x < -1$ or $x > 2$

$x \in (-\infty, -1) \cup (2, \infty)$

b) $\frac{1}{2x+10} \geq \frac{1}{x+3}$

$\frac{1}{2x+10} - \frac{1}{x+3} \geq 0$

$\frac{1(x+3) - 1(2x+10)}{(2x+10)(x+3)} \geq 0$

$\frac{x+3-2x-10}{(2x+10)(x+3)} \geq 0$

$\frac{-x-7}{(2x+10)(x+3)} \geq 0$

x-int: $x = -7$

restrictions: $x \neq -5, -3$

	$-\infty$ to -7 (-8)	-7 to -5 (-6)	-5 to -3 (-4)	-3 to ∞ (0)
$-x-7$	+	−	−	−
$2x+10$	−	−	+	+
$x+3$	−	−	−	+
overall	+	−	+	−

Solutions:

$x \leq -7$ or $-5 < x < -3$

$x \in (-\infty, -7] \cup (-5, -3)$

c) $\frac{2x-3}{x+5} \geq \frac{2x+7}{x-3}$

$\frac{2x-3}{x+5} - \frac{2x+7}{x-3} \geq 0$

$\frac{(2x-3)(x-3) - (2x+7)(x+5)}{(x+5)(x-3)} \geq 0$

$\frac{2x^2 - 6x - 3x + 9 - (2x^2 + 10x + 7x + 35)}{(x+5)(x-3)} \geq 0$

$\frac{2x^2 - 9x + 9 - 2x^2 - 17x - 35}{(x+5)(x-3)} \geq 0$

$\frac{-26x - 26}{(x+5)(x-3)} \geq 0$

$\frac{-26(x+1)}{(x+5)(x-3)} \geq 0$

x-int: $x = -1$

restrictions: $x \neq -5, 3$

	$-\infty$ to -5 (-6)	-5 to -1 (-2)	-1 to 3 (0)
-26	−	−	−
$x+1$	−	−	+
$x+5$	−	+	+
$x-3$	−	−	−
overall	+	−	+

Solution:

$x < -5$ or $-1 \leq x < 3$

$x \in (-\infty, -5) \cup [-1, 3)$

d) $\frac{7}{x-3} \geq \frac{2}{x+4}$

$\frac{7}{x-3} - \frac{2}{x+4} \geq 0$

$\frac{7(x+4) - 2(x-3)}{(x-3)(x+4)} \geq 0$

$\frac{7x + 28 - 2x + 6}{(x-3)(x+4)} \geq 0$

$\frac{5x + 34}{(x-3)(x+4)} \geq 0$

x-int: $x = -6.8$

restrictions: $x \neq -4, 3$

	$-\infty$ to -6.8 (-7)	-6.8 to -4 (-5)	-4 to 3 (0)	3 to ∞ (4)
$5x+34$	−	+	+	+
$x-3$	−	−	−	+
$x+4$	−	−	+	+
overall	−	+	−	+

SOLUTION:

$-6.8 \leq x < -4$ or $x > 3$

$x \in [-6.8, -4) \cup (3, \infty)$

e) $\frac{x^2-x-12}{x-1} < 0$

$$\frac{(x-4)(x+3)}{x-1} < 0$$

x-int: $x = -3, 4$

restrictions: $x \neq 1$

	$-\infty$ -4	-3 0	1 2	4 5 ∞
$x-4$	−	−	−	+
$x+3$	−	+	+	+
$x-1$	−	−	+	+
overall	−	+	−	+

Solution:

$x < -3$ or $1 < x < 4$

$x \in (-\infty, -3) \cup (1, 4)$

f) $\frac{6x^2-5x+1}{2x+1} < 0$

P: 6, S: -5

$-\frac{1}{3} = -\frac{2}{6}$ $-\frac{3}{6} = -\frac{1}{2}$

$$\frac{(3x-1)(2x-1)}{2x+1} < 0$$

x-int: $x = \frac{1}{3}, \frac{1}{2}$

restrictions: $x \neq -\frac{1}{2}$

	$-\infty$ -1	$-\frac{1}{2}$ 0	$\frac{1}{3}$ 0.4	$\frac{1}{2}$ 1 ∞
$3x-1$	−	−	+	+
$2x-1$	−	−	−	+
$2x+1$	−	+	+	+
overall	−	+	−	+

Solution:

$x < -\frac{1}{2}$ or $\frac{1}{3} < x < \frac{1}{2}$

$x \in (-\infty, -\frac{1}{2}) \cup (\frac{1}{3}, \frac{1}{2})$

g) $\frac{2x-10}{x} > x - 5$

$$\frac{2x-10}{x} - x + 5 > 0$$

$$\frac{2x-10}{x} - \frac{x^2}{x} + \frac{5x}{x} > 0$$

$$\frac{-x^2+7x-10}{x} > 0$$

$$\frac{-1(x^2-7x+10)}{x} > 0$$

$$\frac{-1(x-5)(x-2)}{x} > 0$$

x-int: $x = 2, 5$

restrictions: $x \neq 0$

	$-\infty$ -1	0 1	2 3	5 6 ∞
-1	−	−	−	−
$x-5$	−	−	−	+
$x-2$	−	−	+	+
x	−	+	+	+
overall	+	−	+	−

SOLUTION:

$x < 0$ or $2 < x < 5$

$x \in (-\infty, 0) \cup (2, 5)$

Section 2: Average and Instantaneous Rates of Change

3) Consider the data below for a car tire with a leak:

Minutes after the leak began	0	5	10	15	20	25	30
Pressure of air in the tire in kilopascals (kPa)	400	335	295	255	225	195	170

a) Calculate the average rate of change over the 30 minute interval. Explain the meaning of this rate using proper units.

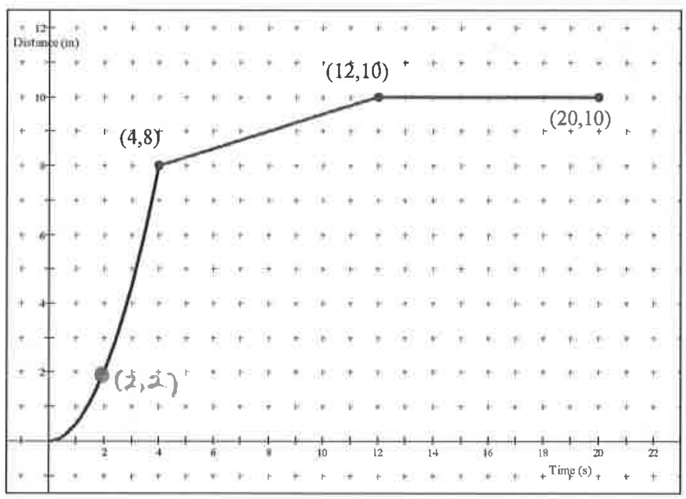
$$m = \frac{\Delta y}{\Delta x} = \frac{170-400}{30-0} = \frac{-230}{30} \simeq -7.67 \text{ kPa/min}$$

Over the 30 minute span, the tire lost 7.67 kPa of air each minute, on average.

b) Estimate the instantaneous rate of change at 5 minutes using a surrounding interval.

$$\left.\frac{dy}{dx}\right|_{x=5} \approx \frac{295-400}{10-0} = \frac{-105}{10} = -10.5 \text{ kPa/min.}$$

4) The graph to the right represents the escape of a vole that was frightened by a hawk flying by. Describe the motion of the vole as suggested by the graph.



a) What is the average speed of the vole on the intervals…

i) $[0, 4]$ $m = \frac{8-0}{4-0} = \frac{8}{4} = 2$ m/s

ii) $[4, 12]$ $m = \frac{10-8}{12-4} = \frac{2}{8} = 0.25$ m/s

4
iii) $[\cancel{12}, 20]$ $m = \frac{10-8}{20-4} = \frac{2}{16} = 0.125$ m/s

b) Estimate the instantaneous rate of change (speed) of the vole at 2 seconds. Use an average of 2 secant lines.

m for [0,2]

$m = \frac{2-0}{2-0} = \frac{2}{2} = 1$ m/s

m for [2,4]

$m = \frac{8-2}{4-2} = \frac{6}{2} = 3$ m/s

$\left.\frac{dy}{dx}\right|_{x=2} \approx \frac{1+3}{2} = 2$ m/s.

5) For the function $f(x) = x^2 - 3x + 2$

a) Calculate the average rate of change for the following intervals

i) $-1 \leq x \leq 2$

$m = \frac{f(2) - f(-1)}{2-(-1)}$

$= \frac{0-6}{3}$

$= -\frac{6}{3}$

$= -2$

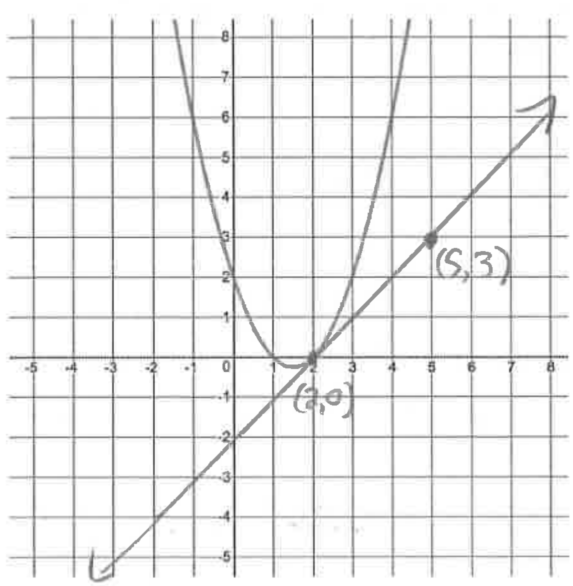
ii) $4 \leq x \leq 8$

$m = \frac{f(8) - f(4)}{8-4}$

$= \frac{42-6}{4}$

$= 9$

b) Use the graph of the function to estimate the instantaneous rate of change at $x = 2$ by drawing a tangent line and calculating it's slope.



$m = \frac{\Delta y}{\Delta x}$

$= \frac{3-0}{5-2}$

$= \frac{3}{3}$

$= 1$

$\therefore \left.\frac{dy}{dx}\right|_{x=2} \approx 1$

c) Verify your answers from part b) by calculating the LIMIT of the secant slopes as you approach $x = 2$.

Interval	Δy	Δx	Slope of secant $= \frac{\Delta y}{\Delta x}$
$2 \le x \le 2.5$	$= f(2.5) - f(2)$ $= 0.75 - 0$ $= 0.75$	$= 2.5 - 2$ $= 0.5$	$= \frac{0.75}{0.5}$ $= 1.5$
$2 \le x \le 2.1$	$= f(2.1) - f(2)$ $= 0.11 - 0$ $= 0.11$	$= 2.1 - 2$ $= 0.1$	$= \frac{0.11}{0.1}$ $= 1.1$
$2 \le x \le 2.01$	$= f(2.01) - f(2)$ $= 0.0101 - 0$ $= 0.0101$	$= 2.01 - 2$ $= 0.01$	$= \frac{0.0101}{0.01}$ $= 1.01$
$2 \le x \le 2.001$	$= f(2.001) - f(2)$ $= 0.001001 - 0$ $= 0.001001$	$= 2.001 - 2$ $= 0.001$	$= \frac{0.001001}{0.001}$ $= 1.001$

Estimate of instantaneous rate of change at $x = 2$ … $\left.\frac{dy}{dx}\right|_{x=2} \simeq 1$

6) Use the data below for the temperature in degrees Celsius for a wood fire oven.

Time in Minutes	0	5	8	10	13	15	19	21	25
Temp (°C)	25	120	205	250	290	280	290	285	285

a) Find the average rate of change of the temperature between 0 and 25 minutes. Show proper units and notation.

$$m = \frac{\Delta y}{\Delta x} = \frac{285 - 25}{25 - 0} = \frac{260}{25} = 10.4 \text{ °C/min}$$

b) Estimate the instantaneous rate of change of the temperature at 10 minutes. Use 2 methods.

Method 1: surrounding interval

$$\left.\frac{dy}{dx}\right|_{x=10} \simeq \frac{290-205}{13-8} = \frac{85}{5} = 17 \text{ °C/min}$$

Method 2: Average preceding & following

m for [8,10]

$$m = \frac{250-205}{10-8} = \frac{45}{2} = 22.5 \text{ °C/min}$$

m for [10,13]

$$= \frac{290-250}{13-10} = \frac{40}{3} = 13.3 \text{ °C/min}$$

$$\left.\frac{dy}{dx}\right|_{x=10} \simeq \frac{22.5+13.3}{2} = 17.9 \text{ °C/min}$$

Section 3: Newton Quotient

7) Find the equation of the derivative for each of the following functions. Also, find the instantaneous rate of ange for the function when $x = -2$ and $x = 3$.

a) $f(x) = 4x - 1$

$$f'(x) = \lim_{h\to 0} \frac{4(x+h) - 1 - (4x-1)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{4x + 4h - 1 - 4x + 1}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{4h}{h}$$

$$\boxed{f'(x) = 4}$$

$$\boxed{f'(-2) = 4}$$

$$\boxed{f'(3) = 4}$$

b) $f(x) = 3x^2 - 5x + 2$

$$f'(x) = \lim_{h\to 0} \frac{3(x+h)^2 - 5(x+h) + 2 - (3x^2 - 5x + 2)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{3(x^2 + 2xh + h^2) - 5x - 5h + 2 - 3x^2 + 5x - 2}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{5x} - 5h + \cancel{2} - \cancel{3x^2} + \cancel{5x} - \cancel{2}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{6xh + 3h^2 - 5h}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{h}(6x + 3h - 5)}{\cancel{h}}$$

$$f'(x) = 6x + 3(0) - 5$$

$$\boxed{f'(x) = 6x - 5}$$

$$\boxed{f'(-2) = -17}$$

$$\boxed{f'(3) = 13}$$

[redacted]

c) $f(x) = -2x^3 + 3x^2$

$$f'(x) = \lim_{h\to 0} \frac{-2(x+h)^3 + 3(x+h)^2 - (-2x^3 + 3x^2)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{-2(x+h)(x^2 + 2xh + h^2) + 3(x^2 + 2xh + h^2) + 2x^3 - 3x^2}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{-2(x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3) + \cancel{3x^2} + 6xh + 3h^2 + 2x^3 - \cancel{3x^2}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{-2(x^3 + 3x^2h + 3xh^2 + h^3) + 6xh + 3h^2 + 2x^3}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{-\cancel{2x^3} - 6x^2h - 6xh^2 - 2h^3 + 6xh + 3h^2 + \cancel{2x^3}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{h}(-6x^2 - 6xh - 2h^2 + 6x + 3h)}{\cancel{h}}$$

$$f'(x) = -6x^2 - 6x(0) - 2(0)^2 + 6x$$

$$\boxed{f'(x) = -6x^2 + 6x}$$

$$f'(-2) = -6(-2)^2 + 6(-2)$$

$$\boxed{f'(-2) = -36}$$

$$f(3) = -6(3)^2 + 6(3)$$

$$\boxed{f(3) = -36}$$

8) Determine the equation of the tangent line at $x = -2$ for the function in part $f(x) = 3x^2 - 5x + 2$

$f(-2) = 3(-2)^2 - 5(-2) + 2$

$f(-2) = 24$

∴ $(-2, 24)$ is on tangent line

$f'(-2) = -17$

∴ slope of tangent line is -17

$y = mx + b$

$24 = -17(-2) + b$

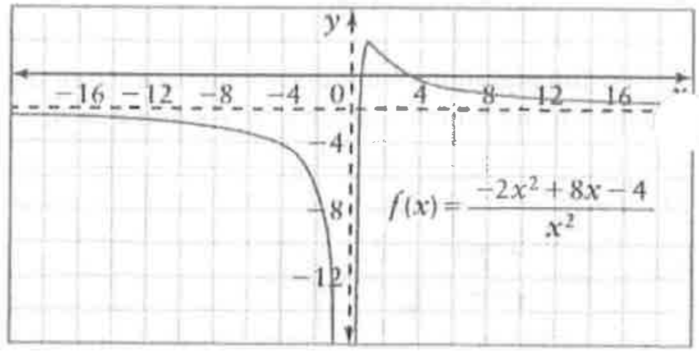
$24 = 34 + b$

$b = -10$

$$\boxed{y = -17x - 10}$$

Section 4: Limits

9) Use the graph to find the following limits



a) $\lim_{x\to\infty} f(x)$ $= -2$

b) $\lim_{x\to-\infty} f(x)$ $= -2$

c) $\lim_{x\to 0^+} f(x)$ $= -\infty$

d) $\lim_{x\to 0^-} f(x)$ $= -\infty$

e) $\lim_{x\to 0} f(x)$ $= -\infty$

10) Evaluate each limit if it exists

a) $\lim_{x\to3} \frac{-x^2+8x}{2x+1}$

$= \dfrac{-(3)^2+8(3)}{2(3)+1}$

$= \dfrac{15}{7}$

b) $\lim_{x\to-2} \frac{3x^2+5x-2}{x^2-2x-8}$

P

$\left(\frac{2}{1}\right) = \frac{6}{3}$ -6 $\left(\frac{-1}{3}\right)$ 5 5

$= \lim_{x\to-2} \dfrac{\cancel{(x+2)}(3x-1)}{\cancel{(x+2)}(x-4)}$

$= \lim_{x\to-2} \dfrac{3x-1}{x-4}$

$= \dfrac{3(-2)-1}{-2-4}$

$= \dfrac{-7}{-6}$ $= \dfrac{7}{6}$

c) $\lim_{x\to7} \frac{x^2-49}{x-7}$

$= \lim_{x\to7} \dfrac{\cancel{(x-7)}(x+7)}{\cancel{x-7}}$

$= \lim_{x\to7} x+7$

$= 7+7$

$= 14$

d) $\lim_{x\to0} \frac{9x}{2x^2-5x}$

$= \lim_{x\to0} \dfrac{9\cancel{x}}{\cancel{x}(2x-5)}$

$= \lim_{x\to0} \dfrac{9}{2x-5}$

$= \dfrac{9}{2(0)-5}$

$= \dfrac{9}{-5}$

$= -\dfrac{9}{5}$

Answer Key

1)a) -3, 4 **b)** -1, 2 **c)** -4 **d)** $-\frac{1}{2}$ **e)** 3 **f)** $\frac{3}{4}$ **g)** $3, -0.5$

2)a) $x < -1$ or $x > 2$ **b)** $-7 < x < -5$ or $x > -3$ **c)** $x < -5$ or $-1 \le x < 3$
d) $6.8 \le x < -4$ or $x > 3$ **e)** $x < -3$ or $1 < x < 4$ **f)** $-\frac{1}{2} < x < \frac{1}{3}$ or $x > \frac{1}{2}$ **g)** $x < -5$ or $-2 < x < 0$

3)a) $m = -7.67$ kPa/min, which means over the 30-minute interval, the tire lost 7.67 kPa of air pressure every minute on average. **b)** $\left.\frac{dy}{dx}\right|_{t=5} \approx -10.5$ kPa/min

4)a)i) $m = 2$ m/s **ii)** $m = 0.25$ m/s **iii)** $m = 0.125$ m/s
b) $\left.\frac{dy}{dx}\right|_{t=2} \approx 2$ m/s

5)a)i) $m = -2$ **ii)** $m = 9$ **b)c)** $\left.\frac{dy}{dx}\right|_{x=2} \approx 1$

6)a) $m = 10.4$ °C/min **b)** surrounding interval: $\left.\frac{dy}{dx}\right|_{x=10} \approx 17$ °C/min, average intervals: $\left.\frac{dy}{dx}\right|_{x=10} \approx 17.9$ °C/min

7)a) $f'(x) = 4, f(-2) = 4, f(3) = 4$ **b)** $f'(x) = 6x - 5, f(-2) = -17, f(3) = 13$
c) $f'(x) = -6x^2 + 6x, f(-2) = -36, f(3) = -36$ **d)** $f'(x) = x^2 - 10x, f(-2) = 24, f(3) = -21$

8) $y = -17x - 10$

9)a) -2 **b)** -2 **c)** $-\infty$ **d)** $-\infty$ **e)** $-\infty$

10)a) $\frac{15}{17}$ **b)** $\frac{7}{6}$ **c)** 14 **d)** $-\frac{9}{5}$